REVIEW SHEET FOR MIDTERM 1: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.

1. Formula summary

1.1. Parametric.
Set $x = f(t)$, $y = g(t)$, parametric curve in $\mathbb{R}^2$.

- $dy/dt = g'(t)$ and $dx/dt = f'(t)$.
- $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$.
- $\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$
- Arc length: $\int \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

1.2. Polar.
Set $r = F(\theta)$, polar equation of a curve.

- $y = F(\theta)\sin\theta$ and $x = F(\theta)\cos\theta$.
- $dy/d\theta = F'(\theta)\sin\theta + F(\theta)\cos\theta$ and $dx/d\theta = F'(\theta)\cos\theta - F(\theta)\sin\theta$.
- $\frac{dy}{dx} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$
- Arc length: $\int \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$

1.3. Three-dimensional geometry.

- Distance formula between (x_1, y_1, z_1) and (x_2, y_2, z_2): $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center having coordinates (h, k, l) and radius r is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

1.4. Vectors.

- Vector dot product: $\langle v_1, v_2, \dots, v_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle = v_1w_1 + v_2w_2 + \dots + v_nw_n$.
- Length of vector $\langle v_1, v_2, \dots, v_n \rangle$ is $\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.
- Unit vector in the direction of a vector v is $v/|v|$. Unit vector in opposite direction but along same line (so parallel) is $-v/|v|$.
- Vector cross product: $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$.
- For nonzero vectors v and w in three dimensions, we have $|v \times w| = |v||w|\sin\theta$ where θ is the angle between v and w.
- Scalar triple product is $a \cdot (b \times c)$.
- Angle between nonzero vectors v and w is $\arccos\left(\frac{v \cdot w}{|v||w|}\right)$.
- Scalar projection of b onto a is $(a \cdot b)/|a|$. *Note: Be careful what is being projected onto what.*
- Vector projection of b onto a is $((a \cdot b)/|a|^2)a$.
- Area of triangle with vertices P, Q and R is $(1/2)|PQ \times PR|$. Need to: (i) compute difference vectors, (ii) take cross product, (iii) compute length of the cross product, (iv) divide by 2.
- Area of parallelogram with vertices P, Q, R, S is $|PQ \times PR|$ or $|PQ \times PS|$ (same number). Steps (i)-(iii) of above.
- Volume of parallelepiped is *absolute value of* scalar triple product of vectors for adjacent triple of edges.

2. Quickly: what you should know from one-variable calculus

You need to be able to do the following from one-variable calculus and before:

(1) Finding domains of functions
(2) Basic algebraic manipulation and trigonometric identities

(3) Graphing: Know equation of circle centered at origin, graph linear functions, sine, cosine.
(4) Differentiation and integration: Everything you saw in one-variable calculus. However, for this midterm, you will get only simple integrations that rely on the very basic formulas and not, for instance, those that use integration by parts.

3. Parametric stuff

Error-spotting exercises ...

(1) Consider the parametric curve given by $x = \sin^3 t$, $y = t^3$. We want to calculate dy/dx at $t = 0$. We note that $dy/dt = 3t^2$, and at $t = 0$, this takes the value 0. Thus:

$$\frac{dy}{dx}|_{t=0} = \frac{dy/dt}{dx/dt}|_{t=0} = \frac{3t^2}{dx/dt}|_{t=0} = \frac{0}{dx/dt} = 0$$

(2) Consider the curve $x = (\cos t)^{2/3}$ and $y = (\sin t)^{2/3}$, $t \in \mathbb{R}$. This curve is described by the relation $x^3 + y^3 = 1$.
(3) Consider the curve given by $x = e^t$, $y = e^{t^2}$, $t \in \mathbb{R}$. Then, the graph of this function is the part of the parabola $y = x^2$ for $x \geq 0$.
(4) Consider the curve given parametrically by $x = \cos(t^2)$, $y = \sin(t^2)$. To calculate the length of the arc of this curve from $t = 0$ to $t = 5$, we calculate:

$$\int_0^5 \sqrt{(\cos(t^2))^2 + (\sin(t^2))^2}\, dt = \int_0^5 \sqrt{\cos^2(t^2) + \sin^2(t^2)}\, dt = \int_0^5 dt = 5$$

4. Polar coordinates

Error-spotting exercises ...

(1) Consider the parametric description $x = \cos^2\theta$, $y = \sin^2\theta$. To convert to a polar description, we set $x = r\cos\theta$, $y = r\sin\theta$, so we get $r\cos\theta = \cos^2\theta$ and $r\sin\theta = \sin^2\theta$. Simplifying, we get either $r = \cos\theta = \sin\theta$ or $r = \cos\theta$, $\sin\theta = 0$, or $r = \sin\theta$, $\cos\theta = 0$.

5. Three-dimensional geometry

Error-spotting exercises ...

(1) Suppose A and B are points in $\mathbb{R}^3$. Suppose λ is a fixed positive real number. Then, the set of points C such that $|AC|/|BC| = \lambda$ is a plane whose intersection with the line segment AB divides it into the ratio $\lambda : 1$. The case $\lambda = 1$ is a case in point: in this case, the plane is the perpendicular bisector of AB.

6. Introduction to vectors and relation with geometry

6.1. n-dimensional generality.

Error-spotting exercises ...

(1) The product of the vectors $\langle 1, 2, 3\rangle$ and $\langle 3, 4, 5\rangle$ is the vector $\langle 3, 8, 15\rangle$.
(2) If a is a scalar and $v = \langle v_1, v_2, \dots, v_n\rangle$ is a vector, the length of $av = \langle av_1, av_2, \dots, av_n\rangle$ is a times the length of v.
(3) The dot product of the three vectors $\langle 1, 2, 3\rangle$, $\langle 4, 5, 6\rangle$, and $\langle 7, 8, 9\rangle$ is $\langle 28, 80, 162\rangle$.

6.2. Three-dimensional geometry.

Error-spotting exercises ...

(1) The cross product of the vectors $\langle 2, 3, 0\rangle$ and $\langle 4, 5, 0\rangle$ is $\langle (2)(5)-(4)(3), (3)(0)-(0)(5), (0)(4)-(0)(2)\rangle$ which simplifies to $\langle -2, 0, 0\rangle$.
(2) We can compute the angle between vectors v and w by using the formula $\arcsin(|v \times w|/(|v||w|))$.
(3) Because the dot product of two vectors a and b is symmetric in a and b, the scalar projection of a on b is the same as the scalar projection of b on a.
(4) To check whether three points are coplanar, we take the scalar triple product of the vectors giving their coordinates and check if the scalar triple product is zero.

7. Vector-valued functions

7.1. Vector-valued functions, limits, and continuity. Error-spotting exercises ...

(1) Consider the vector-valued function $\langle 1/t, 1/(t-1), 1/(t+1)\rangle$. The domain is all real numbers, because at every real number, at least one of the coordinates is defined.
(2) Consider the vector-valued functions $\langle t, 1, t\rangle$ and $\langle t, -2t^2, t\rangle$. The dot product of these vector-valued functions is identically the 0 function. Thus, the corresponding parametric curves for these functions are orthogonal curves, i.e., they intersect at right angles.

7.2. Top-down and bottom-up descriptions. Error-spotting exercises ...

(1) If S_1 and S_2 are two surfaces in $\mathbb{R}^3$ given as the solutions to $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$ respectively, then $S_1 \cap S_2$ is given by the equation $F_1(x, y, z) + F_2(x, y, z) = 0$ and $S_1 \cup S_2$ is given by the equation $F_1(x, y, z)F_2(x, y, z) = 0$.
(2) The intersection of finitely many two-dimensional subsets of $\mathbb{R}^3$ is generically expected to be one-dimensional. For instance, the intersection of two planes (each two-dimensional) is expected to be a line (one-dimensional).
(3) Surfaces in $\mathbb{R}^3$ have dimension 2 and codimension 1. So, the intersection of two surfaces should have codimension $1 + 1 = 2$ and dimension $3 - 2 = 1$, hence should be a curve. This means that the intersection of any surface with itself should be a curve. In other words, every surface should be a curve.
(4) The intersection of the surfaces $x^2 + y^2 + z^2 = 1$ and $x^4 + y^4 + z^4 = 1/2$ is the surface $(x^2 + y^2 + z^2 - 1)(x^4 + y^4 + z^4 - (1/2)) = 0$.
(5) $x^2 + y^2 = 1$ defines a circle in the xy-plane in $\mathbb{R}^3$ centered at the origin and with radius 1. Hence, the solution set in $\mathbb{R}^3$ to $(x^2 + y^2 - 1)(y^2 + z^2 - 1)(z^2 + x^2 - 1) = 0$ is the union of the three circles in the xy-plane, yz-plane, and xz-plane, with center at the origin and radius 1.

7.3. Differentiation, tangent vectors, integration. Error-spotting exercises ...

(1) The indefinite integral of the vector-valued function $t \mapsto \langle 2t, 3t^2, 4t^3\rangle$ is $t \mapsto \langle t^2 + C, t^3 + C, t^4 + C\rangle$.
(2) Suppose f and g are vector-valued functions. Then:

$$\int (f(t) \cdot g(t))\, dt = f(t) \cdot \left(\int g(t)\, dt\right) + \left(\int f(t)\, dt\right) \cdot g(t)$$