REVIEW SHEET FOR MIDTERM 1: BASIC

MATH 195, SECTION 59 (VIPUL NAIK)

We will not be going over this sheet, but rather, we'll be going over the advanced review sheet in the session. Please review this sheet on your own time.

1. Formula summary

1.1. Parametric.
Set $x = f(t)$, $y = g(t)$, parametric curve in $\mathbb{R}^2$.

- $dy/dt = g'(t)$ and $dx/dt = f'(t)$.
- $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$.
- $\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$
- Arc length: $\int \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

1.2. Polar.
Set $r = F(\theta)$, polar equation of a curve.

- $y = F(\theta)\sin\theta$ and $x = F(\theta)\cos\theta$.
- $dy/d\theta = F'(\theta)\sin\theta + F(\theta)\cos\theta$ and $dx/d\theta = F'(\theta)\cos\theta - F(\theta)\sin\theta$.
- $\frac{dy}{dx} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$
- Arc length: $\int \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$

1.3. Three-dimensional geometry.

- Distance formula between (x_1, y_1, z_1) and (x_2, y_2, z_2): $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center having coordinates (h, k, l) and radius r is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

1.4. Vectors.

- Vector dot product: $\langle v_1, v_2, \dots, v_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle = v_1w_1 + v_2w_2 + \dots + v_nw_n$.
- Length of vector $\langle v_1, v_2, \dots, v_n \rangle$ is $\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.
- Unit vector in the direction of a vector v is $v/|v|$. Unit vector in opposite direction but along same line (so parallel) is $-v/|v|$.
- Vector cross product: $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$.
- For nonzero vectors v and w in three dimensions, we have $|v \times w| = |v||w|\sin\theta$ where θ is the angle between v and w.
- Scalar triple product is $a \cdot (b \times c)$.
- Angle between nonzero vectors v and w is $\arccos\left(\frac{v \cdot w}{|v||w|}\right)$.
- Scalar projection of b onto a is $(a \cdot b)/|a|$. *Note: Be careful what is being projected onto what.*
- Vector projection of b onto a is $((a \cdot b)/|a|^2)a$.
- Area of triangle with vertices P, Q and R is $(1/2)|PQ \times PR|$. Need to: (i) compute difference vectors, (ii) take cross product, (iii) compute length of the cross product, (iv) divide by 2.
- Area of parallelogram with vertices P, Q, R, S is $|PQ \times PR|$ or $|PQ \times PS|$ (same number). Steps (i)-(iii) of above.
- Volume of parallelepiped is *absolute value of* scalar triple product of vectors for adjacent triple of edges.

2. Quickly: what you should know from one-variable calculus

You need to be able to do the following from one-variable calculus and before:

(1) Finding domains of functions
(2) Basic algebraic manipulation and trigonometric identities

(3) Graphing: Know equation of circle centered at origin, graph linear functions, sine, cosine.
(4) Differentiation and integration: Everything you saw in one-variable calculus. However, for this midterm, you will get only simple integrations that rely on the very basic formulas and not, for instance, those that use integration by parts.

3. Parametric stuff

Words ...

(1) A parametric description of a curve is one where both coordinates are expressed as functions of of a parameter, typically denoted t. Parametric descriptions offer an alternative to functional and implicit (relational) descriptions of curves. Here, t varies over some subset of the real numbers. In symbols, we have something like $x = f(t)$, $y = g(t)$, where t varies over some subset D of the real numbers.
(2) Descriptions where x is a function of y or y is a function of x are special cases of parametric descriptions.
(3) The same curve may admit multiple parametrizations, and different parameterizations may correspond to different speeds and different orderings of traversal of the point. The curve itself only contains the information of *what* points were traversed, not the information of the *sequence* and *pace* in which they were traversed.
(4) The curve traced by a parameterization depends not only on the coordinate functions but also the domain for the parameter. The larger the domain, in general, the larger the curve traced. However, in some cases, expanding the domain may not make the curve strictly larger. This happens in cases where both coordinate functions are even or have commensurable periods.
(5) A parameterization of a curve may involve self-intersections, retracings (e.g., tracing back for even function pairs), or even wrapping around itself (for periodic function pairs).
(6) Function composition allows us to switch between multiple parameterizations.
(7) In some cases, it is possible to move back and forth between parametric and relational descriptions.
(8) Parametric differentiation: if $x = f(t)$ and $y = g(t)$, then $dy/dx = (dy/dt)/(dx/dt) = g'(t)/f'(t)$. This can also be used to differentiate repeatedly. Note that the derivative is a function of t rather than of (x, y), so to find the derivative given the point (x, y) we need to go back and determine t.
(9) Higher derivatives can be computed iteratively using parametric differentiation. But note that it is *not* true that $d^2y/dx^2 = (d^2y/dt^2)/(d^2x/dt^2)$. The actual formula/procedure is more complicated (see lecture notes or formula summary).
(10) Arc length: The formula for arc length from $t = a$ to $t = b$ (with $a < b$) is $\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$.

Actions ...

(1) *Parametric to relational: elimination of parameter*: In many cases, it is possible to eliminate a parameter from a parametric description. The idea is to use some well known identities or manipulation techniques to try to directly relate x and y by finding some equation between them that is true for all t. However, this is not the full story. We next need to see if there are additional restrictions on x and y deducible from the fact that they arose as function of t, also keeping in mind the domain restrictions on t.

    For instance, the parameterization $x = t^2, y = t^4$ for $t \in \mathbb{R}$ can be rewritten as $y = x^2$, but we need the additional condition that $x \geq 0$.

    See more examples in the lecture notes, quizzes, and homeworks.
(2) *Relational to parametric*: Here, we see a relation between x and y, and try to choose a parametric description that would give rise to the relation. Again, the domain of choice for the parameter needs to be chosen wisely.

    See more examples in the lecture notes and quizzes.
(3) *Parametric differentiation and geometric consequences*: We use the formula $(dy/dt)/(dx/dt)$. If $x = f(t)$ and $y = g(t)$, then this becomes $g'(t)/f'(t)$. This is valid for all t in the interior of the domain of definition where both f' and g' are defined and $f' \neq 0$. If $f'(t) = 0$ but $g'(t) \neq 0$, we have a vertical tangent situation. If $g'(t) = 0$ but $f'(t) \neq 0$, we have a horizontal tangent situation.

4. Polar coordinates

Words ...

(1) *Specifying a polar coordinate system*: To specify a polar coordinate system, we need to choose a point (called the *origin* or *pole*), a half-line starting at the point (called the *polar axis* or *reference line*) and an orientation of the plane (chosen counter-clockwise in the usual depictions).
(2) *Finding the polar coordinates of a point and vice versa*: The radial coordinate r is the distance between the point and the pole. The angular coordinate θ is the angle (measured in the counter-clockwise direction) from the polar axis to the line segment from the pole to the point. Note that θ is uniquely defined up to addition of multiples of 2π, and it becomes truly unique if we restrict it to a half-open half-closed interval of length 2π. *The exception is the pole itself, for which θ is undefined* in the sense that any value of θ could be chosen.
(3) *Converting between Cartesian and polar coordinates*: If we take the polar axis as the positive x-axis and the axis at an angle of $+\pi/2$ from it as the positive y-axis, we get a Cartesian coordinate system. The point defined by polar coordinates (r, θ) has Cartesian coordinates $(r\cos\theta, r\sin\theta)$. Conversely, given a point with Cartesian coordinates (x, y) the corresponding polar coordinates are $r = \sqrt{x^2 + y^2}$ and θ is the unique angle (up to addition of multiples of 2π) such that $x = r\cos\theta$, $y = r\sin\theta$.

Actions ...

(1) A functional description of the form $r = F(\theta)$ gives rise to a parametric description in Cartesian coordinates: $x = F(\theta)\cos\theta$ and $y = F(\theta)\sin\theta$. We can do the usual things (like find slopes of tangent lines) using this parametric description. Note that here, θ is typically allowed to vary over all of $\mathbb{R}$ rather than simply being restricted to an interval of length 2π. The slope of the tangent line in Cartesian terms is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d(F(\theta)\sin\theta)/d\theta}{d(F(\theta)\cos\theta)/d\theta} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$$

(2) The arc length is given by integrating $\sqrt{(F(\theta))^2 + (F'(\theta))^2}$ for θ in the suitable interval. (See quiz question on this).
(3) An implicit (relational) description in Cartesian coordinates can be converted to a description in polar coordinates by replacing x by $r\cos\theta$ and y by $r\sin\theta$.
(4) An implicit (relational) description in polar coordinates can sometimes be converted to a description in Cartesian coordinates, but with some ambiguity. General idea: replace r by $\sqrt{x^2 + y^2}$, $\cos\theta$ by $x/\sqrt{x^2 + y^2}$, and $\sin\theta$ by $y/\sqrt{x^2 + y^2}$.

5. Three-dimensional geometry

Words ...

(1) Three-dimensional space is coordinatized using a Cartesian coordinate system by selecting three mutually perpendicular axes passing through a point called the origin: the x-axis, y-axis, and z-axis. These satisfy the right-hand rule. The coordinates of a point are written as a 3-tuple (x, y, z).
(2) There are $2^3 = 8$ octants based on the signs of each of the coordinates. There are three coordinate planes, each corresponding to the remaining coordinate being zero (the xy-plane corresponds to $z = 0$, etc.). There are three axes, each corresponding to the other two coordinates being zero (e.g., the x-axis corresponds to $y = z = 0$).
(3) The distance formula between points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is similar to the formula in two dimensions and the squares and square root arises from the Pythagorean theorem.
(4) The equation of a sphere with center having coordinates (h, k, l) and radius r is $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. Given an equation, we can try completing the square to see if it fits the model for the equation of a sphere.

6. Introduction to vectors and relation with geometry

6.1. ***n*-dimensional generality.** Words ...

(1) A vector is an ordered n-tuple of real numbers (or quantities measured using real numbers). The space of such n-tuples is a n-dimensional vector space over the real numbers. Vectors can be used to store tuples of prices, probabilities, and other kinds of quantities.
(2) There is a zero vector. We can add vectors and we can multiply a vector by a scalar. Note that these operations may or may not have an actual meaning based on the thing we are storing using the vector.
(3) We can take the dot product $v \cdot w$ of two vectors v and w in n-dimensional space. if $v = \langle v_1, v_2, \dots, v_n \rangle$ and $w = \langle w_1, w_2, \dots, w_n \rangle$, then $v \cdot w = \sum_{i=1}^n v_i w_i$. The dot product is a real number (though if we put units to the coordinates of the vector, it gets corresponding squared units).
(4) The length or norm of a vector v, denoted $|v|$, is defined as $\sqrt{v \cdot v}$. It is a nonnegative real number.
(5) The correlation between two vectors v and w is defined as $(v \cdot w)/(|v||w|)$. It is in $[-1, 1]$. (For geometric interpretation, see the three-dimensional case).
(6) *Properties of the dot product*: The dot product is symmetric, the dot product of any vector with the zero vector is 0, the dot product is additive (distributive) in each coordinate and scalars can be pulled out.
(7) *Properties of length*: The only vector with length zero is the zero vector, all other vectors have positive length. The length of λv is $|\lambda|$ times the length of v. We also have $|v + w| \le |v| + |w|$ for any vectors v and w, with equality occurring if either is a positive scalar multiple of the other or one of them is the zero vector.

6.2. **Three-dimensional geometry.** Words ...

(1) We can identify points in three-dimensional space with three-dimensional vector as follows: the vector corresponding to a point (x, y, z) is the vector $\langle x, y, z \rangle$. Physically, this can be thought of as a directed line segment or arrow from the origin to the point (x, y, z).
(2) We can also consider vectors starting at any point in three-dimensional space and ending at any point. The corresponding vector can be obtained by subtracting the coordinates of the points. The vector from point P to point Q is denoted $\overline{PQ}$.
(3) There are unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. These are thus the vectors of length 1 along the positive x, y, and z directions respectively. A vector $\langle x, y, z \rangle$ can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
(4) Vectors can be added geometrically using the *parallelogram law*. This procedure gives the same answer as the usual coordinate-wise addition.
(5) Scalar multiplication also has a geometric interpretation − the length gets scaled by the scalar multiple, and the direction remains the same or is reversed depending on the scalar's sign.
(6) For vectors v and w, we have $v \cdot w = |v||w| \cos \theta$ where θ is the angle between v and w. We can use this procedure to find the angle between two vectors. The correlation between the vectors is thus $\cos \theta$. We can interpret this specifically for $\theta = 0$, θ an acute angle, $\theta = \pi/2$, θ an obtuse angle, and $\theta = \pi$ (see the table in the lecture notes).
(7) We can define the vector cross product $v \times w$ using a matrix determinant. Equivalently, if $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$, then $v \times w = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$. *This is a specifically three-dimensional construct.*
(8) The cross product has the property that cross product of any two collinear vectors is zero, cross product of any vector with the zero vector is zero, the cross product is skew-symmetric, distributive in each variable, and allows scalars to be pulled out. It is not associative in general. There is an identity relating cross product and dot product: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. Also, the cross product satisfies the relation:
$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$
(9) The cross product of a and b satisfies $|a \times b| = |a||b| \sin \theta$ where θ is the angle between a and b, and further, the cross product vector is perpendicular to both a and b, and its direction is given by the right hand rule.

(10) There is a scalar triple product. The scalar triple product of vectors a, b, and c is defined as the number $a \cdot (b \times c)$. It can also be viewed as the determinant of a matrix whose rows are the coordinates of a, b, and c respectively. The scalar triple product is preserved under cyclic permutations of the input vectors and gets negated under flipping two of the input vectors. It is linear in each input variable (i.e., distributive and pulls out scalars). The scalar triple product is zero if and only if the three input vectors can all be made to lie in the same plane.

(11) *Added for clarification*: In particular, $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$ for any vectors a and b in three dimensions.

Actions ...

(1) Vector and scalar projections: Given vectors a and b, the *vector projection* of b onto a, denoted $\operatorname{proj}_a b$, is given by the vector $\frac{a \cdot b}{|a|^2} a$. The scalar projection or component of b along a, denoted $\operatorname{comp}_a b$, is given by $\frac{a \cdot b}{|a|}$. The vector projection is what we obtain by taking the vector from the origin to the foot of the perpendicular from the head of b to the line of a. The scalar projection is the *directed* length of this vector, measured positive in the direction of a.

(2) Finding the angle between vectors: This is done using the dot product. The angle between vectors v and w is $\arccos((v \cdot w)/|v||w|)$.

(3) Finding the area of a triangle or a parallelogram: We first find two adjacent sides as vectors both with the same starting vertex (by taking the differences of coordinates of endpoints). For the parallelogram, we take the length of the cross product of these two vectors. For the triangle, we take *half* the length.

(4) Finding the volume of a parallelopiped: We find three sides as vectors, all with the same starting vertex. Then we take the *absolute value of* the scalar triple product of these sides.

(5) Finding a vector orthogonal to two given vectors: Simply take the cross product if they are linearly independent. Otherwise, just pick anything that dots with one of them to zero.

(6) Testing orthogonality: We check whether the dot product is zero.

(7) Testing coplanarity of points: We take one point as the basepoint, compute difference vectors to it from the other three points. We then take the scalar triple product of these three vectors. If we get zero, then the four points are coplanar, otherwise they are not.

7. Vector-valued functions

7.1. **Vector-valued functions, limits, and continuity.**

(1) *Not for review discussion*: A vector-valued function is a function from $\mathbb{R}$, or a subset of $\mathbb{R}$, to a vector space $\mathbb{R}^n$. It comprises n scalar functions, one for each of the coordinates. For instance, given scalar functions $f_1, f_2, \dots, f_n$, we can construct a vector-valued function $f = \langle f_1, f_2, \dots, f_n \rangle$ defined by $t \mapsto \langle f_1(t), f_2(t), \dots, f_n(t) \rangle$.

(2) *Not for review discussion*: A vector-valued function in n dimensions corresponds to a parametric description of a curve in $\mathbb{R}^n$ whose points are just the heads of the corresponding vectors. The vector-valued function from the previous observation has corresponding curve $\{(f_1(t), f_2(t), \dots, f_n(t)) : t \in D\}$ where D is the appropriate domain.

(3) To add two vector-valued functions in n dimensions, we add them coordinate-wise, where the corresponding scalar functions are added pointwise as usual. This sum is also a vector-valued function in n dimensions.

(4) We can multiply a scalar function and a vector-valued function to get a new vector-valued function. At each point in the domain, this is just multiplication of the corresponding scalar number and the corresponding vector.

(5) We can take the dot product of two vector-valued functions in n dimensions. The dot product is a scalar-valued function. At each point in the domain, it is obtained by taking the dot product of the corresponding vector values.

(6) For $n = 3$, we can take the cross product of two vector-valued functions and get a vector-valued function. This cross product is taken pointwise.

(7) To calculate the limit of a vector-valued function at a point, we calculate the limit separately for each coordinate. We use this idea to define the *limit*, *left hand limit*, and *right hand limit* at any point in the domain.
(8) Limit theorems: Limit of sum is sum of limits, constant scalars pull out of limits, limit of scalar-vector product is product of scalar limit and vector limit, limit of dot product is dot product of limits, limit of cross product (case $n = 3$) is cross product of limits.
(9) A vector-valued function is *continuous* at a point in its domain if each coordinate function is continuous, or equivalently, if the limit equals the value. We say it is continuous on its interval if it is continuous at every point in the interior of the interval and has one-sided continuity at one of the endpoints.
(10) Continuity theorems: Sum of continuous vector-valued functions is continuous, product of continuous scalar function and continuous vector-valued function is continuous, dot product of continuous vector-valued functions is continuous, cross product (case $n = 3$) of continuous vector-valued functions is continuous.
(11) There is no n-dimensional analogue of the intermediate value theorem, multiple things fail.

Actions ...

(1) If no domain is specified, the domain of a vector-valued function is the intersection of the domains of all the constituent scalar functions.

7.2. **Top-down and bottom-up descriptions.** Words ...

(1) A top-down description of a subset of $\mathbb{R}^n$ is in terms of a system of equations and inequality constraints. Each equation (equality constraint) is expected to reduce the dimension by 1 (we start from n) whereas inequality constraints usually have no effect on the dimension. So if there are k independent equality constraints describing a subset of $\mathbb{R}^n$, we expect the subset to have dimension $n - k$.
(2) A bottom-up description is a parametric description with possibly more than one parameter. The number of parameters needed is the dimension of the subset. The parametric descriptions we have seen so far are 1-parameter descriptions and hence they describe curves – 1-dimensional subsets.
(3) The codimension of a m-dimensional subset is $n - m$.
(4) When intersecting, codimensions are expected to add. If the total codimension we get after adding is greater than the dimension of the space, the intersection is expected to be empty.
(5) In $\mathbb{R}^3$, curves are one-dimensional, surfaces are two-dimensional. Thus, curves are not expected to intersect each other, but curves and surfaces are expected to intersect at finite collections of points (in general).

Actions ...

(1) Strategy for finding intersection of subsets in $\mathbb{R}^n$ (specifically, curves and surfaces in $\mathbb{R}^3$) given with top-down descriptions: Take all the equations together and solve simultaneously.
(2) Strategy for finding intersection of curve given parametrically and curve or surface given by top-down description: Plug in the functions of the parameter for the coordinates in the top-down description.
(3) Strategy for finding intersection of curves given parametrically: Choose different letters for parameter values, and then equate coordinate by coordinate. We get a bunch of equations in two variables (the two parameter values).
(4) Strategy for finding collision of curves given parametrically: Just equate coordinates, using the same letter for parameter values. Get a bunch of equations all in one variable.

7.3. **Differentiation, tangent vectors, integration.**

(1) The derivative of a n-dimensional vector-valued function is again a n-dimensional vector-valued function. It can be defined by differentiating each coordinate with respect to the parameter, or by using a difference quotient expression. These definitions are equivalent.
(2) This derivative operation satisfies the sum rule, pulling out constant scalars, and product rules for scalar-vector multiplication, dot product, and cross product (case $n = 3$).

(3) As a free vector, the tangent vector at $t = t_0$ to a parametric description of a curve is just the derivative vector for the corresponding vector-valued function. As a localized vector, it starts off at the corresponding point in $\mathbb{R}^n$.
(4) The tangent vector for a curve with parametric description depends on the choice of parameterization. The *unit tangent vector* does not, apart from the issue of direction (forward or backward). The unit tangent vector is a unit vector (i.e., length 1 vector) in the direction of the tangent vector. It is unique for a given curve (independent of parameterization) up to forward-backward issues.
(5) To perform definite or indefinite integration of a vector-valued function, we perform the integration coordinate-wise.