SEMINAR TALK

VIPUL NAIK

0.1. Lazard correspondence. *Say time*: 2 minutes

The global Lazard correspondence is a correspondence:

Some groups (p-groups of class less than p) $\leftrightarrow$ Some Lie rings (p-Lie rings of class less than p)

Put arrow-marked exp *and* log

The group and Lie ring have the same underlying set, and there are formulas for the group operations in terms of the Lie ring operations (and vice versa) that are inverses of each other.

- Baker-Campbell-Hausdorff formula: group multiplication in terms of Lie ring operations
- Inverse Baker-Campbell-Hausdorff formulas: Lie ring addition and Lie bracket in terms of group multiplication

Baker-Campbell-Hausdorff formula:

$$xy + x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] - [y, [x, y]]) + \dots$$

Say, don't write: Note that the formula involves division by some numbers. The weight c term in the formula involves division by a number all of whose prime divisors are less than or equal to c. In other words, the larger the class we have, the larger the primes we need to divide by (loosely speaking).

TIME OR INTEREST PERMITTING ONLY:

Write: π_c = the set of primes less than or equal to c.

π_c-powered group = group where every element has a unique p^{th} root for all $p \in \pi_c$

0.2. Isoclinism. *Say time*: 2 minutes

For any group G, the commutator map in G descends to a map of sets:

$$\omega_G : \mathrm{Inn}(G) \times \mathrm{Inn}(G) \to G'$$

An *isoclinism* from G_1 to G_2 is a pair of isomorphisms (ζ, φ) where ζ is an isomorphism from $\mathrm{Inn}(G_1)$ to $\mathrm{Inn}(G_2)$ and φ is an isomorphism from G_1' to G_2', satisfying the condition that:

$$\varphi(\omega_{G_1}(x, y)) = \omega_{G_2}(\zeta(x), \zeta(y)) \tag{1}$$

Pictorially:

$$\begin{array}{ccc} \mathrm{Inn}(G_1) \times \mathrm{Inn}(G_1) & \stackrel{\zeta\times\zeta}{\to} & \mathrm{Inn}(G_2) \times \mathrm{Inn}(G_2) \\ \downarrow^{\omega_{G_1}} & & \downarrow^{\omega_{G_2}} \\ G_1' & \stackrel{\varphi}{\to} & G_2' \end{array}$$

Abelian $\iff$ Isoclinic to the trivial group

Say, don't write: Note that both the inner automorphism group and the derived subgroup are quantitative measurements of the "non-abelianness" of the group. The notion of isoclinism can thus properly be thought of as saying "equivalent modulo the subvariety of abelian groups." In particular, a group is abelian if and only if it is isoclinic to the trivial group.

0.3. **Global Lazard correspondence up to isoclinism.** *Write + say together time*: 3 minutes

Global Lazard correspondence up to isoclinism from a Lie ring L to a group G is a pair of isomorphisms (ζ, φ) where ζ is an isomorphism from $\mathrm{Inn}(L)$ to $\log(\mathrm{Inn}(G))$ and φ is an isomorphism from L' to $\log(G')$ such that:

$$\varphi(\omega_L(x, y)) = \omega_G^{\mathrm{Lie}}(\zeta(x), \zeta(y))$$

This is equivalent to the requirement that:

$$\varphi(\omega_L^{\mathrm{Group}}(x, y)) = \omega_G(\zeta(x), \zeta(y))$$

(make the diagrams, discuss)

It's a correspondence:

Equivalence classes up to isoclinism of p-groups of class at most $p \leftrightarrow$ Equivalence classes up to isoclinism of p-Lie rings of class at most p

0.4. **Key difficulty: showing existence.** *Write + say time*: 2 minutes

We'll apply ideas from group extension theory and Lie ring extension theory to:

$$0 \to Z(L) \to L \to L/Z(L) \to 0$$

and

$$0 \to Z(G) \to G \to G/Z(G) \to 1$$

Key idea (say, don't write, rest of this section): We begin by viewing L as an extension with central subring $Z(L)$ and quotient ring $L/Z(L) \cong \mathrm{Inn}(L)$. We obtain the corresponding Lie bracket map $\mathrm{Inn}(L) \times \mathrm{Inn}(L) \to L'$. We then obtain a desired commutator map $\exp(\mathrm{Inn}(L)) \times \exp(\mathrm{Inn}(L)) \to \exp(L')$ by using the formula describing the commutator map in terms of the Lie bracket map. Finally, we demonstrate the existence of a group G that realizes this commutator map.

We will show that equivalence classes of groups up to isoclinism can be described by storing the commutator structure in an abstract fashion, without reference to an actual group in that equivalence class.

This will be useful to the final step of our proof of existence established above: instead of directly trying to construct the groups in the equivalence class up to isoclinism, we construct the commutator structure. In the notation above, we construct the desired commutator map $\exp(\mathrm{Inn}(L)) \times \exp(\mathrm{Inn}(L)) \to \exp(L')$.

Below, we provide a few more details about how we store the commutator structure abstractly. This discussion may be accessible only to people familiar either with group cohomology or with some other type of cohomology theory that is structurally similar. Note also that the group G that we use here is not the same as the group G used above.

0.5. **Option bifurcation.** *Say + write time*: 2 minutes

We could either:

- go over the Baer correspondence up to isoclinism (the case $c = 1$ and $c + 1 = 2$) in detail. The advantage here is that we could delve deeply into examples and understand everything explicitly, or
- go over the general case, but we will have to hand-wave quite a bit and will not be able to cover examples.

1. First option: the Baer correspondence up to isoclinism

1.1. **The case $c = 1$: the Baer correspondence up to isoclinism.** For now, G and A are abelian groups (A also viewed as an abelian Lie ring by abuse of notation), and L is an abelian Lie ring.

The short exact sequence classifying extensions of abelian groups is:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(G; A) \to H^2(G; A) \to \mathrm{Hom}(G \wedge G, A) \to 0 \tag{2}$$

(explain in detail each of the maps and what it means)

Say some version of this: The map:

$$\mathrm{Ext}^1_{\mathbb{Z}}(G;A) \to H^2(G;A)$$

can be interpreted as follows. The underlying set of the group on the left is canonically identified with the set of all *abelian* group extensions with subgroup A and quotient group G. The group on the right is the group whose elements are all the *central* extensions with central subgroup A and quotient group G. Every abelian group extension is a central extension, and there is therefore a canonical injective set map from $\mathrm{Ext}^1_{\mathbb{Z}}(G;A)$ to $H^2(G;A)$.

The map:

$$H^2(G;A) \to \mathrm{Hom}(G \wedge G, A)$$

can be described as follows. For any group extension E, the commutator map $E \times E \to E$ descends to a set map:

$$\omega_{E,G} : G \times G \to A$$

Our earlier definition of $\omega_{E,G}$ defined it as a map to $[E,E]$, but $[E,E]$ lies in the image of A (under the inclusion of A in E), so it can be viewed as a map to A.

Note that the image of the map is in A *because* G is abelian. Further, $\omega_{E,G}$ is bilinear, because the image of the map is central. It thus defines a group homomorphism $G \wedge G \to A$.

The homomorphism above can also be described in terms of the how it operates at the level of 2-cocycles (this description requires understanding the explicit description of the second cohomology group using the bar resolution, as given in Section **??**). Explicitly, the map:

$$H^2(G;A) \to \mathrm{Hom}(G \wedge G, A)$$

arises from a homomorphism:

$$Z^2(G;A) \to \mathrm{Hom}(G \wedge G, A)$$

given by:

$$f \mapsto \mathrm{Skew}(f)$$

where $\mathrm{Skew}(f)$ is the map $(x,y) \mapsto f(x,y) - f(y,x)$.

Intuitively, this is because the commutator of two elements represents the distance between their products in both possible orders, i.e., $[x,y]$ is the quotient $(xy)/(yx)$. Whether we use left or right quotients does not matter because the group has class two.

Based on the discussion in Section **??**, the homomorphism:

$$H^2(G;A) \to \mathrm{Hom}(G \wedge G, A)$$

Write: Similarly for Lie rings:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(L;A) \to H^2_{\mathrm{Lie}}(L;A) \to \mathrm{Hom}(L \wedge L, A) \to 0 \tag{3}$$

The short exact sequence splits *canonically*, and we get a canonical isomorphism:

$$H^2_{\mathrm{Lie}}(L;A) \cong \mathrm{Ext}^1_{\mathbb{Z}}(L;A) \oplus \mathrm{Hom}(L \wedge L, A)$$

Say: We can describe the splitting either by specifying the projection $H^2_{\mathrm{Lie}}(L;A) \to \mathrm{Ext}^1_{\mathbb{Z}}(L;A)$ or by specifying the inclusion $\mathrm{Hom}(L \wedge L, A) \to H^2_{\mathrm{Lie}}(L;A)$. We do both.

The projection:

$$H^2_{\mathrm{Lie}}(L;A) \to \mathrm{Ext}^1_{\mathbb{Z}}(L;A)$$

is defined as follows. For any extension Lie ring M, map it to the extension Lie ring that is *abelian* as a Lie ring and has the same additive group as M. In other words, keep the additive structure intact, but "forget" the Lie bracket.

The inclusion:

$$\mathrm{Hom}(L \wedge L, A) \to H^2_{\mathrm{Lie}}(L; A)$$

is defined as follows. Given a bilinear map $b : L \times L \to A$, define the extension Lie ring as a Lie ring M whose additive group is $L \oplus A$, and where the Lie bracket is:

$$[(x_1, y_1), (x_2, y_2)] = [0, b(x_1, x_2)]$$

In other words, we use the direct sum for the additive structure, and use the bilinear map to define the Lie bracket.

In light of this, we can think of the direct sum decomposition as follows:

$$H^2_{\mathrm{Lie}}(L; A) \cong \mathrm{Ext}^1_{\mathbb{Z}}(L; A) \oplus \mathrm{Hom}(L \wedge L, A)$$

The projection onto the first component stores the additive structure of the Lie ring, while destroying, or forgetting, the Lie bracket. The projection onto the second component preserves the Lie bracket while replacing the additive structure with a direct sum of L and A. Note also that the latter projection is equivalent to passing to the associated graded Lie ring.

1.2. **Baer correspondence up to isoclinism for extensions.** *Write*: We have demonstrated the existence of canonical isomorphisms between the left groups and between the right groups in the two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}^1_{\mathbb{Z}}(G; A) & \to & H^2(G; A) & \to & \mathrm{Hom}(G \wedge G, A) & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & \mathrm{Ext}^1_{\mathbb{Z}}(L; A) & \to & H^2_{\mathrm{Lie}}(L; A) & \to & \mathrm{Hom}(L \wedge L, A) & \to & 0 \end{array}$$

Say: As described in Sections **??** and **??**, both short exact sequences split. Therefore, it is possible to find an isomorphism $H^2(G; A) \to H^2_{\mathrm{Lie}}(L; A)$ that establishes an isomorphism of the short exact sequences.

Do: Mark canonical, non-canonical isomorphisms.

Say: Note, however, that the middle isomorphism is not canonical. In fact, choosing a middle isomorphism is equivalent to choosing a splitting of the top sequence. This is because the bottom sequence splits canonically, as we just described.

When we have the actual Baer correspondence, that gives us a canonical middle isomorphism, or equivalently, a canonical splitting of the universal coefficient theorem short exact sequence.

(can also explain this in the context of the BCH formula if necessary).

1.3. **Cocycle-level description of the Baer correspondence.** Suppose G and A are abelian groups (we will soon restrict to the case that one or both of G and A is 2-powered). Consider the following two short exact sequences. The first is the short exact sequence relating the coboundary, cocycle and cohomology groups, originally described in Section **??**:

$$0 \to B^2(G; A) \to Z^2(G; A) \to H^2(G; A) \to 0$$

The second is the universal coefficient theorem short exact sequence, originally described in Section **??** and described specifically for abelian G in Section **??**:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(G; A) \to H^2(G; A) \to \mathrm{Hom}(G \wedge G, A) \to 0$$

The first short exact sequence need not split. An example where it does not split was discussed in Section **??**. The second short exact sequence does always split but the splitting need not be canonical (see Section **??**).

The right parts of these short exact sequences give surjective homomorphisms, which we can compose:

$$Z^2(G; A) \to H^2(G; A) \to \mathrm{Hom}(G \wedge G, A)$$

As we discussed in Section **??**, the composite of these maps is the skew map. Explicitly, the composite is the map $f \mapsto \mathrm{Skew}(f)$, that sends a function f to the function:

$$\mathrm{Skew}(f) = (x, y) \mapsto f(x, y) - f(y, x)$$

Note that the function $\operatorname{Skew}(f)$ is a $\mathbb{Z}$-bilinear map $G \times G$ to A, which can be interpreted as a homomorphism $G \wedge G \to A$.

Now, suppose that G and A are both 2-powered abelian groups.[1]

In that case, there is a canonical splitting of the composite map, given as follows:

$$f \mapsto \frac{1}{2} f$$

In other words, a $\mathbb{Z}$-bilinear map $f : G \times G \to A$ is sent to $\frac{1}{2}f : G \times G \to A$. Note that any $\mathbb{Z}$-bilinear map is a 2-cocycle (in general, any n-linear map is a n-cocycle) so this works.

In particular, *both* the short exact sequences split, and we get canonical direct sum decompositions:

$$Z^2(G;A) \cong B^2(G;A) \oplus H^2(G;A), \text{ splitting is } H^2(G;A) \to Z^2(G;A)$$

$$H^2(G;A) \cong \operatorname{Ext}^1_{\mathbb{Z}}(G;A) \oplus \operatorname{Hom}(G \wedge G, A), \text{ splitting is } \operatorname{Hom}(G \wedge G, A) \to H^2(G;A)$$

Note that the first short exact sequence need not split for all G and A (see the discussion in Section **??**) and the existence of a splitting is itself a piece of information. The second short exact sequence does split for all G and A, but the splitting is not in general canonical, as discussed in Section **??**, so the case where G and A are both 2-powered is special in that we obtain a *canonical* splitting.

The splitting map $\operatorname{Hom}(G \wedge G, A) \to H^2(G;A)$ is the same as the one arising from the Baer correspondence. Explicitly, as noted in Section **??**, specifying the splitting map $\operatorname{Hom}(G \wedge G, A) \to H^2(G;A)$ is equivalent to specifying an isomorphism of $H^2(G;A)$ and $H^2_{\text{Lie}}(L;A)$ such that the diagram below commutes:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ext}^1_{\mathbb{Z}}(G;A) & \to & H^2(G;A) & \to & \operatorname{Hom}(G \wedge G, A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Ext}^1_{\mathbb{Z}}(L;A) & \to & H^2_{\text{Lie}}(L;A) & \to & \operatorname{Hom}(L \wedge L, A) & \to & 0 \end{array}$$

This isomorphism can be described in an alternative way. Let E be an extension group corresponding to an element of $H^2(G;A)$. Let $N = \log(E)$ via the Baer correspondence. We can relate two short exact sequences via a log functor.

$$\begin{array}{ccccc} 0 \to & A \to & E \to & G \to & 1 \\ & \downarrow^{\log} & \downarrow^{\log} & \downarrow^{\log} & \\ 0 \to & A \to & N \to & L \to & 0 \end{array}$$

Note that we abuse notation again, using the same letter A for A as a group and as a Lie ring.

Then, the element of $H^2_{\text{Lie}}(L;A)$ that corresponds to the second row is the same as the image of the element of $H^2(G;A)$ under the isomorphism described earlier.

1.4. **A setting where the Baer correspondence works only up to isoclinism.** In the case that G and A are odd-order abelian groups, the *original* Baer correspondence works. To obtain finite examples where the Baer correspondence works only up to isoclinism, we need to look at 2-groups. Further, our examples must be cases where the quotient $\operatorname{Hom}(G \wedge G, A)$ is nontrivial, so that there is at least some non-abelian extension.[2]

The smallest sized example is: $A = \mathbb{Z}_2$ is the cyclic group of order 2 and $G = V_4$ is the Klein four-group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The short exact sequences discussed in Sections **??** and **??**, along with the canonical isomorphisms discussed in Section **??**, give the following:

[1]The assumption can be modified to requiring that any one of G and A be 2-powered, in which case we will need to use one of the generalizations of the Baer correspondence described in Section **??**, but we do not describe it here since it is not necessary for our purpose.

[2]The abelian extensions can be put in correspondence based on the correspondence between abelian groups and abelian Lie rings, which, although not strictly part of the Baer correspondence as have defined it, falls under the generalization (1) of it described in Section **??**

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ext}^1(V_4;\mathbb{Z}_2) & \to & H^2(V_4;\mathbb{Z}_2) & \to & \operatorname{Hom}(V_4 \wedge V_4, \mathbb{Z}_2) & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & \operatorname{Ext}^1(V_4;\mathbb{Z}_2) & \to & H^2_{\text{Lie}}(V_4;\mathbb{Z}_2) & \to & \operatorname{Hom}(V_4 \wedge V_4, \mathbb{Z}_2) & \to & 0 \end{array}$$

Recall that both short exact sequences split, and, as per the discussion in Section **??**, the Lie ring short exact sequence splits canonically (with the splitting separating out the addition and Lie bracket parts).

It turns out that:

- $\operatorname{Ext}^1(V_4;\mathbb{Z}_2)$ is itself isomorphic to V_4, the Klein four-group.
- $(V_4 \wedge V_4)$ is isomorphic to $\mathbb{Z}_2$, and thus, $\operatorname{Hom}(H_2(V_4;\mathbb{Z}),\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$.
- Thus, both of the second cohomology groups (the group and Lie ring side) are isomorphic to the elementary abelian group of order eight.

On the group side, we have the following eight extensions (eight being the order of the cohomology group):

(a) Elementary abelian group of order eight (1 time).
(b) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ (3 times).
(c) D_8 (3 times).
(d) Q_8 (1 time).

(a) and (b) together form the image of Ext^1 (total size 4) while (c) and (d) form the non-identity coset of that image.

On the Lie ring side, the eight extensions (eight being the order of the cohomology group) are:

(a) Abelian Lie ring, additive group elementary abelian of order eight (1 time)
(b) Abelian Lie ring, additive group direct product of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ (3 times).
(c) The niltriangular matrix Lie ring (3×3 strictly upper triangular matrices) over the field of two elements. (1 time)
(d) The semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ as Lie rings. (3 times).

(a) and (b) together form the image of Ext^1 (total size 4) while (c) and (d) form the non-identity coset of that image.

Note that there is no canonical bijection between the set of eight group extensions and the set of eight Lie ring extensions, but we can naturally correspond the images of Ext^1 in both. The problem arises when attempting an element-to-element identification of the non-identity cosets in the two cases. In other words, we have a correspondence at a coset level:

$$\{\text{The four non-abelian Lie ring extensions}\} \leftrightarrow \{D_8, D_8, D_8, Q_8\}$$

But there is no clear-cut way of making sense of *which* Lie ring extension to correspond to *which* group. This is an example of a situation where the Baer correspondence up to isoclinism does not seem to have any natural refinement to a correspondence up to isomorphism.

Note that in this case, it so happens that we can use an automorphism-invariance criterion and get a unique automorphism-invariant bijection. This would map the niltriangular matrix Lie ring to the quaternion group and the semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ to the dihedral group. However, this does not give a meaningful bijection at the level of elements. For instance, one feature that holds in all generalizations described so far for the Baer correspondence is that the correspondence restricts to isomorphism between cyclic subgroups and cyclic subrings. In particular, the multiset of the orders of the elements in the group must match the multiset of the orders of the elements in the additive group of the Lie ring. However, the multiset of the orders of the elements of D_8 does not match the multiset of the orders of elements in any abelian group of order 8. The same is true of Q_8.

2. Second option: going over the general case, but in a handwavy fashion

2.1. Universal coefficient theorem: technical details.

Say + write time: 2 minutes

Central extensions with central subgroup A and quotient group G:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(G^{\text{ab}}, A) \to H^2(G;A) \to \operatorname{Hom}(M(G), A) \to 0$$

It splits, but not canonically:

$$H^2(G;A) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}}, A) \oplus \mathrm{Hom}(M(G), A)$$

The map:

$$H^2(G;A) \to \mathrm{Hom}(M(G), A)$$

classifies extensions up to "isoclinism of extensions."

Surjectivity is key.

Say, don't write: Surjectivity tells us: every homomorphism from $M(G)$ to A can be "realized" via an equivalence class up to isoclinism of extensions.

We can do something similar on the Lie ring side.

2.2. **Boils down to isomorphism of Schur multipliers.** *Say + write time*: 1 minute

Schur multiplier $M(G)$ "classifies" equivalence classes up to isoclinism of extensions of G.

Schur multiplier $M(L)$ "classifies" equivalence classes up to isoclinism of extensions of L.

We show that if G and L are in global Lazard correspondence, then $M(G) \cong M(L)$ canonically.

This allows us to prove results for groups that are extensions of groups in Lazard correspondence, i.e., π_c

2.3. **Optional extra: explain about the exterior square and Schur multiplier.** *Write header*: Exterior square

Say, don't write: Recall how we define exterior square of a *vector space*. We have a concept of alternating bilinear maps from the vector space. The exterior square is an object such that homomorphisms from it correspond to alternating bilinear maps from the vector space.

Write: Exterior square of G is the "freest" group $G \wedge G$ generated by formal symbols $x \wedge y$, $x, y \in G$, such that for any central extension E of G, if we consider the map:

$$\omega_{E,G} : G \times G \to [E, E]$$

(slight variant of the ω_E described above)

this extends to a group homomorphism:

$$\Omega_{E,G} : G \wedge G \to [E, E]$$

If there's time: Originally introduced by Miller (1952), later described by Brown and Loday (1987) and Graham Ellis (1987).

There is an explicit presentation with generators and relations.

Say (don't write): There is a similar definition of the exterior square of a Lie ring. Note that the exterior square of a Lie ring as a Lie ring is not the same as its exterior square as an abelian group. It is a quotient of that, however.

Write: There's a canonical short exact sequence (central extension):

$$0 \to M(G) \to G \wedge G \to [G, G] \to 1$$

Right map, on a generating set:

$$x \wedge y \mapsto [x, y]$$

$M(G) =$ Schur multiplier of G

Say (don't write): We can think of $M(G)$ as the formal products of expressions of the form $x \wedge y$ that, when we evaluate in G, become trivial. This has the flavor of looking at central extensions. Indeed:

Write:

$$\begin{array}{ccccccccc} 0 & \to & M(G) & \to & G \wedge G & \to & [G,G] & \to & 1 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & E & \to & G & \to & 1 \end{array}$$

Do: make dashed arrow $M(G) \to A$. This is the map $\beta : M(G) \to A$, same as the element of $\mathrm{Hom}(M(G), A)$ that appears in the universal coefficient theorem short exact sequence.

Write: Let B be the image of β and $\beta' : M(G) \to B$ be the restriction. We have a map:

$$\begin{array}{ccccccccc} 0 & \to & M(G) & \to & G \wedge G & \to & [G,G] & \to & 1 \\ & & \downarrow^{\beta'} & & \downarrow^{\Omega_{E,G}} & & \downarrow^{\mathrm{id}} & & \\ 0 & \to & B & \to & [E,E] & \to & [G,G] & \to & 1 \end{array}$$

Say or write: Knowing the map $M(G) \to B$ determines the map $\Omega_{E,G}$. Latter classifies extensions up to isoclinism, hence so does the former.

Say: Analogous results hold for the Lie ring. The key aspect we need to prove is that the Schur multiplier, which classifies extensions up to isoclinism, is the same for the group and the Lie ring, if the group and the Lie ring are in Lazard correspondence. This will allow us to draw conclusions about the central extension groups and Lie rings, that has class one more than the original group and Lie ring. We will in fact show that the short exact sequences:

Write:

$$0 \to M(L) \to L \wedge L \to [L,L] \to 0$$

and

$$0 \to M(G) \to G \wedge G \to [G,G] \to 1$$

are in Lazard correspondence up to (canonical) isomorphism.

Say: *Why this is plausible*: Note that if G has class c, $G \wedge G$ has class at most $\lfloor (c+1)/2 \rfloor$, so if G is in the domain of the Lazard correspondence, so is $G \wedge G$.

Why this is nontrivial: The intermediate formulas that we would use to calculate $G \wedge G$ and $L \wedge L$ are inside groups and Lie rings that are not in Lazard correspondence (e.g., if we use the Hopf formula). We need to show that despite this, the subgroups that end up getting used in the formula are in Lazard correspondence.

2.4. **The finite case.** Global Lazard correspondence:

finite p-groups of class $\leq p-1 \leftrightarrow$ finite p-Lie rings of class $\leq p-1$ (write below: additive group is a finite p-group)

Global Lazard correspondence up to isoclinism:

equivalence classes up to isoclinism of finite p-group of class $\leq p \leftrightarrow$ equivalence classes up to isoclinism of finite p-Lie rings of class $\leq p$