VECTOR STUFF

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Sections 12.2, 12.3, 12.4.

What students should definitely get: Vector as a n-tuple, special attention to case $n = 3$, geometric interpretation of vectors, free and localized vectors, computation and properties of dot product, computation and properties of length, computation and properties of cross product, computation and properties of scalar triple product. Geometric applications to computing angles, areas of triangles and parallelograms, finding a vector orthogonal to two vectors, testing coplanarity and orthogonality.

What students should hopefully get: The idea of correlation, cosine as measuring closeness, the significance of n-dimensional vectors, the notion of vector operations and meaningfulness depending on the kind of scale from which the coordinates are drawn, the notions of ordinal scale, difference scale, and ratio scale.

EXECUTIVE SUMMARY

0.1. n-dimensional generality. Words ...

(1) A vector is an ordered n-tuple of real numbers (or quantities measured using real numbers). The space of such n-tuples is a n-dimensional vector space over the real numbers. Vectors can be used to store tuples of prices, probabilities, and other kinds of quantities.

(2) There is a zero vector. We can add vectors and we can multiply a vector by a scalar. Note that these operations may or may not have an actual meaning based on the thing we are storing using the vector.

(3) We can take the dot product $v \cdot w$ of two vectors v and w in n-dimensional space. if $v = \langle v_1, v_2, \ldots, v_n \rangle$ and $w = \langle w_1, w_2, \ldots, w_n \rangle$, then $v \cdot w = \sum_{i=1}^{n} v_i w_i$. The dot product is a real number (though if we put units to the coordinates of the vector, it gets corresponding squared units; if we use different units for the different vectors, the units are the product units).

(4) The length or norm of a vector v, denoted $|v|$, is defined as $\sqrt{v \cdot v}$. It is a nonnegative real number.

(5) The correlation between two vectors v and w is defined as $(v \cdot w)/(|v||w|)$. It is in $[-1, 1]$. (For geometric interpretation, see the three-dimensional case).

(6) *Properties of the dot product*: The dot product is symmetric, the dot product of any vector with the zero vector is 0, the dot product is additive (distributive) in each coordinate and scalars can be pulled out.

(7) *Properties of length*: The only vector with length zero is the zero vector, all other vectors have positive length. The length of λv is $|\lambda|$ times the length of v. We also have $|v + w| \le |v| + |w|$ for any vectors v and w, with equality occurring if either is a positive scalar multiple of the other or one of them is the zero vector.

0.2. Three-dimensional geometry. Words ...

(1) We can identify points in three-dimensional space with three-dimensional vector as follows: the vector corresponding to a point (x, y, z) is the vector $\langle x, y, z \rangle$. Physically, this can be thought of as a directed line segment or arrow from the origin to the point (x, y, z).

(2) We can also consider vectors starting at any point in three-dimensional space and ending at any point. The corresponding vector can be obtained by subtracting the coordinates of the points. The vector from point P to point Q is denoted $\overline{PQ}$.

(3) There are unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. These are thus the vectors of length 1 along the positive x, y, and z directions respectively. A vector $\langle x, y, z \rangle$ can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(4) Vectors can be added geometrically using the *parallelogram law*. This procedure gives the same answer as the usual coordinate-wise addition.

(5) Scalar multiplication also has a geometric interpretation – the length gets scaled by the scalar multiple, and the direction remains the same or is reversed depending on the scalar's sign.

(6) For vectors v and w, we have $v \cdot w = |v||w| \cos \theta$ where θ is the angle between v and w. We can use this procedure to find the angle between two vectors. The correlation between the vectors is thus $\cos \theta$. We can interpret this specifically for $\theta = 0$, θ an acute angle, $\theta = \pi/2$, θ an obtuse angle, and $\theta = \pi$ (see the table in the lecture notes).

(7) We can define the vector cross product $v \times w$ using a matrix determinant. Equivalently, if $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$, then $v \times w = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$. *This is a specifically three-dimensional construct.*

(8) The cross product has the property that cross product of any two qcollinear vectors is zero, cross product of any vector with the zero vector is zero, the cross product is skew-symmetric, distributive in each variable, and allows scalars to be pulled out. It is not associative in general. There is an identity relating cross product and dot product: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. Also, the cross product satisfies the relation:
$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$

(9) The cross product of a and b satisfies $|a \times b| = |a||b| \sin \theta$ where θ is the angle between a and b, and further, the cross product vector is perpendicular to both a and b, and its direction is given by the right hand rule.

(10) There is a scalar triple product. The scalar triple product of vectors a, b, and c is defined as the number $a \cdot (b \times c)$. It can also be viewed as the determinant of a matrix whose rows are the coordinates of a, b, and c respectively. The scalar triple product is preserved under cyclic permutations of the input vectors and gets negated under flipping two of the input vectors. It is linear in each input variable (i.e., distributive and pulls out scalars). The scalar triple product is zero if and only if the three input vectors can all be made to lie in the same plane.

Actions ...

(1) Vector and scalar projections: Given vectors a and b, the *vector projection* of b onto a, denoted $\text{proj}_a b$, is given by the vector $\frac{a \cdot b}{|a|^2} a$. The scalar projection or component of b along a, denoted $\text{comp}_a b$, is given by $\frac{a \cdot b}{|a|}$. The vector projection is what we obtain by taking the vector from the origin to the foot of the perpendicular from the head of b to the line of a. The scalar projection is the *directed* length of this vector, measured positive in the direction of a.

(2) Finding the angle between vectors: This is done using the dot product. The angle between vectors v and w is $\arccos((v \cdot w)/|v||w|)$.

(3) Finding the area of a triangle or a parallelogram: We first find two adjacent sides as vectors both with the same starting vertex (by taking the differences of coordinates of endpoints). For the parallelogram, we take the length of the cross product of these two vectors. For the triangle, we take *half* the length.

(4) Finding the volume of a parallelopiped: We find three sides as vectors, all with the same starting vertex. Then we take the scalar triple product of these sides.

(5) Finding a vector orthogonal to two given vectors: Simply take the cross product if they are linearly independent. Otherwise, just pick anything that dots with one of them to zero.

(6) Testing orthogonality: We check whether the dot product is zero.

(7) Testing coplanarity of points: We take one point as the basepoint, compute difference vectors to it from the other three points. We then take the scalar triple product of these three vectors. If we get zero, then the four points are coplanar, otherwise they are not.

1. Vectors – what's the big deal?

In this lecture (and the corresponding notes) I will take a somewhat different approach to vector spaces from what is to be found in the book. The reasons are multi-fold. First, the highly geometric approach to vectors that the book takes, although standard fare, is somewhat misleading for applications of vector space

theory to the social sciences. The approach taken in the book *is* right for applying vector spaces to the physical sciences, particularly physics. But the intended as well as actual audience for this course comprises economics majors, so I've determined the focus accordingly.

1.1. **Vector as a tuple of data, and vector-valued functions.**
A *real number* can be used to store a single quantitative variable. What happens when we need to store more than one variable? In this case, we need to store a bunch of real numbers. One way of storing a bunch of data is as a tuple, or a system of comma-separated values. This kind of bunch of data can be thought of as a vector.

For instance, let's think of the share prices of five different companies as a function of time. At any given time, each share price is a number (measured in fixed units). The collection of five share prices is a tuple of 5 numbers, which can be represented as comma-separated values. If the prices are p_1, p_2, p_3, p_4, and p_5, then the tuple in question is $\langle p_1, p_2, p_3, p_4, p_5 \rangle$.

Since the prices vary *with time*, each of the prices is actually a function of time, so what we really have is a *vector-valued function* whose coordinates are p_1, p_2, p_3, p_4, and p_5, given by:

$$t \mapsto \langle p_1(t), p_2(t), p_3(t), p_4(t), p_5(t) \rangle$$

We say that this is a vector-valued function in a five-dimensional vector space.

Vector-valued functions are a new perspective on the parametric descriptions we saw a little while ago. In fact, these parametric descriptions of curves in the plane can be thought of as vector-valued functions in two dimensions.

Here are some other examples of vectors and vector-valued functions:

- To keep track of prices, a bundle of commodities is chosen and unit prices for these commodities are tracked as a function of time. If 15 commodities are chosen, each of the unit prices is a function of time. The collection of all unit prices at a given time is a vector in 15 dimensions and the function that sends a time to the collection of these 15 unit prices is a vector-valued function.
- In a given market, the quantity demanded and quantity supplied are both functions of price, i.e., at a given price, there is a certain quantity demanded and a certain quantity supplied for a commodity. The pair of these quantities thus gives a 2-dimensional vector-valued function of price.
- The scores of 20 students on a given test are stored as a 20-dimensional tuple, or a vector in 20-dimensional space.

1.2. **The theory of measurement: a little aside.**
When we talk of scales for measuring stuff, there are three different kinds of scales we typically use:

- *Ordinal scales* or *comparison scales* where we can only compare things as greater or less – the quantification of *how* different two things are doesn't really make sense.
- *Difference scales* where we can measure how different two things are – but there need not be an absolute zero quantity.
- *Ratio scales* where there is an absolute zero and hence we can perform operations like scalar multiplication, addition, subtraction, ratios.

Ordinal scales or comparison scales are the weakest, in the sense that they reflect very little understanding of what is being measured. Difference scales reflect a better local understanding but a poor global understanding. Ratio scales are pretty good.

Consider, for instance, the example of temperature. In the olden days, the human understanding of temperature was simply as a means of comparison – some things were hotter than others, some things were colder than others. We didn't have an idea how to address the question "Is the hotness difference between A and B more than the hotness difference between C and D?" Then, temperature scales such as the Celsius and Fahrenheit scale were invented/discovered, and it was now possible to address the question of *difference in temperature* precisely. One could say that the temperature difference between 20 degrees Celsius and 30 degrees Celsius is more than the temperature difference between 40 degrees Celsius and 45 degrees Celsius. Note that the conclusions are the same whether we measure temperatures in degrees Celsius or degrees Fahrenheit.

With a difference scale, we can also talk of *average* temperatures. The average of the temperatures 20 degrees Celsius and 30 degrees Celsius is 25 degrees Celsius. Note that we arrive at the same average whether we measure temperatures in Celsius or Fahrenheit.

However, what does *not* make sense is the *addition* of temperatures. We cannot add 20 degrees Celsius and 30 degrees Celsius to get 50 degrees Celsius. We can, but it does not make sense. Also, this addition does not give the same answers in Celsius and Fahrenheit. Similarly, it does not make sense to multiple a temperature of 20 degrees Celsius by 1.7.

The reason these fail is that in neither Celsius nor Fahrenheit is the zero temperature an *absolute zero*.

Prior to the discovery of absolute zero by Kelvin and the creation of the Kelvin scale for temperature, a difference scale was the best one could do. Now that the Kelvin scale exists, and we have an absolute zero, we can measure temperatures in degrees Kelvin. Measured in these units, temperatures can be added and the ratios of temperatures can be taken.

1.3. **Return to vectors: vector operations coordinate-wise.** Recall that a vector is just a bunch of scalars stored side by side. Now, we can define some vector operations, but *those operations may not have any meaning for some interpretations of vectors*. The basic operations are:

- The *zero vector*, denoted 0, is defined as the vector all of whose coordinates are zero.
- We can multiply a vector by a scalar (i.e., a real number). Given a real number λ and a vector $v = \langle v_1, v_2, \ldots, v_n \rangle$, we achieve this multiplication by multiplying λ by each of the coordinates v_i, to get the new vector $\langle \lambda v_1, \lambda v_2, \ldots, \lambda v_n \rangle$.
- We can add two vectors in the same vector space $\langle v_1, v_2, \ldots, v_n \rangle$ added to $\langle w_1, w_2, \ldots, w_n \rangle$ is $\langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$.

Both addition and scalar multiplication are carried out *coordinate-wise*. In particular, we can use these to compute the *difference* of two vectors (by subtracting the values in each coordinate) and the *average* of two or more vectors (by averaging the values in each coordinate). Note that it is imperative that the vectors live in the same space in order for us to perform the operations.

Further, note that although we can perform these operations arithmetically, whether they make sense depends on whether the quantities being measured are on the right kind of scale. If the quantities are merely on an ordinal scale, and the quantification is purely for ease of understanding, then *none of the vector operations have any meaning*. If the quantities are on a difference scale, then taking differences and averages makes sense but taking sums and scalar multiples does not. If the quantities are on a ratio scale, then all the vector operations make sense.

1.4. **Dot product or scalar product.** Given two vectors v and w in the same vector space, say $v = \langle v_1, v_2, \ldots, v_n \rangle$ and $w = \langle w_1, w_2, \ldots, w_n \rangle$, their *dot product* is defined as:

$$\sum_{i=1}^{n} v_i w_i$$

or, in long form, as:

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

As such, this number is a *scalar*, not a vector, because it is a single number obtained as a sum of products of real numbers.

However, in the real world interpretation where the coordinates of the vector represent quantities that have units attached to them, we can interpret this dot product too as having a unit. Note that:

- Quantities of different kinds cannot be added. Thus, for a dot product to make sense, it is necessary that all coordinates of the vector be measuring the same kind of quantity in the same units. In other words, all coordinates of v should use the same units as each other, and all coordinates of w should use the same units as each other.
- It is not necessary that the vectors v and w be measured in the same units as each other. The units for the dor product are the units for v times the units for w. For instance, if the units for v are meters per second, and the units for w are seconds, the units for $v \cdot w$ are meters. In case both v and w are measured in the same units, the units for $v \cdot w$ are the corresponding squared units.

Here's one (somewhat contrived?) application to *total money value* calculations (as come up when calculating GDP). The *money value* of a commodity is the product of its unit price p and its quantity q. If we have a bunch of commodities and the price vector (i.e., the collection of unit prices) is given by $p = \langle p_1, p_2, \ldots, p_n \rangle$ while the quantity vector (i.e., the collection of quantities) is given by $q = \langle q_1, q_2, \ldots, q_n \rangle$, then the total money value is the dot product $p \cdot q$ of the two vectors. Explicitly, it is the sum $\sum_{i=1}^{n} p_i q_i$.

Here's another example. Suppose you make a gamble which has n possible outcomes. The pay-off to you for outcomes i, with $1 \leq i \leq n$, is m_i. The probability of outcome i is p_i. The *expected value of pay-off* for this gamble is $\sum_{i=1}^{n} p_i m_i$. We can think of this as a dot product of the *probabilities vector* $\langle p_1, p_2, \ldots, p_n \rangle$ and the *pay-offs vector* $\langle m_1, m_2, \ldots, m_n \rangle$.

Another example is calorie consumption: your overall calorie consumption in a day is the dot product of the vector giving the caloric contents of the foods you eat, with the vector describing how much of each food you eat.

1.5. **Length of a vector and correlation between vectors.** The *length* or *norm* of a vector $v = \langle v_1, v_2, \ldots, v_n \rangle$ is denoted $|v|$ and defined as:

$$|v| = \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Note that it makes sense to take the square root of the expression because a sum of squares is nonnegative. Also note that, by definition, $|v|$ is a nonnegative real number.

When we study the geometric interpretation of vectors, we will see that the length or norm is the same as the length of the corresponding geometric notion of vector when we are in the familiar three-dimensional setting.

The *correlation* between two vectors v and w is defined as:

$$\frac{v \cdot w}{|v||w|}$$

This correlation basically measures how similar the *directions* of the vectors v and w are to each other. In particular, this correlation is in the interval $[-1, 1]$, where -1 occurs if v and w are *negative* scalar multiples of each other, and 1 occurs if v and w are *positive* scalar multiples of each other. We will see the relation with some geometry and trigonometry a little later.

1.6. **Why square up to measure the length?** There is a real-world reason for defining the norm or length by taking squares, adding, and then taking the square root, rather than just adding up the absolute values. The reason is that we want each coordinate to be weighed *by itself*, so that larger magnitude coordinates get a higher weighting. An example is when considering population densities. If we average the population density between an area with high density and an area with zero density, the effective population density comes out to be a lot less than that of the area with high density. However, the population density *lived/experienced* by people is only the density in the high density area, not in the zero density area. Ideally, we want to weight the zero density area by zero, because nobody actually lives there. More generally, we want to weight low density areas by their low densities or low populations, reflecting the fact that since fewer people actually live there, they should count for less in the average. The use of squares in the norm concept captures this idea.

1.7. **Properties of the dot product and length.** Here are some key properties of the dot product:
- The dot product of any vector with the zero vector is the number zero: $a \cdot 0 = 0 \cdot a = 0$ for all a. Note that the right-most 0 is the *scalar* 0 and the other two 0s are vector 0s.
- The dot product is *symmetric*, i.e., $a \cdot b = b \cdot a$ for all vectors a and b.
- The dot product is *distributive* in both coordinates, i.e., $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.
- Scalars can be pulled out of the dot product: $(\lambda a) \cdot b = \lambda(a \cdot b)$.

Note: The last two properties *together* constitute what is called *linearity* in each variable, and hence, *bilinearity*. This may make more sense to you if/when you become familiar with linear algebra.

Here are the key properties of length:

- The length of a vector is 0 if and only if the vector is the zero vector.
- For any vector a and scalar λ, the length of λa is $|\lambda|$ times the length of a.
- *Triangle inequality*: For any vectors a and b, we have $|a + b| \le |a| + |b|$ with equality holding if and only if a and b are positive scalar multiples of each other (i.e., the angle between them is 0) or one of the vectors a and b is 0.

2. Vectors in the physical world: in three dimensions

2.1. **Vectors as arrows or directed line segments.** We now shift attention to the notion of vectors in the physical world. This concept is not important as an end in itself, but some of these ideas are necessary for later applications to vector integration and differentiation, as well as developing the theory of vector-valued functions.

Consider a three-dimensional space equipped with a coordinate system. Geometrically, a *vector based at the origin* in the space can be thought of as an arrow (a directed line segment) with tail at the origin and head at some point in the space. Thus, there is a correspondence:

Points in three-dimensional space $\leftrightarrow$ Vectors based at the origin in three-dimensional space

More generally, we can talk of a vector *based at any point*. This is just a directed line segment starting at one point and ending at another (possibly the same) point. Given a vector based at a point, there is a unique vector based at the origin that is parallel to it and has the same length and same direction. Basically, this is obtained by taking the original vector and translating it till its tail becomes the origin.

Consequently, there is a notion of a *free vector*. A free vector can be thought of as an equivalence class of vectors, with different starting points, all of which have the same length and direction. Given any free vector, there is a unique way of representing it using a vector based at the origin.

Some notation: the vector with tail at a point A and head at a point B is denoted $\overline{AB}$, and we sometimes put an arrow in the left-to-right direction.

2.2. **Vectors and coordinates.** In a coordinate system, the vector from the origin to the point with coordinates (x, y, z) is denoted as the vector $\langle x, y, z \rangle$. The operations of vector addition, scalar multiplication, and dot product, are all done in the context of this description.

Before describing these, I will briefly note the **i,j,k** notation. We define the three unit vectors **i**, **j**, and **k** as follows:

- **i** is the unit vector $\langle 1, 0, 0 \rangle$, i.e., it is a vector of length 1 along the positive x-axis.
- **j** is the unit vector $\langle 0, 1, 0 \rangle$, i.e., it is a vector of length 1 along the positive y-axis.
- **k** is the unit vector $\langle 0, 0, 1 \rangle$, i.e., it is a vector of length 1 along the positive z-axis.

We can thus write:

$$\langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

In other words, the vector $\langle x, y, z \rangle$ is obtained by traveling x in the x-axis direction, y in the y-axis direction, z in the z-axis direction.

2.3. **Vector operations: scalar multiplication and addition.** We can do a bunch of vector operations, as discussed earlier, and relate it to geometric operations:

- *Scalar multiplication*: Given a vector $\langle x, y, z \rangle$ and a scalar λ, we get the vector $\langle \lambda x, \lambda y, \lambda z \rangle$. Geometrically, this new vector is parallel to the old vector, its length is $|\lambda|$ times the length of the old vector, and its direction (forward/backward) is the same as or reverse of the old vector depending on whether $\lambda > 0$ or $\lambda < 0$.
- *Addition*: Given a vector $v_1 = \langle x_1, y_1, z_1 \rangle$ and a vector $v_2 = \langle x_2, y_2, z_2 \rangle$, the sum of the vectors is $\langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$. Geometrically, this can be obtained as follows: we take the vector v_2, treat it as a free vector, and translate it so that its tail is at the head of v_1. Now, we make a vector from the origin to the *head* of this translated v_2. This new vector is $v_1 + v_2$. Note that we could interchange the roles of v_1 and v_2 in this process, and get the same answer. When we make the whole picture, we get a parallelogram, so this method sometimes goes under the name of the *parallelogram law of addition*.

2.4. Terminological note. We say that two vectors are *collinear* or *parallel* or *linearly dependent* if either is a scalar multiple of the other.

The *zero vector* is the vector all of whose coordinates are zero. It corresponds to the arrow from the origin to itself.

2.5. Bold face conventions. Note that the book uses a bold face letter convention for vectors, e.g., the vector with name v will be written as $\mathbf{v}$ instead of just v.

I will not be bothered too much about this convention, and it's anyway hard to make bold face conventions clear when writing things with pen, pencil, or chalk. So you don't have to bother much about it.

2.6. Vector operations: dot product. Recall the general definition of dot product, $\langle v_1, v_2, \ldots, v_n \rangle \cdot \langle w_1, w_2, \ldots, w_n \rangle$ is given by $\sum_{i=1}^{n} v_i w_i$.

In particular, if $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$, we get:

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

We also define:

$$|v| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

This is the *length* of the vector v and is the same as the length of the actual line segment corresponding to the vector.

It turns out that if v and w are vectors, then:

$$v \cdot w = |v||w| \cos \theta$$

where θ is the angle between v and w.

2.7. Thinking about dot product: what does it measure? As noted above:

$$v \cdot w = |v||w| \cos \theta$$

This can be rearranged to give:

$$\cos \theta = \frac{v \cdot w}{|v||w|}$$

In other words, taking θ as the angle between 0 and π:

$$\theta = \arccos \left[\frac{v \cdot w}{|v||w|} \right]$$

The quantity $\cos \theta$, which is also equal to the right side, is the *correlation* between the vectors v and w, or, in other words, it is the degree of similarity in the direction of the vectors v and w. This is a number in the interval $[-1, 1]$. Let's think more clearly about what this means:

θ	$\cos \theta$	Interpretation on the relation between v and w
0	1	w is a *positive* scalar multiple of v, i.e., they are in the same direction
In $(0, \pi/2)$	In $(0, 1)$	v and w are positively correlated, form an acute angle
$\pi/2$	0	v and w are *uncorrelated*. They are perendicular or orthogonal.
In $(\pi/2, \pi)$	In $(-1, 0)$	v and w are negatively correlated, form an obtuse angle.
π	-1	w is a negative scalar multiple of v, i.e., they are in exactly opposite directions.

2.8. Rethinking trigonometry: components and projections. The preceding discussion can give you a deeper perspective on trigonometry. The cos function measures the *closeness* or *correlation* or *degree of agreement* between two things separated by the given angle. The sin function measures how big the *rest of the stuff is*. The relation:

$$\cos^2 \theta + \sin^2 \theta = 1$$

basically says that if you add up the parts that agree and the rest of the stuff you get 100%.

Let's discuss a related notion that helps make this clearer. Given vectors a and b, both based at the origin, the *vector projection* of b onto a is the vector starting at the origin and ending at the foot of the perpendicular from the head of b to the line of a. The *component* or *scalar projection* of b along a is the length of this vector projection measured in the direction of a (so it could be negative). Specifically:

- The vector projection of b onto a, denoted $\text{proj}_a b$, is $\frac{a \cdot b}{|a|^2} a$.
- The scalar projection of b onto a, denoted $\text{comp}_a b$, is $\frac{a \cdot b}{|a|}$.

2.9. **Sidenote: orthogonal as unrelated.** People who are mathematically and statistically literate have an interesting new adjective to add to their vocabulary for daily use: "orthogonal" in the sense of being unrelated or irrelevant. This usage stems from the above discussion of dot products and correlation. However, this usage is not common among people without a math/stats background, and it is a way for you to show off your mathematical/statistical erudition. An amusing exchange was recorded in Supreme Court proceedings in the United States in 2006:

`http://www.volokh.com/2010/01/11/orthogonal-ooh/`

Short version: Supreme Court justices were flummoxed when a law professor used the word "orthogonal" in the sense of unrelated.

3. Vagaries of three dimensions: cross product and scalar triple product

The constructions we will discuss now are specific to three dimensions. Although they do have some analogues or generalizations in higher dimensions, the analogues are not obvious or intuitive and require considerable mathematical machinery.

3.1. **The cross product.** The cross product of two vectors $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ is defined as follows:

$$a \times b = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

The formula looks a bit cryptic. Here is the idea: we arrange the coordinates in the cyclic ordering $1 \to 2 \to 3 \to 1$. A particular coordinate in $a \times b$ is then the next coordinate of a times the next to next coordinate of b minus the next coordinate of b minus the next-to-next coordinate of a. We can also write this as a *determinant* as follows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

For those of you who have seen determinants, this should make sense. Basically, we evaluate the determinant by expansion using the first row. Recall that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, are the unit vectors in the x, y, and z directions, so their coefficients are the coordinates of the vector we finally want.

3.2. **Properties of the cross product.** Here are some key properties of the cross product:

- The cross product of any vector with the zero vector is the zero vector: $a \times 0 = 0 \times a = 0$ for any vector a.
- The cross product is *anti-symmetric* or *skew-symmetric*: For any vectors a and b, $a \times b = -(b \times a)$.
- The cross product of two vectors that are scalar multiples of each other is zero: $a \times b = 0$ if $b = \lambda a$ for some scalar λ.
- The cross product is *distributive* in both vectors: $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$.
- Scalars can be pulled out of cross product: $(\lambda a) \times b = \lambda(a \times b)$ and $a \times (\lambda b) = \lambda(a \times b)$.
- The cross product is *not associative*: $a \times (b \times c) \neq (a \times b) \times c$.
- We have the following relation: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.
- The cross product $a \times b$ is orthogonal to a and to b. In particular, we get that $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$.

- We have the following identity involving the cross product:

$$(a \times (b \times c)) + (b \times (c \times a)) + (c \times (a \times b)) = 0$$

Also:

$$((a \times b) \times c) + ((b \times c) \times a) + ((c \times a) \times b) = 0$$

3.3. **Geometric interpretation of the cross product.** The cross product of non-collinear vectors a and b is a vector perpendicular to both a and b (and hence, perpendicular to the plane spanned by a and b). Its direction is given by the right hand rule: curl the fingers of the right hand from a to b, make the thumb point away from the fingers, then the thumb points in the direction of $a \times b$. Its magnitude is given by the formula:

$$|a \times b| = |a||b|\sin\theta$$

where θ is the angle between a and b.

In particular, we note some important relations between the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

$$\begin{aligned}
\mathbf{i} \times \mathbf{j} &= \mathbf{k} \\
\mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\
\mathbf{j} \times \mathbf{k} &= \mathbf{i} \\
\mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\
\mathbf{k} \times \mathbf{i} &= \mathbf{j} \\
\mathbf{i} \times \mathbf{k} &= -\mathbf{j}
\end{aligned}$$

3.4. **Scalar triple product.** There is a special construction in three dimensions called the *scalar triple product*. Given vectors $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$, $c = \langle c_1, c_2, c_3 \rangle$. The scalar triple product, denoted $[a\ b\ c]$, is defined as the number:

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1$$

It can be defined also as the determinant of the matrix:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Finally, $[a\ b\ c]$ can be defined in terms of dot and cross product:

$$[a\ b\ c] = a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$$

The scalar triple product has the following properties:

- If a, b, and c are in a common plane (i.e., there is a plane containing the heads of all the vectors, when their tails are at the origin), then the scalar triple product is 0. In particular, if any two of the vectors are collinear (i.e., one is a scalar multiple of the other), the scalar triple product is 0.
- The scalar triple product is invariant under cyclic permutations, i.e., $[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$.
- The scalar triple product gets negated if we just switch two of the pieces: $[a\ b\ c] = -[b\ a\ c]$.
- The scalar triple product is distributive in each input. For instance, here's distributivity in the third input: $[a\ b\ (c + d)] = [a\ b\ c] + [a\ b\ d]$.
- Scalars can be pulled out from any of the three inputs to the outside.

4. Geometric applications of dot product and cross product

Examples are not included in the lecture notes, but you can see some in the book and you will also do more practice of examples when doing your homework.

4.1. **Area of triangle, parallelogram, volume of parallelopided.** To compute the area of a triangle or parallelogram, we need to first treat the sides as free vectors. To do this, we subtract the coordinates of the endpoints of the sides from each other.

For instance, consider a triangle with vertices $P(p_1, p_1, p_3)$, $Q(q_1, q_2, q_3)$, and $R(r_1, r_2, r_3)$. The side PQ, viewed as a vector, is $\langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$. Similarly, the side PR, viewed as a vector, is $\langle r_1 - p_1, r_2 - p_2, r_3 - p_3 \rangle$. The area of the triangle is given by *half* the length of the cross product:

$$\text{Ar}(\triangle PQR) = \frac{1}{2}|\overline{PQ} \times \overline{PR}|$$

For a parallelogram, we compute two of its adjacent sides as vectors, and the area of the parallelogram is then the length of the cross product of these sides. Thus, for a parallelogram $PQRS$, the area is given by:

$$\text{Ar}(PQRS) = |\overline{PQ} \times \overline{PR}|$$

A parallelopiped (sometimes written parallelepided) is a three-dimensional analogue of a parallelogram. The volume of a parallelopiped is the absolute value of the scalar triple product of three adjacent vectors emanating from a single vertex.

4.2. **Determining the angle between vectors.** This can be done using the dot product (see earlier). Note that *we cannot use the cross product approach* – the cross product approach is sufficient to find $\sin \theta$, but there could be two different values of θ (an acute and an obtuse angle) giving rise to the same value of $\sin \theta$.

4.3. **Finding a vector orthogonal to two given vectors.** If the two given vectors are not collinear, we simply take the cross product.

If they are collinear, we can choose any vector orthogonal to either of them, and we basically do this by looking at the dot product.

4.4. **Testing coplanarity and orthogonality.** We can use the dot product, cross product, and scalar triple product to do various kinds of tests:

- To test whether three points are collinear, take the difference vectors of the first and second point, and then of the second and third point. Then, check if these difference vectors are scalar multiples of each other.
- To test whether four points are coplanar, pick one as basepoint, and calculate difference vectors from it of the other three points. The four points are coplanar if and only if this scalar triple product is zero.