TANGENT PLANES AND LINEAR APPROXIMATIONS

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 14.4. *Note: We are, for now, omitting the topic of differentials, which is the second half of this section (Pages 931–934). We may return to it later in the course, if we get time after completing the rest of Chapters 14 and 15.*

What students should definitely get: Finding the tangent plane at a point, the concept of best linear approximation.

Executive summary

Words ...

(1) For a d-dimensional subset of $\mathbb{R}^n$, it (occasionally) makes sense to talk of the tangent space and the normal space at a point. The tangent space is a linear/affine d-dimensional space and the normal space is a linear/affine $(n-d)$-dimensional space. Both pass through the point and are mutually orthogonal.

(2) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that f is differentiable at the point* (x_0, y_0) is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

(3) It may be the case that a function f of two variables is not differentiable at a point in its domain but the partial derivatives exist. In this case, although the *above formula makes sense as a formula*, the plane it gives *is not the tangent plane* – in fact, *no tangent plane exists*. Similarly, *no linearization exists*, and the linear function given by the above formula *is not a close approximation to the function near the point*.

1. Approximation theory: recall of one variable

1.1. **Taylor polynomials.** For a function f of one variable, we can define some Taylor polynomials of f. If f is n times differentiable around a point $x = c$, we can define the n^{th} Taylor polynomial of f about c as the polynomial:

$$P_n(f, c)(x) := \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

If f is $(n+1)$ times differentiable, it turns out that $f - P_n(f, c)$ is a function with a zero of order at least $n+1$ at c. This means that we can approximate f by a *polynomial function* of degree at most n such that the error term is very zeroey (order at least $n+1$). Note that the Taylor polynomial has degree at most n, but the degree may be less than n if the n^{th} derivative takes the value 0 at c.

1.2. **Special cases: degrees zero and one.** The zeroth degree Taylor polynomial is a constant function with the same value as $f(c)$. This is a very crude description of the function around the point and ignores any change in the value.

The first degree Taylor polynomial is the function whose graph gives the tangent line, i.e., the line:

$$y = f(c) + f'(c)(x - c)$$

Note that this is the tangent line in point-slope form.

The first degree Taylor polynomial, or the tangent line, captures the rate of change of the function *at* the point. However, it fails to capture second derivative and higher derivative behavior, i.e., how this rate of change itself is changing.

Geometrically, the tangent line to a point on a curve is the "best linear approximation" to the curve locally around the point, i.e., the line that comes closest to describing the curve near the point. Note that it is *not* true that the tangent line intersects the curve at a unique point, or that no other line has this property.

2. The multivariable situation

2.1. **Geometric notion of tangent space and normal space.** Given a subset of dimension d in n-dimensional space, we can try talking of a tangent space at a point on the subset to the subset. This attempt at talk succeeds only when that subset has some particularly nice properties. Anyway, the point is that this tangent space looks like a *flat* d-dimensional space, i.e., a line, or a plane, or a higher-dimensional analogue thereof.

For instance, the "tangent" to a curve (which is a one-dimensional construct) is a line (the linear one-dimensional construct). Similarly, the tangent to a surface in $\mathbb{R}^3$ is a plane (a linear two-dimensional construct). To take the example of a sphere, think of a sphere resting on a floor. The floor is the tangent plane to the sphere through the point of contact.

The "normal space" at a point to a d-dimensional subset in n-dimensional space is a $(n - d)$-dimensional linear space through the same point, such that the two spaces intersect orthogonally, i.e., every direction in one space is orthogonal to every direction in the other space. Thus, for instance:

- For a curve in $\mathbb{R}^2$, the tangent space and normal space are both one-dimensional.
- For a curve in $\mathbb{R}^3$, the tangent space is one-dimensional and the normal space is two-dimensional.
- For a surface in $\mathbb{R}^3$, the tangent space is two-dimensional and the normal space is one-dimensional.

Note that for geometric subsets, we can only make sense of tangent and normal spaces, rather than specific tangent and normal vectors. However, given a *parametric* description of a curve, we can make sense of the tangent *vector*, with the length and direction of the vector determined by the speed and direction of motion along the curve as per the parameterization.

2.2. **Non-existent tangent planes.** There are many possible reasons why the tangent plane to a surface at a point on the surface may not exist. First, the surface may be broken at that point, i.e., it may not locally look like a plane in a small neighborhood of the point. In such a case, it might be an abuse of language to call it a surface.

Second, the surface may have a lot of variation around the point – too many hills and valleys to make sense of a meaningful tangent plane. This is a surface analogue of the function $x \sin(1/x)$ which, if extended to the value 0 at 0, becomes continuous but not differentiable at 0.

Third, the surface may be sharp and pointy at the point. Think, for instance, of the curved surface of a right circular cone. At most points on this curved surface, the tangent plane exists. However, at the vertex of the point, the tangent plane does not exist. This is a surface analogue of a function that has one-sided derivatives that are not equal, i.e., it takes a sharp turn.

2.3. **How should the tangent plane relate geometrically to the surface?** In the one-dimensional situation, we recall that *generally speaking*, the tangent line *locally* lies entirely to one side of the curve, i.e., the curve rests against the tangent line. In fact, it is the only line that does not *cut through* the curve.

But this is not always true. Two notable kinds of exceptions are:

- *Points of inflection*: Here, the tangent line *cuts through* the curve, i.e., the curve and the tangent line cross each other. An example is $y = x^3$ at $(0, 0)$. The tangent line is the x-axis, and it crosses the curve.
- Points where the curve keeps crossing above and below the tangent line arbitrarily close to the point of tangency: For this to occur, the second derivative must change sign infinitely often close to the point. Examples include functions such as $x^2 \sin(1/x)$ (with the value 0 at 0) about the origin $(0, 0)$. The tangent line is the x-axis, and it intersects the curve at points arbitrarily close to $(0, 0)$, hence fails to be *on one side* of the curve.

Does something similar happen for planes? Yes. *Generically*, we expect that *locally*, the tangent plane lies to one side of the surface, and this can be a reasonable characterization of the tangent plane. That's the reason why for a sphere "resting" on a floor, the floor is the tangent plane to the sphere at the point of contact.

However, there is an analogue of point of inflection, where the tangent plane *cuts through* the surface at the point. This type of two-dimensional analogue of point of inflection is termed a *saddle point*. We will deal with saddle points when we cover the topic of maxima and minima for functions of many variables later in the course.

2.4. **Relation between tangents for curve on surface.** If a curve is in/on a surface in $\mathbb{R}^3$, then the tangent *line* to the curve at a point on the curve lies in the tangent *plane* to the surface at the point. Similarly, the normal plane to the curve at a point contains the normal line to the surface at that point.

In particular, if we have two different curves on a surface intersecting at a point, and the tangent lines to these curves at the point do not coincide, then these lines together determine the tangent plane to the surface, if it exists: it is the unique plane containing both the tangent lines.

If, on the other hand, we find a situation where there are three curves intersecting at a point, all in a surface, and the tangent lines at the point to these three curves do not lie in the same plane, then the tangent plane at the point does not exist.

2.5. **Partial derivatives of functions of many variables.** Consider a function $z = f(x, y)$, a function of two variables. The graph of this function is a surface in $\mathbb{R}^3$. Recall that the partial derivatives can be interpreted as slopes of tangent lines as follows:

- $f_x(x_0, y_0)$: Consider the plane $y = y_0$. This is a plane parallel to the xz-plane, and the intersection of the surface with this plane can be thought of as the graph of the function $z = f(x, y_0)$ of one variable. The partial derivative is the slope of the tangent line in this plane to the graph at the point $(x_0, y_0, f(x_0, y_0))$. A free vector along this tangent line, viewed in $\mathbb{R}^3$, is $\langle 1, 0, f_x(x_0, y_0) \rangle$.
- $f_y(x_0, y_0)$: Here, we fix the plane $x = x_0$, parallel to the yz-plane, and a similar interpretation follows. A free vector along this tangent line, viewed in $\mathbb{R}^3$, is $\langle 0, 1, f_y(x_0, y_0) \rangle$.

Thus, computing the partial derivatives allows us to compute tangent vectors to two curves in the surface that's the graph of this function. Since we know the basepoint, we can also compute parametric descriptions of the corresponding tangent lines.

If both the partial derivative f_x and f_y are continuous, then we can make sense of the notion of tangent plane, to which we now turn.

These two vectors are both parallel to the tangent plane, so we can take their cross product and find a normal vector. A quick computation of the cross product shows that $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$ is a normal vector. Thus, the tangent plane passes through the point $(x_0, y_0, f(x_0, y_0))$ and has normal vector $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$. Working out the scalar equation from the vector equation, we get:

$$(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) + (z - f(x_0, y_0))(-1) = 0$$

Rearranging, we get:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We can rewrite this as:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This is the equation of the tangent plane, and is the most convenient form for applications.

2.6. **Tangent plane as good linear approximation.** Just as the tangent line is a good linear approximation to the graph of a function in one variable, the tangent plane is a good linear approximation to the graph of a function in two variables. Roughly, it can be thought of as a first-order approximation, so that any "error term" will be zeroey of order two or higher. In particular, for points close to the point at which we are computing the tangent plane, the function value arising from the linear approximation is pretty close to the actual function value.

More concretely, with the above setup, we have the plane:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function is the right side of this equation:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

2.7. **Continuity of partial derivatives necessary.** Recall the earlier example we studied of a function that is separately continuous in each variable but is not jointly continuous. The function is $f(x,y) = (xy)/(x^2 + y^2)$ except at the origin, and $f(0,0) = 0$. This can also be described as $(1/2)\sin(2\theta)$ with respect to polar coordinates. At the origin, the function is separately continuous in each variable, but not jointly continuous.

We can further note that in fact, since the function is a constant zero function along both the axes, all its partial derivatives exist and equal zero at the origin. However, the function is not jointly continuous, and hence, we should not expect it to have anything like a tangent plane. In fact, it does not. Although we can blindly apply the above formula to obtain the equation of a plane, this is not a "tangent plane" to the surface that is the graph of the function. (See Page 930 of the book).

2.8. **Tangent plane and total derivative.** Recall an earlier conundrum:

Separate continuity:Joint continuity::Partial derivatives:?

In other words, what is the "joint" equivalent of partial derivatives? Unfortunately, the proper way of thinking of this joint equivalent requires the use of linear algebra, and we will therefore not be able to cover it. I will simply state the following result:

> If all the first-order partial derivatives of a function exist around a point, and each first-order partial derivative is jointly continuous at the point, then the function is differentiable at the point. However, the converse is not true.

In other words:

Partials exist around the point, are continuous $\implies$ Function is differentiable (total derivative exists) $\implies$ Partials exist *at* the point