# TAKE-HOME CLASS QUIZ: DUE WEDNESDAY NOVEMBER 27: SIMILARITY OF LINEAR TRANSFORMATIONS

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

**PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.**

This quiz corresponds to material discussed in the lecture notes titled `Coordinates`. It also corresponds to Section 3.4 of the text.

Recall that $n \times n$ matrices $A$ and $B$ are termed *similar* if there exists an invertible $n \times n$ matrix $S$ such that $A = SBS^{-1}$. The relation of matrices being similar is an *equivalence relation* (please refer to the notes for an explanation of the terminology).

For these questions, assume $n > 1$, because a lot of phenomena are much simpler in the case $n = 1$ and you may be misled if you look only at that case. In other words, just because an equality is true for $1 \times 1$ matrices, do not assume it is always true. On the other hand, if you can find *counterexamples* to a statement for $1 \times 1$ matrices, you can probably use that to construct counterexamples for all sizes of matrices by using scalar matrices.

(1) Which of the following can we say about two (possibly equal, possibly distinct) similar $n \times n$ matrices $A$ and $B$? Please see Options (D) and (E) before answering.
(A) $A$ is invertible if and only if $B$ is invertible.
(B) $A$ is nilpotent if and only if $B$ is nilpotent.
(C) $A$ is idempotent if and only if $B$ is idempotent.
(D) All of the above.
(E) None of the above.

Your answer: ______________________

(2) Which of the following can we say about two (possibly equal, possibly distinct) similar $n \times n$ matrices $A$ and $B$? Please see Options (D) and (E) before answering.
(A) $A$ is scalar if and only if $B$ is scalar.
(B) $A$ is diagonal if and only if $B$ is diagonal.
(C) $A$ is upper triangular if and only if $B$ is upper triangular.
(D) All of the above.
(E) None of the above.

Your answer: ______________________

(3) Suppose $A_1, A_2, B_1, B_2$ are $n \times n$ matrices such that $A_1$ is similar to $B_1$ and $A_2$ is similar to $B_2$. Which of the following is *definitely* true? Please see Options (D) and (E) before answering.
(A) $A_1 + A_2$ is similar to $B_1 + B_2$.
(B) $A_1 - A_2$ is similar to $B_1 - B_2$.
(C) $A_1A_2$ is similar to $B_1B_2$.
(D) All of the above.
(E) None of the above.

Your answer: ______________________

(4) Suppose $A_1, A_2, B_1, B_2$ are $n \times n$ matrices such that $A_1$ is similar to $B_1$ and $A_2$ is similar to $B_2$. Which of the following is *definitely* true? Please see Options (D) and (E) before answering.
(A) $A_1 + B_1$ is similar to $A_2 + B_2$.
(B) $A_1 - B_1$ is similar to $A_2 - B_2$.

(C) $A_1B_1$ is similar to $A_2B_2$.
(D) All of the above.
(E) None of the above.

Your answer: ____________________________

(5) Suppose $A$ and $B$ are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
(A) $A$ is similar to $B$ if and only if $-A$ is similar to $-B$.
(B) If $A$ is similar to $B$, then $-A$ is similar to $-B$. However, $-A$ being similar to $-B$ does not imply that $A$ is similar to $B$.
(C) If $-A$ is similar to $-B$, then $A$ is similar to $B$. However, $A$ being similar to $B$ does not imply that $-A$ is similar to $-B$.
(D) $A$ being similar to $B$ does not imply that $-A$ is similar to $-B$. Also, $-A$ being similar to $-B$ does not imply that $A$ is similar to $B$.

Your answer: ____________________________

(6) Suppose $A$ and $B$ are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
(A) $A$ is similar to $B$ if and only if $2A$ is similar to $2B$.
(B) If $A$ is similar to $B$, then $2A$ is similar to $2B$. However, $2A$ being similar to $2B$ does not imply that $A$ is similar to $B$.
(C) If $2A$ is similar to $2B$, then $A$ is similar to $B$. However, $A$ being similar to $B$ does not imply that $2A$ is similar to $2B$.
(D) $A$ being similar to $B$ does not imply that $2A$ is similar to $2B$. Also, $2A$ being similar to $2B$ does not imply that $A$ is similar to $B$.

Your answer: ____________________________

(7) Suppose $A$ and $B$ are both invertible $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
(A) $A$ is similar to $B$ if and only if $A^{-1}$ is similar to $B^{-1}$.
(B) If $A$ is similar to $B$, then $A^{-1}$ is similar to $B^{-1}$. However, $A^{-1}$ being similar to $B^{-1}$ does not imply that $A$ is similar to $B$.
(C) If $A^{-1}$ is similar to $B^{-1}$, then $A$ is similar to $B$. However, $A$ being similar to $B$ does not imply that $A^{-1}$ is similar to $B^{-1}$
(D) $A$ being similar to $B$ does not imply that $A^{-1}$ is similar to $B^{-1}$. Also, $A^{-1}$ being similar to $B^{-1}$ does not imply that $A$ is similar to $B$.

Your answer: ____________________________

(8) Suppose $A$ and $B$ are both $n \times n$ matrices (but they are not given to be similar). Which of the following holds?
(A) $A$ is similar to $B$ if and only if $A^2$ is similar to $B^2$.
(B) If $A$ is similar to $B$, then $A^2$ is similar to $B^2$. However, $A^2$ being similar to $B^2$ does not imply that $A$ is similar to $B$.
(C) If $A^2$ is similar to $B^2$, then $A$ is similar to $B$. However, $A$ being similar to $B$ does not imply that $A^2$ is similar to $B^2$.
(D) $A$ being similar to $B$ does not imply that $A^2$ is similar to $B^2$. Also, $A^2$ being similar to $B^2$ does not imply that $A$ is similar to $B$.

Your answer: ____________________________

(9) Suppose $A$ and $B$ are $n \times n$ matrices (but they are not given to be similar and they are not given to be invertible). We say that $A$ and $B$ are *quasi-similar* (not a standard term!) if there exist $n \times n$

matrices $C$ and $D$ such that $A = CD$ and $B = DC$. What can we say is the relation between being similar and being quasi-similar?

(A) $A$ and $B$ are similar if and only if they are quasi-similar.

(B) If $A$ and $B$ are similar, they are quasi-similar. However, the converse is not necessarily true: $A$ and $B$ may be quasi-similar but not similar.

(C) If $A$ and $B$ are quasi-similar, they are similar. However, the converse is not necessarily true: $A$ and $B$ may be similar but not quasi-similar.

(D) Neither implies the other. $A$ and $B$ may be similar but not quasi-similar. Also, $A$ and $B$ may be quasi-similar but not similar.

Your answer: ____________________________

(10) With the notion of quasi-similar as defined in the preceding question, what can we say about the relation between being similar and being quasi-similar for $n \times n$ matrices $A$ and $B$ that are both given to be *invertible*?

(A) $A$ and $B$ are similar if and only if they are quasi-similar.

(B) If $A$ and $B$ are similar, they are quasi-similar. However, the converse is not necessarily true: $A$ and $B$ may be quasi-similar but not similar.

(C) If $A$ and $B$ are quasi-similar, they are similar. However, the converse is not necessarily true: $A$ and $B$ may be similar but not quasi-similar.

(D) Neither implies the other. $A$ and $B$ may be similar but not quasi-similar. Also, $A$ and $B$ may be quasi-similar but not similar.

Your answer: ____________________________

(11) Suppose $A$ and $B$ are two $n \times n$ matrices. Which of the following best describes the relation between similarity and having the same rank?

(A) $A$ and $B$ are similar if and only if they have the same rank.

(B) If $A$ and $B$ are similar, then they have the same rank. However, it is possible for $A$ and $B$ to have the same rank but not be similar.

(C) If $A$ and $B$ have the same rank, then they are similar. However, it is possible for $A$ and $B$ to be similar but not have the same rank.

(D) $A$ and $B$ may be similar but have different ranks. Also, $A$ and $B$ may have the same rank but not be similar.

Your answer: ____________________________

(12) Suppose $A$ and $B$ are two $n \times n$ matrices. Which of the following best describes the relation between quasi-similarity and having the same rank?

(A) $A$ and $B$ are quasi-similar if and only if they have the same rank.

(B) If $A$ and $B$ are quasi-similar, then they have the same rank. However, it is possible for $A$ and $B$ to have the same rank but not be quasi-similar.

(C) If $A$ and $B$ have the same rank, then they are quasi-similar. However, it is possible for $A$ and $B$ to be quasi-similar but not have the same rank.

(D) $A$ and $B$ may be quasi-similar but have different ranks. Also, $A$ and $B$ may have the same rank but not be quasi-similar.

Your answer: ____________________________