# REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 196, SECTION 57 (VIPUL NAIK)

**Please bring a copy (print or readable electronic) of this sheet to the review session.**

There is also a basic review sheet that contains executive summaries of the lecture notes. You should review that on your own time.

I've kept the error-spotting exercises brief, because I intend to concentrate more on reviewing some of the techniques covered in the quizzes.

## 1. Matrix multiplication and inversion

Error-spotting exercises ...

(1) Suppose $A$ and $B$ are $n \times n$ matrices, with $B$ invertible. Suppose $r$ is a positive integer. Then, $(BAB^{-1})^r = B^r A^r (B^{-1})^r = B^r A^r B^{-r}$. Note that since $A$ and $B$ do not in general commute, we must write the terms in precisely this order.

(2) Suppose $A$ and $B$ are $n \times n$ matrices and $r$ is a positive integer such that $(AB)^r = 0$. Then, we can conclude that $(BA)^r = 0$ as follows: we can write $(BA)^r = (BA)^r BB^{-1} = BABA \dots BABB^{-1} = B(AB)^r B^{-1} = B(0)B^{-1} = 0$.

(3) Suppose $A$ and $B$ are $n \times n$ matrices. Then, $AB$ is nilpotent if and only if at least one of the matrices $A$ or $B$ is nilpotent. To see this, suppose $r$ is a positive integer such that $(AB)^r = 0$. Then, we know that $(AB)^r = A^r B^r$, so $A^r B^r = 0$, forcing that either $A^r = 0$ or $B^r = 0$. The argument also works in reverse: if either of the matrices is nilpotent, there exists $r$ such that one of the matrices $A^r$ and $B^r$ is 0. Thus, $A^r B^r = 0$, so $(AB)^r = 0$, so $AB$ is nilpotent.

(4) Suppose $A$ and $B$ are invertible $n \times n$ matrices. Then, the sum $A + B$ is also an invertible $n \times n$ matrix, and $(A + B)^{-1} = A^{-1} + B^{-1}$.

(5) Suppose $A$ and $B$ are invertible $n \times n$ matrices. Then, the product $AB$ is also an invertible $n \times n$ matrix, and $(AB)^{-1} = A^{-1}B^{-1}$.

(6) Suppose $A$ and $B$ are matrices with real entries, with $A$ a single row matrix and $B$ a single column matrix. Then, $AB$ makes sense if and only if $BA$ makes sense, and if so, we must have that $AB = BA$.

(7) Suppose $A$ is a $m \times n$ matrix and $B$ is a $n \times p$ matrix. The product $C = AB$ is a $m \times p$ matrix. For $1 \le i \le m$ and $1 \le k \le p$, the value $c_{ik}$ is the product $a_{ij}b_{jk}$, with $1 \le j \le n$.

## 2. Geometry of linear transformations

Error-spotting exercises ... (this section is not too important, so we will probably do it last)

(1) Suppose $D$ is a $2 \times 2$ diagonal matrix and $T$ is the linear transformation corresponding to $D$. Let's say the two diagonal entries of $D$ are $a$ and $d$. The necessary and sufficient condition for $T$ to be area-preserving is that the total effect on the $x$ and $y$ directions add up to 1 (the ratio of change of areas). Thus, $T$ is area-preserving if and only if $a + d = 1$.

(2) The composite of the reflection maps about two lines through the origin that make an angle of $\theta$ with each other is the rotation map by the angle of $\theta$. Moreover, the order of composition does not matter, i.e., the composite for both orders of composition is the same.

(3) The composite of two rotations in $\mathbb{R}^2$ is always a rotation, even if the centers of rotation differ.

(4) A bijective function $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine linear automorphism of $\mathbb{R}^n$ if and only if it sends lines to lines.

## 3. Image and kernel

**3.1. Injectivity, surjectivity, and bijectivity.** Error-spotting exercises ...

(1) Suppose $f_1, f_2, f_3 : A \to A$ are set maps. Suppose the composite $f_1 \circ f_2 \circ f_3$ is bijective. Then, $f_1$ must be injective (because it's the one done first, so it cannot create any collision), $f_2$ must be bijective, and $f_3$ must be surjective (because it's the one done last, so it must hit everything).
(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree equal to the natural number $n \geq 3$. If $n$ is even, $f$ is surjective but not injective (e.g., $f(x) = x^4$). If $n$ is odd, $f$ is injective but not surjective (e.g., $f(x) = x^3$).
(3) Suppose $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that the restriction of $f$ to $\mathbb{Z}$ maps $\mathbb{Z}$ to inside $\mathbb{Z}$ (i.e., $f$ takes integer values at integer inputs). Let $g : \mathbb{Z} \to \mathbb{Z}$ be the function obtained by restricting $f$ to $\mathbb{Z}$. Then:
   (a) $f$ is injective if and only if $g$ is injective.
   (b) $f$ is surjective if and only if $g$ is surjective.
   (c) $f$ is bijective if and only if $g$ is bijective.

3.2. **Linear transformation and rank.** Error-spotting exercises ...

(1) If $T_1$ and $T_2$ are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, then the kernel of $T_1 + T_2$ equals the intersection of the kernels of $T_1$ and $T_2$. Here's a proof. Suppose a vector $\vec{u}$ is in the kernel of $T_1$ as well as the kernel of $T_2$. Then, $T_1(\vec{u}) = T_2(\vec{u}) = 0$. Thus, $(T_1 + T_2)(\vec{u}) = T_1(\vec{u}) + T_2(\vec{u}) = 0 + 0 = 0$.
   In particular, this means that if both $T_1$ and $T_2$ are invertible, then $T_1 + T_2$ is invertible.
(2) Suppose $T_1 : \mathbb{R}^a \to \mathbb{R}^b$ and $T_2 : \mathbb{R}^b \to \mathbb{R}^c$ are linear transformations. Then, the composite $T_1 \circ T_2$ is a linear transformation from $\mathbb{R}^a$ to $\mathbb{R}^c$. In terms of matrices, the matrix for $T_1$ is an $a \times b$ matrix and the matrix for $T_2$ is a $b \times c$ matrix. So the matrix for $T_1 \circ T_2$ is an $a \times c$ matrix, and is given by the matrix product of those two matrices.
   Suppose the kernel of $T_1$ has dimension $m$ and the kernel of $T_2$ has dimension $n$. A vector is in the kernel of $T_1 \circ T_2$ if and only if it is in the kernel of either $T_1$ or $T_2$. Thus, the dimension of the kernel of $T_1 \circ T_2$ is the maximum of the dimensions of the kernels of $T_1$ and $T_2$, which is $\max\{m, n\}$.
(3) Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation with matrix $A$. Then $A$ is a $m \times n$ matrix and the following are true:
   (a) The rows of $A$ form a spanning set for the image of $T$.
   (b) The columns of $A$ form a spanning set for the kernel of $T$.
(4) Consider the linear transformation:

$$\nu = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} (y+z)/2 \\ (z+x)/2 \\ (x+y)/2 \end{bmatrix}$$

   The kernel of $T$ is precisely those vectors where $x = -y = z$, i.e., each coordinate is the negative of the next one. The image of $T$ is the set where $x = y = z$.

3.3. **The linear operation of differentiation.** Error-spotting exercises ...

We will get to this only if we have enough time, since we will not see these topics outside the MCQs on the test.

(1) Denote by $C(\mathbb{R})$ the vector space of all continuous functions with the usual addition and scalar multiplication of functions. Denote by $C^1(\mathbb{R})$ the vector space of all continuously differentiable functions with the usual addition and scalar multiplication of functions. Differentiation defines a linear transformation from $C(\mathbb{R})$ to $C^1(\mathbb{R})$. The image of this linear transformation is precisely the set of constant functions.
(2) For every positive integer $k$, denote by $C^k(\mathbb{R})$ the subspace of $C(\mathbb{R})$ comprising those polynomials that are at least $k$ times continuously differentiable. Then, $C^k(\mathbb{R}) \subseteq C^{k+1}(\mathbb{R})$ and the union of all the spaces $C^k(\mathbb{R})$ for $k$ varying over the positive integers is the space $C^\infty(\mathbb{R})$ of infinitely differentiable functions.
(3) The set of polynomials of degree at most $k$ form a vector subspace of $C^k(\mathbb{R})$ but not of $C^{k+1}(\mathbb{R})$.