TAKE-HOME CLASS QUIZ: DUE MONDAY NOVEMBER 25: STOCHASTIC MATRICES

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz can be viewed as a continuation of the quiz on linear dynamical systems. The book defines column-stochastic matrices using the jargon "transition matrix" on Page 53 (Definition 2.1.4) and uses them throughout the text when describing (a simplified version of) Google's PageRank algorithm. The quiz questions are self-contained and do not require you to read the book, but you may benefit from skimming through the book's discussion of PageRank to complement these questions. Note that the 4th Edition does not include the discussion of transition matrices and PageRank.

In this quiz, we discuss the dynamics of a very special type of linear transformation. A $n \times n$ matrix A is termed a *row-stochastic matrix* if all its entries are in the interval $[0, 1]$ and all the row sums are equal to 1. A $n \times n$ matrix is termed a *column-stochastic matrix* if all its entries are in the interval $[0, 1]$ and all the column sums are equal to 1. A $n \times n$ matrix A is termed a *doubly stochastic matrix* if it is both row-stochastic and column-stochastic, i.e., all the entries are in the interval $[0, 1]$, all the row sums are equal to 1, and all the column sums are equal to 1.

(1) Suppose A and B are two $n \times n$ row-stochastic matrices. Which of the following is *guaranteed* to be row-stochastic? Please see Options (D) and (E) before answering.
(A) $A + B$
(B) $A - B$
(C) AB
(D) All of the above
(E) None of the above

Your answer: ______________________

(2) Suppose A and B are two $n \times n$ column-stochastic matrices. Which of the following is *guaranteed* to be column-stochastic? Please see Options (D) and (E) before answering.
(A) $A + B$
(B) $A - B$
(C) AB
(D) All of the above
(E) None of the above

Your answer: ______________________

(3) Suppose A and B are two $n \times n$ doubly stochastic matrices. Which of the following is *guaranteed* to be doubly stochastic? Please see Options (D) and (E) before answering.
(A) $A + B$
(B) $A - B$
(C) AB
(D) All of the above
(E) None of the above

Your answer: ______________________

We now consider the case $n = 2$. In this case, the doubly stochastic matrices have the form:

$$\begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix}$$

where $a \in [0, 1]$. Denote this matrix by $D(a)$ for short.

(4) Suppose $a, b \in [0, 1]$ (they are allowed to be equal). The product $D(a)D(b)$ equals $D(c)$ for some $c \in [0, 1]$. What is that value of c?
(A) $a + b$
(B) ab
(C) $2ab + a + b$
(D) $(1 - a)(1 - b)$
(E) $1 - a - b + 2ab$

Your answer: ____________________________

(5) For what value(s) of a is the matrix $D(a)$ non-invertible? Note that when judging invertibility, we do not insist that the inverse matrix also be doubly stochastic.
(A) $a = 0$ only
(B) $a = 1/2$ only
(C) $a = 1$ only
(D) $0 < a < 1$ (i.e., $D(a)$ is invertible only at $a = 0$ and $a = 1$)
(E) $a \neq 1/2$

Your answer: ____________________________

(6) For what value(s) of a is it true that the matrix $D(a)$ does not have an inverse that is a doubly stochastic matrix? In other words, either $D(a)$ should be non-invertible or it should be invertible but the inverse is not a doubly stochastic matrix.
(A) $a = 0$ only
(B) $a = 1/2$ only
(C) $a = 1$ only
(D) $0 < a < 1$ (i.e., $D(a)$ has an inverse that is also doubly stochastic only if $a = 0$ or $a = 1$)
(E) $a \neq 1/2$

Your answer: ____________________________

For the next few questions, denote by T_a the linear transformation whose matrix is $D(a)$. For any vector $\vec{x} \in \mathbb{R}^2$, we can consider the sequence:

$$\vec{x}, T_a(\vec{x}), T_a^2(\vec{x}), \ldots$$

Note that if we were to start with a vector $\vec{x} \in \mathbb{R}^2$ with both coordinates equal, it would be invariant under T_a.

Thus, for the questions below, assume that we start with a nonzero vector $\vec{x} \in \mathbb{R}^2$ for which the two coordinates are not equal to each other.

(7) For what value of a is it the case that $\lim_{r \to \infty} T_a^r(\vec{x})$ does *not* exist?
(A) $a = 0$ only
(B) $a = 1/2$ only
(C) $a = 1$ only
(D) $0 < a < 1$
(E) $a \neq 1/2$

Your answer: ____________________________

(8) For what value of a is it the case that the sequence

$$\vec{x}, T_a(\vec{x}), T_a^2(\vec{x}), \ldots$$

is a constant sequence?

(A) $a = 0$ only
(B) $a = 1/2$ only
(C) $a = 1$ only
(D) $0 < a < 1$
(E) $a \neq 1/2$

Your answer: ____________________________

(9) For what value of a is it the case that the sequence

$$\vec{x}, T_a(\vec{x}), T_a^2(\vec{x}), \ldots$$

is not a constant sequence but becomes constant from $T_a(\vec{x})$ onward?

(A) $a = 0$ only
(B) $a = 1/2$ only
(C) $a = 1$ only
(D) $0 < a < 1$
(E) $a \neq 1/2$

Your answer: ____________________________

(10) For a other than 0, 1/2, or 1, what is the limit $\lim_{r \to \infty} (D(a))^r$? Here, when we talk of taking the limit of a sequence of matrices, we are taking the limit entry-wise.

(A) The matrix $D(0)$
(B) The matrix $D(1/2)$
(C) The matrix $D(1)$
(D) The matrix $D(a)$
(E) The matrix $D(1 - a)$

Your answer: ____________________________