GEOMETRY OF LINEAR TRANSFORMATIONS

MATH 196, SECTION 57 (VIPUL NAIK)

Corresponding material in the book: Section 2.2.

Executive summary

(1) There is a concept of *isomorphism* as something that preserves essential structure or feature, where the concept of isomorphism depends on what feature is being preserved.
(2) There is a concept of *automorphism* as an isomorphism from a structure to itself. We can think of automrohpisms of a structure as *symmetries* of that structure.
(3) Linear transformations have already been defined. An *affine linear transformation* is something that preserves lines and ratios of lengths within lines. Any affine linear transformation is of the form $\vec{x} \mapsto A\vec{x} + \vec{b}$. For the transformation to be linear, we need $\vec{b}$ to be the zero vector, i.e., the transformation must send the origin to the origin. If A is the identity matrix, then the affine linear transformation is termed a *translation*.
(4) A linear *isomorphism* is an invertible linear transformation. For a linear isomorphism to exist from $\mathbb{R}^m$ to $\mathbb{R}^n$, we must have $m = n$. An affine linear isomorphism is an invertible affine linear transformation.
(5) A linear automorphism is a linear isomorphism from $\mathbb{R}^n$ to itself. An affine linear automorphism is an affine linear isomorphism from $\mathbb{R}^n$ to itself.
(6) A self-isometry of $\mathbb{R}^n$ is an invertible function from $\mathbb{R}^n$ to itself that preserves Euclidean distance. Any self-isometry of $\mathbb{R}^n$ must be an affine linear automorphism of $\mathbb{R}^n$.
(7) A self-homothety of $\mathbb{R}^n$ is an invertible function from $\mathbb{R}^n$ to itself that scales all Euclidean distances by a factor of λ, where λ is the factor of homothety. We can think of self-isometries precisely as the self-homotheties by a factor of 1. Any self-homothety of $\mathbb{R}^n$ must be an affine linear automorphism of $\mathbb{R}^n$.
(8) Each of these forms a group: the affine linear automorphisms of $\mathbb{R}^n$, the linear automorphisms of $\mathbb{R}^n$, the self-isometries of $\mathbb{R}^n$, the self-homotheties of $\mathbb{R}^n$.
(9) For a linear transformation, we can consider something called the *determinant*. For a 2×2 linear transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is $ad - bc$.

We can also consider the *trace*, defined as $a + d$ (the sum of the diagonal entries).
(10) The trace generalizes to $n \times n$ matrices: it is the sum of the diagonal entries. The determinant also generalizes, but the formula becomes more complicated.
(11) The determinant for an affine linear automorphism can be defined as the determinant for its linear part (the matrix).
(12) The sign of the determinant being positive means the transformation is orientation-preserving. The sign of the determinant being negative means the transformation is orientation-reversing.
(13) The magnitude of the determinant gives the factor by which volumes are scaled. In the case $n = 2$, it is the factor by which areas are scaled.
(14) The determinant of a self-homothety with factor of homothety λ is $\pm\lambda^n$, with the sign depending on whether it is orientation-preserving or orientation-reversing.
(15) Any self-isometry is volume-preserving, so it has determinant ± 1, with the sign depending on whether it is orientation-preserving or orientation-reversing.

(16) For $n = 2$, the orientation-preserving self-isometries are precisely the translations and rotations. The ones fixing the origin are precisely rotations centered at the origin. These form groups.
(17) For $n = 2$, the orientation-reversing self-isometries are precisely the reflections and glide reflections. The ones fixing the origin are precisely reflections about lines passing through the origin.
(18) For $n = 3$, the orientation-preserving self-isometries fixing the origin are precisely the rotations about axes through the origin. The overall classification is more complicated.

1. Geometry of linear transformations

1.1. **Geometry is secondary, but helps build intuition.** Our focus on linear transformations so far has been *information-centric*: they help reframe existing pieces of information in new ways, allowing us to extract things that are valuable to us. This information-centric approach undergirds the applicability of linear algebra in the social sciences.

We will now turn to a geometric perspective on linear transformations, restricting largely to the case $n = 2$ for many aspects of our discussion. A lot of this geometry is more parochial than the information-centric approach, and is not too important for the application of linear algebra to the social sciences. The main reason the geometry is valuable to us is that it can help build intuition regarding what is going on with the algebraic operations and hence offer another way of "sanity checking" our algebraic intuitions. As a general rule, always try to come up with reasons that are algebraic and information-centric, but in cases where we can perform sanity checks using geometric intuitions, use them.

1.2. **Composition of transformations.** If $f : A \to B$ is a function and $g : B \to C$ is a function, we can make sense of the composite $g \circ f$. The key feature needed to make sense of the composite is that the co-domain (the target space) of the function applied first (which we write on the right) must equal the domain (the starting space) of the function applied next (which we write later).

This means that if we have a collection of maps all from a space to *itself*, it makes sense to compose any two maps in the collection. We can even compose more than two maps if necessary.

The geometry of linear transformations that we discuss here is in the context of transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Here, we can compose, and if the transformations are bijective, also invert.

A small note is important here. We often see maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ where, even though both the domain and co-domain space have the same dimension, they do not represent the "same space" conceptually. For instance, one side may be measuring masses, while the other side may be measuring prices. In this case, composing multiple such maps does not make sense because even though the domain and co-domain are mathematically $\mathbb{R}^n$, they are conceptually different.

2. Particular kinds of transformations in linear algebra

2.1. **The concept of isomorphism and automorphism.** We will briefly describe two central mathematical concepts called *isomorphism* and *automorphism*. You are not expected to understand these concepts, but they help demystify some of the following discussion.

Understanding the concept of *isomorphism* is central to abstraction and to human intelligence, even though the word is not too well-known outside of mathematics and philosophy. When looking at different structures, we may be interested in whether they have *a certain feature in common*. For instance, when looking at sets, we may be interested in judging them by the number of elements in them. If we care only about size, then in our tunnel vision, all other aspects of the set are irrelevant. For our purposes, then, a set of three rabbits is effectively the same as a set of three lions, or a set of three stars. As another related example, suppose the only thing you care about potential mates in the dating market is their bank balance. In that case, two potential mates with the same bank balance are effectively the same, i.e., they are isomorphic.

This kind of abstraction is crucial to humans being able to understand numbers in the first place. If you crossed that hurdle back at the age of 3, 4, 5, or 6 (or whenever you understood the idea of counting), it's high time you went a step further.

One way of capturing the idea that two sets have the same size is as follows. We say that two sets have the same size if it is possible to construct a bijective function from one set to the other. In fact, roughly speaking, this is the *only* way to define the concept of "same size". In other words, we can *define* the size (technical term: cardinality) of a set as that attribute that is preserved by bijections and is different for sets if there is no bijection between them.

Another way of framing this is to christen bijective functions as *isomorphisms of sets*. In other words, a function from a set to a (possibly same, possibly different) set is termed an isomorphism of sets if it is a bijective function of sets. "Iso+morph" stands for "same shape" and signifies that the bijective function preserves the shape.

Why is this important? Instead of caring only about size, we can ratchet up our caring levels to care about more structural aspects of the sets we are dealing with. If we do so, our definition of "isomorphism" will become correspondingly more stringent, since it will require preserving more of the structure.

A related notion to isomorphism is that of *automorphism*. An automorphism is an isomorphism (in whatever sense the term is being used) from a set to itself.

Whatever our definition of isomorphism and automorphism, the identity map from a set to itself is always an automorphism.

Non-identity automorphisms from a set to itself signify *symmetries* of the set. This will become clearer as we proceed.

2.2. **Linear isomorphisms and automorphisms.** A *linear isomorphism* is defined as a bijective linear transformation. We can think of a linear isomorphism as a map that preserves precisely all the "linear structure" of the set.

Note that for a linear transformation to be bijective, the dimensions of the start and end space are the same. Explicitly, if $T : \mathbb{R}^m \to \mathbb{R}^n$ is a bijective linear transformation, then $m = n$. Also, the matrix for T has full rank. It is also a square matrix, since $m = n$.

In other words, the dimension of a vector space is invariant under linear isomorphisms. In fact, the relationship between linear isomorphisms and dimension is similar to the relationship between set isomorphisms and cardinality (set size).

A *linear automorphism* is defined as a linear isomorphism from a vector space $\mathbb{R}^n$ to itself. Based on the above discussion, you might believe that every linear isomorphism must be a linear automorphism. This is not quite true. The main caveat at the moment (there will be more later) is that $\mathbb{R}^n$ and $\mathbb{R}^n$ could refer to different spaces depending on what kinds of things we are storing using the real numbers (for instance, one $\mathbb{R}^n$ might be masses, the other $\mathbb{R}^n$ might be prices, and so on). A linear isomorphism between a $\mathbb{R}^n$ of one sort and a $\mathbb{R}^n$ of another sort should not rightly be considered a linear automorphism.

2.3. **Affine isomorphisms.** An *affine linear transformation* is a function that preserves collinearity and ratios within lines, $T : \mathbb{R}^m \to \mathbb{R}^n$ is affine if the image of any line (not necessarily through the origin) in $\mathbb{R}^m$ is a line (not necessarily through the origin) in $\mathbb{R}^n$, and moreover, it preserves the ratios of lengths within each line. So in particular, if $\vec{x}, \vec{y} \in \mathbb{R}^m$, then $T(a\vec{x} + (1-a)\vec{y}) = aT(\vec{x}) + (1-a)T(\vec{y})$. In particular, it preserves midpoints.

Note that any linear transformation is affine linear. The main thing we get by allowing "affine" is that the origin need not go to the origin. An important class of affine linear transformations that are not linear is the class of *translations*. Explicitly, for a nonzero vector $\vec{v} \in \mathbb{R}^n$, the function $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $\vec{x} \mapsto \vec{x} + \vec{v}$ is affine linear but not linear. In fact, all affine linear transformations can be obtained by composing translations with linear transformations.

An *affine linear isomorphism* (or *affine isomorphism* for short) is a bijective affine linear transformation.

An *affine linear automorphism* is an affine linear isomorphism from a vector space to itself.

Every linear automorphism is an affine linear automorphism. Nonzero translations give examples of affine linear automorphisms that are not linear. In an important sense, these cover all the affine linear automorphisms: every affine linear automorphism can be expressed as a composite of a translation and a linear automorphism.

2.4. **Linear and affine linear: the special case of one dimension.** In the case of one dimension, a linear isomorphism is a function of the form $x \mapsto mx, m \neq 0$. The matrix of this, viewed as a linear transformation, is $[m]$. In other words, it is a linear function with zero intercept.

An affine linear isomorphism is a function of the form $x \mapsto mx + c$, $m \neq 0$. In other words, it is a linear function with the intercept allowed to be nonzero.

2.5. **The general description of affine linear transformations.** A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ can be written in the form:

$$\vec{x} \mapsto A\vec{x} + \vec{b}$$

where A is a $n \times m$ matrix and $\vec{v}$ is a vector in $\mathbb{R}^n$. Explicitly, the matrix A describes the linear transformation part and the vector $\vec{b}$ describes the translation part. Notice how this general description parallels and generalizes the description in one dimension.

2.6. **Understanding how transformations and automorphisms behave intuitively.** In order to understand transformations and automorphisms using our geometric intuitions, it helps to start off with some geometric picture in the plane or Euclidean space that we are transforming, then apply the transformation to it, and see what we get. It is preferable to not take something *too* symmetric, because the less the symmetry, the more easily we can discern what features the transformation preserves and what features it destroys. Human stick features with asymmetric faces may be a good starting point for intuitive understanding, though more mundane figures like triangles might also be reasonable.

3. Euclidean geometry

3.1. **Euclidean distance and self-isometries.** The geometry of $\mathbb{R}^n$ is largely determined by how we define distance between points. The standard definition of distance is *Euclidean distance*. Explicitly, if $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, then the Euclidean distance between $\vec{x}$ and $\vec{y}$ is:

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

If a transformation preserves Euclidean distance, then it preserves all the geometry that we care about. In particular, it preserves shapes, sizes, angles, and other geometric features.

A bijective function from $\mathbb{R}^n$ to itself that preserves Euclidean distance is termed a *self-isometry* of $\mathbb{R}^n$.

We now proceed to an explanation of why self-isometries of $\mathbb{R}^n$ must necessarily be affine linear automorphisms of $\mathbb{R}^n$. The claim is that if something preserves distance, it must preserve linearity.

One way of characterizing the fact that three points $A, B, C \in \mathbb{R}^n$ are collinear, with B *between* A and C, is that we get the equality case for the triangle inequality. Explicitly:

$$(\text{the distance between } A \text{ and } B) + (\text{the distance between } B \text{ and } C) = (\text{the distance between } A \text{ and } C)$$

Suppose T is a self-isometry of $\mathbb{R}^n$. Then, we have:

$$\begin{aligned} \text{The distance between } T(A) \text{ and } T(B) &= \text{The distance between } A \text{ and } B \\ \text{The distance between } T(B) \text{ and } T(C) &= \text{The distance between } B \text{ and } C \\ \text{The distance between } T(A) \text{ and } T(C) &= \text{The distance between } A \text{ and } C \end{aligned}$$

Combining all these, we get that:

$$(\text{the distance between } T(A) \text{ and } T(B)) + (\text{the distance between } T(B) \text{ and } T(C)) = (\text{the distance between } T(A) \text{ and } T(C))$$

The conclusion is that $T(A)$, $T(B)$, and $T(C)$ are collinear with $T(B)$ between $T(A)$ and $T(C)$. In other words, collinear triples of points get mapped to collinear triples of points, so T preserves collinearity. Further, it obviously preserves ratios of lengths within lines. Thus, T is an affine linear automorphism of $\mathbb{R}^n$. The upshot: every self-isometry is an affine linear automorphism.

Self-isometries preserve not just collinearity, but *all* the geometric structure. What they are allowed to change is the location, angling, and orientation. They do not affect the shape and size of figures. In particular, they send triangles to congruent triangles.

3.2. **Self-homotheties.** A *self-homothety* or *similarity transformation* or *similitude transformation* is a bijective map $T : \mathbb{R}^n \to \mathbb{R}^n$ that scales all distances by a fixed nonzero factor called the *ratio of similitude* or *factor of similitude* of T. For instance, a self-homothety by a factor of $1/2$ will have the property that the distance between $T(A)$ and $T(B)$ is half the distance between A and B.

Self-isometries can be described as self-homotheties by a factor of 1.

Self-homotheties are affine linear automorphisms for roughly the same reason that self-isometries are.

A special kind of self-homothety is a *dilation* about a point. A dilation about the origin, for instance, would simply mean multiplying all the position vectors of points by a fixed nonzero scalar. The absolute value of that scalar will turn out to be the factor of similitude. The matrix for such a dilation is a scalar matrix. For instance, the matrix:

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

represents a dilation by a factor of 5 about the origin.

Self-homotheties send triangles to similar triangles.

3.3. **Other types of transformations.** One important type of linear transformation that is not a self-homothety is a transformation with diagonal matrix where the diagonal entries are not all the same. For instance, consider the linear transformation with matrix the following diagonal matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

This sends $\vec{e_1}$ to itself and sends $\vec{e_2}$ to $2\vec{e_2}$. Pictorially, it keeps the x-axis as is and stretches the y-axis by a factor of 2. It distorts shapes, but preserves linearity. It is not a self-homothety because of the different scaling factors used for the axes.

3.4. **Group structure.** A collection of bijective functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to form a *group* if it satisfies these three conditions:

- The composite of two functions in the collection is in the collection.
- The identity function is in the collection.
- The inverse to any function in the collection is in the collection.

The set of all automorphisms of any structure forms a group. Here is the stylized argument:

- Automorphisms preserve some particular structural feature. Composing two automorphisms will therefore also preserve the structural feature.
- The identity map preserves *everything*. Therefore, it must be an automorphism.
- Since an automorphism preserves a specific structural feature, doing it backwards must also preserve the structural feature.

All the examples we have seen above give groups of linear transformations. Explicitly:

- The set of all affine linear automorphisms of $\mathbb{R}^n$ is a group, because these are precisely the invertible functions that preserve the collinearity and ratios-within-lines structure.
- The set of all linear automorphisms of $\mathbb{R}^n$ is a group, because these are precisely the invertible functions that preserve the linear structure.
- The set of all self-isometries of $\mathbb{R}^n$ is a group, because these are precisely the invertible functions that preserve the Euclidean distance.
- The set of all self-homotheties of $\mathbb{R}^n$ of a group, because these are precisely the invertible functions that preserve the "ratios of Euclidean distances" structure.

Further, these groups have containment relations:

Group of all self-isometries of $\mathbb{R}^n$ $\subseteq$ Group of all self-homotheties of $\mathbb{R}^n$ $\subseteq$ Group of all affine linear automorphisms of $\mathbb{R}^n$

And also, separately:

Group of all linear automorphisms of $\mathbb{R}^n$ $\subseteq$ Group of all affine linear automorphisms of $\mathbb{R}^n$

4. The case of two dimensions

If $n = 2$, we can obtain a relatively thorough understanding of the various types of linear transformations. These are discussed in more detail below.

4.1. Trace and determinant.

Trace and determinant. There are two interesting invariants of 2×2 matrices, called respectively the *trace* and *determinant*. The trace of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as the quantity $a + d$. It is the sum of the diagonal entries. The significance of the trace is not clear right now, but will become so later.

The other important invariant for 2×2 matrices is the *determinant*. The determinant of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as the quantity $ad - bc$. As we noted in a homework exercise, the determinant is nonzero if and only if the matrix is invertible.

Both the trace and the determinant generalize to $n \times n$ matrices. The trace of a $n \times n$ matrix is defined as the sum of all the diagonal entries of the matrix. The determinant is defined in a more complicated fashion.

The trace and determinant of a linear transformation are defined respectively as the trace and determinant of the matrix for the linear transformation.

For an affine linear transformation, the trace and determinant are defined respectively as the trace and determinant of the linear part of the transformation.

The role of the determinant is somewhat hard to describe, but it can be split into two aspects:

- The absolute value of the determinant is the factor by which volumes multiply. In the case $n = 2$, it is the factor by which areas multiply. In particular, the determinant of a 2×2 matrix is ± 1 if and only if the corresponding linear transformation is area-preserving.
- The sign of the determinant describes whether the linear transformation is orientation-preserving or orientation-reversing. A positive sign means orientation-preserving, whereas a negative sign means orientation-reversing. Here, *orientation-preserving* means that left-handed remains left-handed while right-handed remains right-handed. In contrast, *orientation-reversing* means that left-handed becomes right-handed while right-handed becomes left-handed. Note that any transformation that can be accomplished through rigid motions, i.e., through a continuous deformation of the identity transformation, must be orientation-preserving. The reason is that continuous change cannot suddenly change the orientation status.

4.2. Justifying statements about the determinant using diagonal matrices.

Justifying statements about the determinant using diagonal matrices. Consider a linear transformation with diagonal matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

The trace of this matrix is $a + d$ and the determinant is ad. Here is the justification for both observations made above:

- The absolute value of the determinant is the factor by which volumes multiply: Think of a rectangle with sides parallel to the axes. The x-dimension gets multiplied by a factor of $|a|$ and the y-dimension gets multiplied by a factor of $|d|$. The area therefore gets multiplied by a factor of $|a||d|$ which is $|ad|$, the absolute value of the determinant.
- The sign of the determinant describes whether the linear transformation is orientation-preserving or orientation-reversing. The sign of a determines whether the x-direction gets flipped. The sign of d determines whether the y-direction gets flipped. The sign of the product determines whether the overall orientation stays the same or gets reversed.

4.3. **Abstract considerations: determinants of self-homotheties.** Consider a linear transformation that is a self-homothety with factor of similitude λ. This linear transformation scales all lengths by a factor of λ. If $n = 2$ (i.e., we are in two dimensions) then it scales all areas by a factor of λ^2. In particular:

- If it is orientation-preserving, then the determinant is λ^2.
- If it is orientation-reversing, then the determinant is $-\lambda^2$.

In particular, for a self-*isometry*:

- If it is orientation-preserving, then the determinant is 1.
- If it is orientation-reversing, then the determinant is -1.

4.4. **Rotations.** A particular kind of transformation of interest in the two-dimensional case is a *rotation*. A rotation is specified by two pieces of information: a point (called the *center of rotation*) and an angle (called the *angle of rotation*). The angle is defined only up to additive multiples of 2π, i.e., if two rotations have the same center and their angles differ by a multiple of 2π, then they are actually the same rotation.

Note that we set a convention in advance that we will interpret rotations in the counter-clockwise sense.

For a rotation whose angle of rotation is not zero (or more precisely, is not a multiple of 2π), the center of rotation is uniquely determined by the rotation and is the only fixed point of the rotation.

The rotation by an angle of π about a point is termed a *half turn* about the point and can alternatively by thought of as *reflecting* relative to the *point*. This is not to be confused with reflections about lines in $\mathbb{R}^2$.

All rotations are self-isometries of $\mathbb{R}^2$. Thus, they are affine linear automorphisms. They are also area-preserving. They are also orientation-preserving, since they can be obtained through continuous rigid motions. However, unless the center of rotation is the origin, the rotation is not a linear automorphism. Rotations centered at the origin *are* linear transformations. We will now proceed to describe rotations centered at the origin in matrix terms.

To describe a rotation centered at the origin, we need to describe the images of the two standard basis vectors $\vec{e_1}$ and $\vec{e_2}$. These images form the first and second column respectively of the matrix describing the rotation as a linear transformation.

Suppose the rotation is by an angle θ. Then $\vec{e_1}$ goes to the vector with coordinates $(\cos\theta, \sin\theta)$. $\vec{e_2}$ goes to the vector with coordinates $(-\sin\theta, \cos\theta)$. The matrix for the rotation is thus:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note that the inverse to rotation about the origin by θ is rotation by $-\theta$. Using the fact that cos is an even function and sin is an odd function ,the matrix for the inverse operation is:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Some specific rotations of interest are listed below:

Angle	Matrix of rotation
0 (no change)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\pi/4$	$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$
$\pi/2$ (right angle)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
π (half turn)	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$3\pi/2$ (right angle clockwise)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Let us try predicting the determinant of a rotation matrix theoretically, then proceed to verify it computationally.

Theoretically, we know that rotations are both area-preserving (on account of being self-isometries) and orientation-preserving (on account of being realized through rigid motions). The area-preserving nature tells us that the magnitude of the determinant is 1, i.e., the determinant is ± 1. The orientation-preserving nature tells us that the determinant is positive. Combining these, we get that the determinant must be 1. Let's check this. The determinant of

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is $\cos^2\theta + \sin^2\theta$, which is 1.

4.5. **Reflections and orientation-reversal.** A reflection about a line does exactly what it is supposed to do: it sends each point to another point such that the line of reflection is the perpendicular bisector of the line segment joining them.

Every reflection is an affine linear automorphism. A reflection is a linear automorphism if and only if the line of reflection passes through the origin. If that is the case, we can write the matrix of the linear transformation. Let's consider the case of a reflection about the x-axis.

This reflection fixes all points on the x-axis, and sends all points on the y-axis to their mirror images about the origin. In particular, it sends $\vec{e_1}$ to $\vec{e_1}$ and sends $\vec{e_2}$ to $-\vec{e_2}$.

The matrix of the linear transformation is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

More generally, consider a reflection about a line through the origin that makes an angle of $\theta/2$ counter-clockwise from the x-axis. The reflection sends $\vec{e_1}$ to a vector making an angle θ counter-clockwise from the horizontal, and sends $\vec{e_2}$ to the vector making an angle of $\theta - (\pi/2)$ counter-clockwise from the horizontal. The matrix is thus:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

This reflection matrix has trace zero. For the determinant, let us first predict it theoretically, then verify it computationally. Theoretically, we know that reflections are self-isometries, hence they are area-preserving. So, the absolute value of the determinant is 1, and the determinant is ± 1. Reflections are also orientation-reversing, so the determinant is negative. Thus, the determinant must be -1. Let's check this. The determinant of:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

is $-\cos^2\theta - \sin^2\theta = -1$.

4.6. **Shear operations.** A shear operation is an operation where one axis is kept fixed, and the other axis is "sheared" by having stuff from the fixed axis added to it.

For instance, suppose the x-axis is the fixed axis and we add the standard basis vector for the x-axis to the y-axis. Explicitly, $\vec{e_1}$ stays where it is, but $\vec{e_2}$ gets sent to $\vec{e_1} + \vec{e_2}$. The matrix of this looks like:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Note that unlike translations, rotations, and reflections, shear operations are not self-isometries.

More generally, we can have a shear of the form:

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

This sends $\vec{e_1}$ to $\vec{e_1}$ and sends $\vec{e_2}$ to $\lambda\vec{e_1} + \vec{e_2}$.

We could also shear in the other direction:

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

Here, $\vec{e_2}$ is fixed, and $\vec{e_1}$ gets sends to $\vec{e_1} + \lambda\vec{e_2}$.

The trace of a shear operation is 2, and the determinant is 1. Thus, shear operations are both area-preserving and orientation-preserving. This can be verified pictorially.

4.7. **Composites of various types, including glide reflections.** We have considered translations, rotations, and reflections. All of these are self-isometries. Self-isometries form a group, so composing things of these types should also give a self-isometry. Two interesting questions:

- Is every self-isometry a translation, rotation, or reflection?
- Can every self-isometry be obtained by composing translations, rotations, and reflections?

It turns out that the answers are respectively *no* and *yes*. Let's look at various sorts of composites:

(1) *Composite of translations*: Translations form a subgroup of the group of all self-isometries of $\mathbb{R}^2$. In other words, the composite of two translations is a translation, the identity map is a translation (namely, by the zero vector) and the inverse of a translation is a translation.
(2) *Composite of rotations centered at the same point*: The rotations centered at a particular point form a subgroup of the group of all self-isometries of $\mathbb{R}^2$. In other words, the composite of two rotations centered at the same point is a rotation, the identity map is a rotation with any point as center (namely, with zero angle of rotation), and the inverse of a rotation is a rotation with the same center of rotation. However, the set of *all* rotations is not a subgroup, as is clear from the next point.
(3) *Composite of rotations centered at different points*: If two rotations centered at different points are composed, the composite is typically a rotation about yet a third point, with the angle of rotation the sum of the angles. The exception is when the angles add up to a multiple of 2π. In that case, the composite is a translation. It is easy to convince yourself by using human stick figures that the angles of rotation add up.
(4) *Composite of rotation and translation*: The composite is again a rotation with the same angle of rotation, but about a different center of rotation.
(5) *Composite of two reflections*: If the lines of reflection are parallel, the composite is a translation by a vector perpendicular to both. If the lines of reflection intersect, then the composite is a rotation by twice the angle of intersection between the lines.
(6) *Composite of reflection and translation*: This gives rise to what is called a **glide reflection**, which is a new type of self-isometry of $\mathbb{R}^2$.
(7) *Composite of reflection and rotation*: This is trickier. It could be a reflection or a glide reflection, depending on whether the center of rotation lies on the line of reflection.

The upshot is that:

- The orientation-*preserving* self-isometries of $\mathbb{R}^2$ are precisely the translations and rotations. Note that these form a group.
- The orientation-*reversing* self-isometries of $\mathbb{R}^2$ are precisely the reflections and glide reflections. Note that these do not form a group, but *together* with translations and rotations, they form the group of all self-isometries.

4.8. **Self-isometries that are linear.** Let's consider self-isometries that are linear, i.e., they fix the origin. These are subgroups of the group of all self-isometries. Explicitly:

- The orientation-*preserving* linear self-isometries of $\mathbb{R}^2$ are precisely the rotations about the origin, specified by the angle of rotation (determined up to additive multiples of 2π). Composing two such rotations involves adding the corresponding angles. These form a group. This group is denoted $SO(2, \mathbb{R})$ (you don't need to know this!) and is called the *special orthogonal group* of degree two over the reals.
- The orientation-*reversing* linear self-isometries of $\mathbb{R}^2$ are precisely the reflections about lines through the origin. These do not form a group, but *together* with rotations about the origin, they form a group. The whole group is denoted $O(2, \mathbb{R})$ (you don't need to know this!) and is called the *orthogonal group* of degree two over the reals.

In general, an affine linear automorphism is a self-isometry if and only if its linear automorphism part is a self-isometry. In other words:

$$\vec{x} \mapsto A\vec{x} + \vec{b}$$

is a self-isometry if and only if $\vec{x} \mapsto A\vec{x}$ is a self-isometry.

5. The case of three dimensions

The case of three dimensions is somewhat trickier than two dimensions, but we can still come somewhat close to a classification.

5.1. Rotations about axes. The simplest type of orientation-preserving self-isometry is a rotation about an axis of rotation. Euler proved a theorem (called *Euler's rotation theorem*) that every orientation-preserving self-isometry that fixes a point must be a rotation about an axis through that point. In particular, all the orientation-preserving self-isometries of $\mathbb{R}^3$ that are *linear* (in the sense of fixing the origin) are rotations about axes through the origin.

5.2. Rotations composed with translations. We know that translations are orientation-preserving self-isometries of $\mathbb{R}^n$ for any n. So are rotations. We also know that the self-isometries form a group. Thus, composing a rotation about an axis with a translation should yield a self-isometry. For $n = 2$, any such self-isometry would already be a rotation. For $n = 3$, this is no longer the case. It is possible to have orientation-preserving self-isometries that are expressible as composites of translations and rotations but are not translations or rotations themselves. For instance, a rotation about the z-axis followed by a translation parallel to the z-axis works.

5.3. Reflections about planes and their composites. A reflection about a plane in $\mathbb{R}^3$ is an orientation-reversing self-isometry of $\mathbb{R}^3$. For instance, the isometry:

$$(x, y, z) \mapsto (-x, y, z)$$

is a reflection about the yz-plane. It is orientation-reversing, and its matrix is:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have not yet seen how to compute the determinant of a general 3×3 matrix. However, the determinant of a diagonal matrix is simply the product of the diagonal entries. In this case, the determinant is -1, as it should be, since the reflection is orientation-reversing but, on account of being a self-isometry, is *volume*-preserving.

A composite of two reflections about different planes is an orientation-preserving self-isometry. If the planes are not parallel, this is a rotation about the axis of intersection by twice the angle of intersection between the planes. For instance, the transformation:

$$(x, y, z) \mapsto (-x, -y, z)$$

corresponds to the diagonal matrix:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This has determinant 1. It is both orientation-preserving and area-preserving. We can also think of it as a rotation by an angle of π about the z-axis, i.e., a half-turn about the z-axis. The angle is π because the individual planes of reflection are mutually perpendicular (angle $\pi/2$).

Finally, consider a composite of *three* reflections. Consider the simple case where the reflections are about three mutually perpendicular planes. An example is:

$$(x, y, z) \mapsto (-x, -y, -z)$$

This corresponds to the diagonal matrix:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The linear transformation here is orientation-reversing on account of being a composite of an odd number of reflections. It is a self-isometry, so the determinant should be -1, and indeed, the determinant is -1.

More generally, we could have a map of the form:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

This map reflects the x-coordinate and performs a rotation by θ on the yz-plane.

6. Where we're hoping to go with this

In the future, we will build on what we have learned so far in the following ways:

- We will understand the procedure for composing linear transformations (or more generally affine linear transformations) purely algebraically, i.e., in terms of their description using matrices and vectors.
- We will understand criteria for looking at a linear transformation algebraically to determine whether it is a self-isometry, self-homothety, orientation-preserving, and/or area-preserving.
- We will understand more about the group structure of various kinds of groups of linear transformations and affine linear transformations.