TAKE-HOME CLASS QUIZ: DUE WEDNESDAY NOVEMBER 13: MATRIX MULTIPLICATION: ROWS, COLUMNS, ORTHOGONALITY, AND OTHER MISCELLANEA

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The purpose of this quiz is two-fold. First, many of the ideas related to matrix multiplication are at the stage where a bit of review will help prevent their fading out. Drawing from the best research on *spaced repetition* (see for instance `http://en.wikipedia.org/wiki/Spaced_repetition`) we will try to recall some of the stuff. But with a twist, because we consider it from a somewhat different angle.

Second, the new angle will also turn out to be useful for later material.

For Questions 1-5: Given a n-dimensional vector $\langle a_1, a_2, \dots, a_n \rangle \in \mathbb{R}^n$, the vector can be interpreted as a $n \times 1$ matrix (a column vector). This is the default interpretation. But there are also two other interpretations: as a $1 \times n$ matrix (a row vector) and as a diagonal $n \times n$ matrix.

Also note that for Questions 1-5, all the three ways of representing vectors coincide with each other for $n = 1$, so the questions are uninteresting for $n = 1$ because all answer options are equivalent. You may therefore assume that $n > 1$ for these questions, though obviously the correct answers are correct for $n = 1$ as well.

(1) Suppose I want to add two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ to obtain the output vector $\langle a_1 + b_1, a_2 + b_2, \dots, a_n + b_n \rangle$ using matrix addition. What format (row vector, column vector, or diagonal matrix) should I use? Please see Option (D) before answering and select the option that best describes your view.
 (A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors and interpret the sum as a row vector.
 (B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors and interpret the sum as a column vector.
 (C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices and interpret the sum as a diagonal matrix.
 (D) We can use any of the above.

 Your answer: ____________________________

(2) Suppose I want to perform coordinate-wise multiplication on two vectors. Explicitly, I have two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ and I want to obtain the output vector $\langle a_1b_1, a_2b_2, \dots, a_nb_n \rangle$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left and the matrix for $\vec{b}$ written on the right). What format (row vector, column vector, or diagonal matrix) should I use? Please see Option (D) before answering and select the option that best describes your view.
 (A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors and interpret the matrix product as a row vector.
 (B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors and interpret the matrix product as a column vector.
 (C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices and interpret the matrix product as a diagonal matrix.
 (D) We can use any of the above.

 Your answer: ____________________________

(3) Suppose I am given two vectors $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$ and $\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$ and I want to obtain a 1×1 matrix with entry $\sum_{i=1}^n a_ib_i$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left and the matrix for $\vec{b}$ written on the right). What format (row vector, column vector, or diagonal matrix) should I use?
 (A) Represent both $\vec{a}$ and $\vec{b}$ as row vectors.

(B) Represent both $\vec{a}$ and $\vec{b}$ as column vectors.
(C) Represent both $\vec{a}$ and $\vec{b}$ as diagonal matrices.
(D) Represent $\vec{a}$ as a row vector and $\vec{b}$ as a column vector.
(E) Represent $\vec{a}$ as a column vector and $\vec{b}$ as a row vector.

Your answer: ____________________________

(4) Suppose I am given three vectors $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$, $\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$, and $\vec{c} = \langle c_1, c_2, \ldots, c_n \rangle$. I want to obtain a 1×1 matrix with entry $\sum_{i=1}^{n} (a_i b_i c_i)$ using matrix multiplication (with the matrix for $\vec{a}$ written on the left, the matrix for $\vec{b}$ written in the middle, and the matrix for $\vec{c}$ written on the right). What format should I use?
(A) $\vec{a}$ as a row vector, $\vec{b}$ as a column vector, $\vec{c}$ as a diagonal matrix.
(B) $\vec{a}$ as a column vector, $\vec{b}$ as a row vector, $\vec{c}$ as a diagonal matrix.
(C) $\vec{a}$ as a diagonal matrix, $\vec{b}$ as a row vector, $\vec{c}$ as a column vector.
(D) $\vec{a}$ as a column vector, $\vec{b}$ as a diagonal matrix, $\vec{c}$ as a row vector.
(E) $\vec{a}$ as a row vector, $\vec{b}$ as a diagonal matrix, $\vec{c}$ as a column vector.

Your answer: ____________________________

The next few questions rely on the concept of orthogonality (*orthogonal* is a synonym for *perpendicular* or *at right angles*). We say that two vectors (of the same dimension) are orthogonal if their dot product is zero. By this definition, the zero vector of a given dimension is orthogonal to every vector of that dimension. Note that it does not make sense to talk of orthogonality for vectors with different dimensions, i.e., with different numbers of coordinates.

(5) Suppose A is a $n \times m$ matrix. We can think of solving the system $A\vec{x} = \vec{0}$ (where $\vec{x}$ is a $m \times 1$ column vector of unknowns) as trying to find all the vectors orthogonal to all the vectors in a given set of vectors. What set of vectors is that?
(A) The set of row vectors of A, i.e., the rows of A, viewed as m-dimensional vectors.
(B) The set of column vectors of A, i.e., the columns of A, viewed as n-dimensional vectors.

Your answer: ____________________________

(6) Suppose A is a $p \times q$ matrix and B is a $q \times r$ matrix where p, q, and r are positive integers. The matrix product AB is a $p \times r$ matrix. What orthogonality condition corresponds to the condition that the matrix product AB is a zero matrix (i.e., all its entries are zero)?
(A) Every row of A is orthogonal to every row of B.
(B) Every row of A is orthogonal to every column of B.
(C) Every column of A is orthogonal to every row of B.
(D) Every column of A is orthogonal to every column of B.

Your answer: ____________________________

(7) Suppose A is an invertible $n \times n$ square matrix. Which of the following correctly characterizes the $n \times n$ matrix A^{-1} using orthogonality? Recall that AA^{-1} and $A^{-1}A$ are both equal to the $n \times n$ identity matrix.
(A) For every i in $\{1, 2, \ldots, n\}$, the i^{th} row of A is orthogonal to the i^{th} row of A^{-1}. The dot product of the i^{th} row of A and the j^{th} row of A^{-1} for distinct i, j in $\{1, 2, \ldots, n\}$ equals 1.
(B) For every i in $\{1, 2, \ldots, n\}$, the i^{th} column of A is orthogonal to the i^{th} column of A^{-1}. The dot product of the i^{th} row of A and the j^{th} column of A^{-1} for distinct i, j in $\{1, 2, \ldots, n\}$ equals 1.
(C) For every distinct i, j in $\{1, 2, \ldots, n\}$, the i^{th} row of A is orthogonal to the j^{th} row of A^{-1}. The dot product of the i^{th} row of A with the i^{th} row of A^{-1} equals 1.
(D) For every i in $\{1, 2, \ldots, n\}$, the i^{th} row of A is orthogonal to the i^{th} column of A^{-1}. The dot product of the i^{th} row of A and the j^{th} column of A^{-1} for distinct i, j in $\{1, 2, \ldots, n\}$ equals 1.

(E) For every distinct i, j in $\{1, 2, \ldots, n\}$, the i^{th} row of A is orthogonal to the j^{th} column of A^{-1}. The dot product of the i^{th} row of A and the i^{th} column of A^{-1} equals 1.

Your answer: ____________________________

The remaining questions review your skills at abstract behavior prediction.

(8) Suppose n is a positive integer greater than 1. Which of the following is always true for two invertible $n \times n$ matrices A and B?
(A) $A + B$ is invertible, and $(A + B)^{-1} = A^{-1} + B^{-1}$
(B) $A + B$ is invertible, and $(A + B)^{-1} = B^{-1} + A^{-1}$
(C) $A + B$ is invertible, though neither of the formulas of the preceding two options is correct
(D) AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$
(E) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

Your answer: ____________________________

(9) Suppose n is a positive integer greater than 1. For a nilpotent $n \times n$ matrix C, define the *nilpotency* of C as the smallest positive integer r such that $C^r = 0$. Note that the nilpotency is not defined for a non-nilpotent matrix. Given two $n \times n$ matrices A and B, what is the relation between the nilpotencies of AB and BA?
(A) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must be equal.
(B) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must differ by 1.
(C) AB is nilpotent if and only if BA is nilpotent, and if so, their nilpotencies must either be equal or differ by 1.
(D) It is possible for AB to be nilpotent and BA to be non-nilpotent; however, *if* both are nilpotent, then their nilpotencies must be equal.
(E) It is possible for AB to be nilpotent and BA to be non-nilpotent, however *if* both are nilpotent, then their nilpotencies must differ by 1.

Your answer: ____________________________

(10) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Your answer: ____________________________

(11) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^3 = B^3$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Your answer: ____________________________

(12) What is the smallest n for which there exist examples of invertible $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$ and $A^3 = B^3$?
(A) 1

(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Your answer: ____________________________

(13) What is the smallest n for which there exist examples of (not necessarily invertible) $n \times n$ matrices A and B such that $A \neq B$ but $A^2 = B^2$ and $A^3 = B^3$?
(A) 1
(B) 2
(C) 3
(D) 4
(E) This is not possible for any n.

Your answer: ____________________________