REVIEW SHEET FOR FINAL: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to all review sessions.

1. Directional derivatives and gradient vectors

Error-spotting exercises ...

(1) *Partials don't tell the whole story*: Consider the function $f(x, y) := (xy)^{1/5}$. We note that f takes the value 0 identically both on the x-axis and the y-axis, thus, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Hence, the gradient of f at $(0, 0)$ is the zero vector.

(2) *Directional derivatives don't tell the whole story either*: Let

$$f(x, y) := \begin{cases} 0 & \text{if } y \le 0 \text{ or } y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

We note that on any line approaching $(0, 0)$, f becomes constant at 0 close enough to $(0, 0)$. Hence, the directional derivative of f in every direction is 0. Thus, the gradient vector of f is 0.

(3) *Orthogonal to nothing*: Consider the function $f(x, y) := \sin(xy)$ at the point $(\pi, 1/2)$. At this point, we have $f_x(x, y) = y\cos(xy) = (1/2)\cos(\pi/2) = 0$. Thus, the gradient of f is in the y-direction, so the tangent line to the level curve of f for this function is parallel to the x-axis.

(4) *Zero gradient, level curve not smooth?*: Consider the function $f(x, y) := (x-y)^3$. At the point $(1, 1)$, both $f_x(x, y)$ and $f_y(x, y)$ take the value 0, so the gradient vector is 0. Thus, the level curve of f passing through the point $(1, 1)$ does not have a well defined normal direction at $(1, 1)$.

(5) *Misquare*: The maximum magnitude of directional derivative for a function f with a nonzero gradient at a point occurs in the direction of the gradient vector ∇f, and its value is $\nabla f \cdot \nabla f = |\nabla f|^2$.

(6) *False addition*: The directional derivative along the direction of the vector $a + b$ is the sum of the directional derivatives along the direction of a and the direction of b.

2. Max-min values

Error-spotting exercises ...

(1) *Separate versus joint*: Suppose F is a function of two variables denoted x and y, and (x_0, y_0) is a point in the interior of the domain of F. If F has a local maximum at (x_0, y_0) with respect to both the x- and the y-directions, then F must have a local maximum.

(2) *Saddled with wrong ideas*: Suppose F is a function of two variables denoted x and y, and (x_0, y_0) is a point in the interior of the domain of F. If F has a saddle point at (x_0, y_0), then that means it must have a local maximum from one of the x- and y-directions and a local minimum from the other.

(3) *Hessian as second derivative*: The second derivative test for a function f of two variables says the following: define the Hessian determinant $D(a, b)$ at a point as $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. If $D(a, b) > 0$, this means that f has a local minimum at (a, b). If $D(a, b) < 0$, this means that f has a local maximum at (a, b). If $D(a, b) = 0$, the second derivative test is inconclusive.

3. Lagrange multipliers

Error-spotting exercises ...

(1) *Local maximum, minimum*: To determine whether a point on a level curve of g satisfying the Lagrange condition on f (i.e., $\nabla f = \lambda \nabla g$) gives a local maximum or a local minimum for f, we simply need to check whether $\lambda > 0$ or $\lambda < 0$. If $\lambda > 0$, we have a local minimum, and if $\lambda < 0$, we have a local maximum.

(2) *Hessian confusion*: Consider a function f of two variables. Let D denote the Hessian determinant. To maximize f along the constraint curve $g(x, y) = k$, we first find points on the constraint curve where $\nabla f = \lambda \nabla g$ for some suitable choice of λ, i.e., points satisfying the Lagrange condition. At any such point, if $D < 0$, then we have neither a local maximum nor a local minimum with respect to the curve. If $D > 0$ and $f_{xx} > 0$, then we have a local minimum with respect to the curve. If $D > 0$ and $f_{xx} < 0$, then we have a local maximum with respect to the curve.

4. Max-min values: examples

Error-spotting exercises ...

(1) *Absolute maximum folly, thinking in the box*: Suppose $F(x, y) := f(x) + g(y)$ and we want to maximize F over the domain $|x| + |y| \le 1$. We note that in the domain $|x| + |y| \le 1$, we have the constraints $-1 \le x \le 1$ and $-1 \le y \le 1$. Thus, to find the absolute maximum for F, we do the following: maximize f on the interval $[-1, 1]$ (say at x_0 with value a), maximize g on the interval $[-1, 1]$ (say at y_0 with value b), and then take the combined point (x_0, y_0) and get value $a + b$.

(2) *Critical missed types*: Suppose $F(x, y) := f(x)g(y)$. Then, (x_0, y_0) gives a critical point for F if and only if x_0 gives a critical point for f and y_0 gives a critical point for g.

(3) *Ignoring the signs of a pessimistic world*: Suppose $F(x, y) := f(x)g(y)$. If f attains a local maximum value at x_0 and g attains a local maximum value at y_0, then F attains a local maximum value at (x_0, y_0).

(4) *Maximum, minimum*: Suppose f is a continuous quasiconvex function defined on the set $|x|+|y| \le 1$. We know by the definition of quasiconvex that f must attain both its absolute maximum and its absolute minimum at one of its extreme points, i.e., at one of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

(5) *Pointy circles*: Suppose f is a strictly convex function defined on the circular disk $x^2 + y^2 \le 1$. Then, f can attain its absolute maximum only at one of the four extreme points: $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.