REVIEW SHEET FOR MIDTERM 1: ADVANCED

MATH 196, SECTION 57 (VIPUL NAIK)

Please bring a copy (print or readable electronic) of this sheet to the review session.

There is also a basic review sheet that contains executive summaries of the lecture notes. You should review that on your own time.

Many of the error-spotting exercises correspond to ideas that you have already seen in quiz questions. We have parenthetically indicated at the beginning of the question whether it corresponds to material seen in lecture (we use L for that) or in a quiz (we use Q for that). We preface with a $\sim$ if there is a considerable gap between the way the idea was presented in the earlier context and the way it is being tested now.

1. Linear functions: a primer

Error-spotting exercises ...

(1) (L, Q): f is a function of two variables x and y that is postulated to be *affine* linear, i.e., linear with a possibly nonzero intercept. Since there are two unknowns, knowing the value of f at 2 points will help us find f precisely (assuming it is linear) and also provide independent verification of the linearity of f.

(2) (L, Q): f is a function of one variable. It is believed to be a polynomial of degree at most n, where n is a known positive integer. In order to find the function f (assuming that it is indeed such a polynomial) we need to know the values of f at n points. If the goal is not merely to find f conditional to its being a polynomial of degree at most n but also to obtain independent confirmation of the hypothesis, we need to know the values of f at $n + 1$ points.

(3) (Q): Suppose $x \mapsto mx + c$ is a linear function of one variable. We know the value of the function at two points, albeit there are uncorrelated measurement errors with a fixed error distribution at both points. Regardless of what two points we choose, we will end up with the same error range in our estimate of the function.

(4) A function is known to be of the form $t \mapsto A\sin(t + \varphi)$ where A and φ are constants. We need to use input-output pairs to find the function. We can do this by setting up a system of equations that are linear in terms of the parameters A and φ. Given enough input-output pairs, we will be able to determine A and φ uniquely.

2. Equation solving

Error-spotting exercises ...

(1) (L): Consider the system of simultaneous (not necessarily linear) equations:

$$\begin{aligned} f(x) &= 0 \\ g(y) &= 0 \end{aligned}$$

Suppose the set of solutions to the first equation viewed as an equation in x alone is $\{x_1, x_2, \dots, x_n\}$ and the set of solutions to the second equation viewed as an equation in y alone is $\{y_1, y_2, \dots, y_n\}$. Then, the set of solutions to the system as a whole, viewed as a system of equations in two variables, is the following set of points in the xy-plane: (x_1, y_1), (x_2, y_2), (x_3, y_3), $\dots$, (x_n, y_n).

(2) (L): Consider a triangular system of the form:

$$\begin{aligned} f(x) &= 0 \\ g(x,y) &= 0 \\ h(x,y,z) &= 0 \end{aligned}$$

Suppose the following are true:

- The first equation has 2 solutions for x.
- For each solution to the first equation, the second equation has 3 solutions for y.
- For each choice of x and y that solve the first two equations, the third equation has 4 choices for z.

Then, the total number of solutions for the system is $2 + 3 + 4 = 9$.

(3) (L): Consider the system:

$$\begin{aligned} x + e^{x^2-x-y} &= 1 \\ y + xe^{x^2-x-y} &= 0 \end{aligned}$$

Subtract x times the first equation from the second equation. We get:

$$y - x^2 = -x$$

Thus, $y = x^2 - x$. The solution set is thus the set of all points on the parabolic curve $y = x^2 - x$.

(4) (Q, ∼L): If we are trying to find an existing function of one or more variables based on some input-output values with some measurement error, and we believe that the function is a polynomial function but we don't know the degree, it makes sense to try to fit using a polynomial function of as large a degree as we can computationally afford. This is because the larger the degree of the polynomial, the easier it is to get a good fit on a given collection of input-output pairs. For instance, given three input-output pairs for a function of one variable, we can always fit them perfectly (with zero measurement error) using a quadratic, but it may be difficult to fit them using a linear function. Larger degree polynomials are better because we have more parameters to work with and they offer us more flexibility. And more flexibility is always good.

3. Gauss-Jordan elimination

Error-spotting exercises ...

(1) (L, Q): Any process to solve an arbitrary linear system with n equations and n variables must take time of the order at least n^3, because Gauss-Jordan elimination is $\Theta(n^3)$. This is true even if we can pre-process the coefficient matrix of the linear system.

(2) (L, ∼Q): If Gauss-Jordan elimination gives us a row in the coefficient matrix that is all zeros, then the system of linear equations cannot have a solution.

(3) (L, Q): The dimension of the solution space to a system of simultaneous linear equations equals the number of leading variables in the system. This number can be effectively computed by converting the coefficient matrix to reduced row-echelon form and counting the number of pivotal 1s. The columns corresponding to these 1s give us the leading variables.

(4) (L): There is a shorter variant of Gauss-Jordan elimination called Gaussian elimination. The idea here is to skip the step of clearing out entries below the pivotal 1s, but concentrate on clearing out all entries above the pivotal 1s. This gets us to a form where it is easy to read the solutions and answer questions about the rank. The form of matrix we obtain at the end of this process is called the *row-echelon form*.

(5) (L, Q): A system of linear equations is inconsistent if and only if there is a row of the coefficient matrix that is all zeros such that the corresponding augmenting entry is nonzero.

4. Linear systems and matrix algebra

Error-spotting exercises ...

(1) (L): If the rank of the coefficient matrix of a linear system equals the number of rows, then the system is consistent and has a unique solution. Otherwise, it may or may not be consistent, and if it is consistent, it has infinitely many solutions.

(2) (∼L): If the number of columns in the coefficient matrix of a linear system is less than the number of rows, then the system has either no solution or a unique solution. It cannot have infinitely many solutions.

(3) (∼L): If the number of rows in the coefficient matrix of a linear system is less than the number of columns, then the system has either a unique solution or infinitely many solutions. In other words, it must be consistent.

(4) (∼L): A matrix has full row rank if and only if none of its rows is a zero row *and* no row is a multiple of another row. This is because, if both these conditions are satisfied, no two rows are dependent on each other. From the equational perspective, it means that every pair of equations is independent, so the equations are all independent of each other.

(5) (Q): Let m, n, and k be natural numbers with $m \geq 3$. We are given a bunch of numbers $x_0 < x_1 < x_2 < \cdots < x_m$ and another bunch of numbers $y_0, y_1, y_2, \ldots, y_m$. We want to find a continuous function f on $[x_0, x_m]$, such that $f(x_i) = y_i$ for all $0 \leq i \leq m$, and such that the restriction of f to any interval of the form $[x_i, x_{i+1}]$ (for $0 \leq i \leq m-1$) is a polynomial of degree $\leq n$. Further, we want f to be at least k times differentiable on the open interval (x_0, x_m). Let's try to see how many equations we can get from the constraints.

We have $m+1$ equations of the form $f(x_i) = y_i$. In addition, at each point of transition, we know that the first k derivatives have to match up. There are $m-1$ transition points and k derivatives to compare, so we get $k(m-1)$ equations that way. In total we thus have $k(m-1)+m+1$ equations. The number of parameters is mn, because we have m pieces, and polynomials of degree n in each piece, so n parameters for the coefficients in each piece.

Thus, we have $k(m-1)+m+1$ equations in mn variables that we need to solve.

5. Hypothesis testing, rank, and overdetermination

Nothing here (but some of these ideas appear in the error-spotting exercises for earlier sections).

6. Linear transformations

Error-spotting exercises ...

(1) (L, Q): Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. If $m < n$, T is injective but not surjective. If $m > n$, T is surjective but not injective. If $m = n$, T is bijective, and hence invertible.

(2) (L, Q): Let A be a $m \times n$ matrix that defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. Denote by $\vec{e}_i$ the vector in $\mathbb{R}^n$ with a 1 in the i^{th} coordinate and 0s elsewhere. Then, $T(\vec{e}_i)$ is the i^{th} row of A.

(3) A matrix A is termed *self-inverse* if $A = A^{-1}$. The only self-inverse 2×2 matrices are the diagonal matrices where each diagonal entry is either 1 or -1.

PLEASE TURN OVER FOR THE PRACTICE WORKSHEET.

7. Practice worksheet: guiding questions

(1) I have an unknown function f of one variable. I know $f(-1)$, $f(0)$, $f(1)$, and $f(3)$ (but I haven't told you these values). If I assumed that f was a polynomial of degree $\leq d$, I could use input-output pairs to construct a linear system. The coefficient matrix depends only on the inputs.

Suppose $d = 2$. Can you enumerate the parameters, then construct the coefficient matrix? Row reduce the coefficient matrix, and store all your steps. I will then provide you with an augmenting column, and you should be able to quickly determine f (if it exists).

Employ the same procedure for $d = 3$.

(2) I have a function f that is continuous on $[0, 3]$, differentiable on $(0, 3)$, and piecewise quadratic, with the pieces on which it is quadratic being the $[0, 1]$, $[1, 2]$, and $[2, 3]$. I'm going to give you the values $f(0)$, $f(1)$, $f(2)$, and $f(3)$. I will also give you the right hand derivative at 0. How would you use this information to find f? I'll give actual numerical examples in the session (or perhaps you can give each other actual numerical examples in the session).

(3) What are the properties of $n \times m$ matrices with randomly chosen entries in terms of whether the system is consistent, and what the dimension of the solution space is? Start by noting that the rank is almost certainly $\min\{m, n\}$. A general rule of thumb is that anything whose truth requires two independent random values to coincide will almost certainly not happen.