REVIEW SHEET FOR MIDTERM 1: BASIC

MATH 195, SECTION 59 (VIPUL NAIK)

We will not be going over this sheet, but rather, we'll be going over the advanced review sheet in the session. Please review this sheet on your own time.

1. Formula summary

1.1. **Parametric.** Set $x = f(t)$, $y = g(t)$, parametric curve in $\mathbb{R}^2$.

- $dy/dt = g'(t)$ and $dx/dt = f'(t)$.
- $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$.
- $\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$
- Arc length: $\int \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

1.2. **Polar.** Set $r = F(\theta)$, polar equation of a curve.

- $y = F(\theta)\sin\theta$ and $x = F(\theta)\cos\theta$.
- $dy/d\theta = F'(\theta)\sin\theta + F(\theta)\cos\theta$ and $dx/d\theta = F'(\theta)\cos\theta - F(\theta)\sin\theta$.
- $\frac{dy}{dx} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$
- Arc length: $\int \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$

1.3. **Three-dimensional geometry.**

- Distance formula between (x_1, y_1, z_1) and (x_2, y_2, z_2): $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center having coordinates (h, k, l) and radius r is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

1.4. **Vectors.**

- Vector dot product: $\langle v_1, v_2, \dots, v_n\rangle \cdot \langle w_1, w_2, \dots, w_n\rangle = v_1w_1 + v_2w_2 + \dots + v_nw_n$.
- Length of vector $\langle v_1, v_2, \dots, v_n\rangle$ is $\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.
- Unit vector in the direction of a vector v is $v/|v|$. Unit vector in opposite direction but along same line (so parallel) is $-v/|v|$.
- Vector cross product: $\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$.
- For nonzero vectors v and w in three dimensions, we have $|v \times w| = |v||w|\sin\theta$ where θ is the angle between v and w.
- Scalar triple product is $a \cdot (b \times c)$.
- Angle between nonzero vectors v and w is $\arccos\left(\frac{v \cdot w}{|v||w|}\right)$.
- Scalar projection of b onto a is $(a \cdot b)/|a|$. *Note: Be careful what is being projected onto what.*
- Vector projection of b onto a is $((a \cdot b)/|a|^2)a$.
- Area of triangle with vertices P, Q and R is $(1/2)|PQ \times PR|$. Need to: (i) compute difference vectors, (ii) take cross product, (iii) compute length of the cross product, (iv) divide by 2.
- Area of parallelogram with vertices P, Q, R, S is $|PQ \times PR|$ or $|PQ \times PS|$ (same number). Steps (i)-(iii) of above.
- Volume of parallelepiped is *absolute value of* scalar triple product of vectors for adjacent triple of edges.

2. Quickly: what you should know from one-variable calculus

You need to be able to do the following from one-variable calculus and before:

(1) Finding domains of functions
(2) Basic algebraic manipulation and trigonometric identities

(3) Graphing: Know equation of circle centered at origin, graph linear functions, sine, cosine.
(4) Differentiation and integration: Everything you saw in one-variable calculus. However, for this midterm, you will get only simple integrations that rely on the very basic formulas and not, for instance, those that use integration by parts.

3. Parametric stuff

Words ...

(1) A parametric description of a curve is one where both coordinates are expressed as functions of of a parameter, typically denoted t. Parametric descriptions offer an alternative to functional and implicit (relational) descriptions of curves. Here, t varies over some subset of the real numbers. In symbols, we have something like $x = f(t)$, $y = g(t)$, where t varies over some subset D of the real numbers.
(2) Descriptions where x is a function of y or y is a function of x are special cases of parametric descriptions.
(3) The same curve may admit multiple parametrizations, and different parameterizations may correspond to different speeds and different orderings of traversal of the point. The curve itself only contains the information of *what* points were traversed, not the information of the *sequence* and *pace* in which they were traversed.
(4) The curve traced by a parameterization depends not only on the coordinate functions but also the domain for the parameter. The larger the domain, in general, the larger the curve traced. However, in some cases, expanding the domain may not make the curve strictly larger. This happens in cases where both coordinate functions are even or have commensurable periods.
(5) A parameterization of a curve may involve self-intersections, retracings (e.g., tracing back for even function pairs), or even wrapping around itself (for periodic function pairs).
(6) Function composition allows us to switch between multiple parameterizations.
(7) In some cases, it is possible to move back and forth between parametric and relational descriptions.
(8) Parametric differentiation: if $x = f(t)$ and $y = g(t)$, then $dy/dx = (dy/dt)/(dx/dt) = g'(t)/f'(t)$. This can also be used to differentiate repeatedly. Note that the derivative is a function of t rather than of (x, y), so to find the derivative given the point (x, y) we need to go back and determine t.
(9) Higher derivatives can be computed iteratively using parametric differentiation. But note that it is *not* true that $d^2y/dx^2 = (d^2y/dt^2)/(d^2x/dt^2)$. The actual formula/procedure is more complicated (see lecture notes or formula summary).
(10) Arc length: The formula for arc length from $t = a$ to $t = b$ (with $a < b$) is $\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$.

Actions ...

(1) *Parametric to relational: elimination of parameter*: In many cases, it is possible to eliminate a parameter from a parametric description. The idea is to use some well known identities or manipulation techniques to try to directly relate x and y by finding some equation between them that is true for all t. However, this is not the full story. We next need to see if there are additional restrictions on x and y deducible from the fact that they arose as function of t, also keeping in mind the domain restrictions on t.

    For instance, the parameterization $x = t^2, y = t^4$ for $t \in \mathbb{R}$ can be rewritten as $y = x^2$, but we need the additional condition that $x \geq 0$.

    See more examples in the lecture notes, quizzes, and homeworks.
(2) *Relational to parametric*: Here, we see a relation between x and y, and try to choose a parametric description that would give rise to the relation. Again, the domain of choice for the parameter needs to be chosen wisely.

    See more examples in the lecture notes and quizzes.
(3) *Parametric differentiation and geometric consequences*: We use the formula $(dy/dt)/(dx/dt)$. If $x = f(t)$ and $y = g(t)$, then this becomes $g'(t)/f'(t)$. This is valid for all t in the interior of the domain of definition where both f' and g' are defined and $f' \neq 0$. If $f'(t) = 0$ but $g'(t) \neq 0$, we have a vertical tangent situation. If $g'(t) = 0$ but $f'(t) \neq 0$, we have a horizontal tangent situation.

4. Polar coordinates

Words ...

(1) *Specifying a polar coordinate system*: To specify a polar coordinate system, we need to choose a point (called the *origin* or *pole*), a half-line starting at the point (called the *polar axis* or *reference line*) and an orientation of the plane (chosen counter-clockwise in the usual depictions).
(2) *Finding the polar coordinates of a point and vice versa*: The radial coordinate r is the distance between the point and the pole. The angular coordinate θ is the angle (measured in the counter-clockwise direction) from the polar axis to the line segment from the pole to the point. Note that θ is uniquely defined up to addition of multiples of 2π, and it becomes truly unique if we restrict it to a half-open half-closed interval of length 2π. *The exception is the pole itself, for which θ is undefined* in the sense that any value of θ could be chosen.
(3) *Converting between Cartesian and polar coordinates*: If we take the polar axis as the positive x-axis and the axis at an angle of $+\pi/2$ from it as the positive y-axis, we get a Cartesian coordinate system. The point defined by polar coordinates (r, θ) has Cartesian coordinates $(r\cos\theta, r\sin\theta)$. Conversely, given a point with Cartesian coordinates (x, y) the corresponding polar coordinates are $r = \sqrt{x^2 + y^2}$ and θ is the unique angle (up to addition of multiples of 2π) such that $x = r\cos\theta$, $y = r\sin\theta$.

Actions ...

(1) A functional description of the form $r = F(\theta)$ gives rise to a parametric description in Cartesian coordinates: $x = F(\theta)\cos\theta$ and $y = F(\theta)\sin\theta$. We can do the usual things (like find slopes of tangent lines) using this parametric description. Note that here, θ is typically allowed to vary over all of $\mathbb{R}$ rather than simply being restricted to an interval of length 2π. The slope of the tangent line in Cartesian terms is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d(F(\theta)\sin\theta)/d\theta}{d(F(\theta)\cos\theta)/d\theta} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$$

(2) The arc length is given by integrating $\sqrt{(F(\theta))^2 + (F'(\theta))^2}$ for θ in the suitable interval. (See quiz question on this).
(3) An implicit (relational) description in Cartesian coordinates can be converted to a description in polar coordinates by replacing x by $r\cos\theta$ and y by $r\sin\theta$.
(4) An implicit (relational) description in polar coordinates can sometimes be converted to a description in Cartesian coordinates, but with some ambiguity. General idea: replace r by $\sqrt{x^2 + y^2}$, $\cos\theta$ by $x/\sqrt{x^2 + y^2}$, and $\sin\theta$ by $y/\sqrt{x^2 + y^2}$.

5. Three-dimensional geometry

Words ...

(1) Three-dimensional space is coordinatized using a Cartesian coordinate system by selecting three mutually perpendicular axes passing through a point called the origin: the x-axis, y-axis, and z-axis. These satisfy the right-hand rule. The coordinates of a point are written as a 3-tuple (x, y, z).
(2) There are $2^3 = 8$ octants based on the signs of each of the coordinates. There are three coordinate planes, each corresponding to the remaining coordinate being zero (the xy-plane corresponds to $z = 0$, etc.). There are three axes, each corresponding to the other two coordinates being zero (e.g., the x-axis corresponds to $y = z = 0$).
(3) The distance formula between points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is similar to the formula in two dimensions and the squares and square root arises from the Pythagorean theorem.
(4) The equation of a sphere with center having coordinates (h, k, l) and radius r is $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. Given an equation, we can try completing the square to see if it fits the model for the equation of a sphere.

6. Introduction to vectors and relation with geometry

6.1. n-dimensional generality. Words ...

(1) A vector is an ordered n-tuple of real numbers (or quantities measured using real numbers). The space of such n-tuples is a n-dimensional vector space over the real numbers. Vectors can be used to store tuples of prices, probabilities, and other kinds of quantities.
(2) There is a zero vector. We can add vectors and we can multiply a vector by a scalar. Note that these operations may or may not have an actual meaning based on the thing we are storing using the vector.
(3) We can take the dot product $v \cdot w$ of two vectors v and w in n-dimensional space. if $v = \langle v_1, v_2, \ldots, v_n \rangle$ and $w = \langle w_1, w_2, \ldots, w_n \rangle$, then $v \cdot w = \sum_{i=1}^{n} v_i w_i$. The dot product is a real number (though if we put units to the coordinates of the vector, it gets corresponding squared units).
(4) The length or norm of a vector v, denoted $|v|$, is defined as $\sqrt{v \cdot v}$. It is a nonnegative real number.
(5) The correlation between two vectors v and w is defined as $(v \cdot w)/(|v||w|)$. It is in $[-1, 1]$. (For geometric interpretation, see the three-dimensional case).
(6) *Properties of the dot product*: The dot product is symmetric, the dot product of any vector with the zero vector is 0, the dot product is additive (distributive) in each coordinate and scalars can be pulled out.
(7) *Properties of length*: The only vector with length zero is the zero vector, all other vectors have positive length. The length of λv is $|\lambda|$ times the length of v. We also have $|v + w| \leq |v| + |w|$ for any vectors v and w, with equality occurring if either is a positive scalar multiple of the other or one of them is the zero vector.

6.2. Three-dimensional geometry. Words ...

(1) We can identify points in three-dimensional space with three-dimensional vector as follows: the vector corresponding to a point (x, y, z) is the vector $\langle x, y, z \rangle$. Physically, this can be thought of as a directed line segment or arrow from the origin to the point (x, y, z).
(2) We can also consider vectors starting at any point in three-dimensional space and ending at any point. The corresponding vector can be obtained by subtracting the coordinates of the points. The vector from point P to point Q is denoted $\overline{PQ}$.
(3) There are unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. These are thus the vectors of length 1 along the positive x, y, and z directions respectively. A vector $\langle x, y, z \rangle$ can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
(4) Vectors can be added geometrically using the *parallelogram law*. This procedure gives the same answer as the usual coordinate-wise addition.
(5) Scalar multiplication also has a geometric interpretation − the length gets scaled by the scalar multiple, and the direction remains the same or is reversed depending on the scalar's sign.
(6) For vectors v and w, we have $v \cdot w = |v||w| \cos \theta$ where θ is the angle between v and w. We can use this procedure to find the angle between two vectors. The correlation between the vectors is thus $\cos \theta$. We can interpret this specifically for $\theta = 0$, θ an acute angle, $\theta = \pi/2$, θ an obtuse angle, and $\theta = \pi$ (see the table in the lecture notes).
(7) We can define the vector cross product $v \times w$ using a matrix determinant. Equivalently, if $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$, then $v \times w = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$. *This is a specifically three-dimensional construct.*
(8) The cross product has the property that cross product of any two collinear vectors is zero, cross product of any vector with the zero vector is zero, the cross product is skew-symmetric, distributive in each variable, and allows scalars to be pulled out. It is not associative in general. There is an identity relating cross product and dot product: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. Also, the cross product satisfies the relation:

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$

(9) The cross product of a and b satisfies $|a \times b| = |a||b| \sin \theta$ where θ is the angle between a and b, and further, the cross product vector is perpendicular to both a and b, and its direction is given by the right hand rule.

(10) There is a scalar triple product. The scalar triple product of vectors a, b, and c is defined as the number $a \cdot (b \times c)$. It can also be viewed as the determinant of a matrix whose rows are the coordinates of a, b, and c respectively. The scalar triple product is preserved under cyclic permutations of the input vectors and gets negated under flipping two of the input vectors. It is linear in each input variable (i.e., distributive and pulls out scalars). The scalar triple product is zero if and only if the three input vectors can all be made to lie in the same plane.
(11) *Added for clarification*: In particular, $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$ for any vectors a and b in three dimensions.

Actions ...

(1) Vector and scalar projections: Given vectors a and b, the *vector projection* of b onto a, denoted $\text{proj}_a b$, is given by the vector $\frac{a \cdot b}{|a|^2} a$. The scalar projection or component of b along a, denoted $\text{comp}_a b$, is given by $\frac{a \cdot b}{|a|}$. The vector projection is what we obtain by taking the vector from the origin to the foot of the perpendicular from the head of b to the line of a. The scalar projection is the *directed* length of this vector, measured positive in the direction of a.
(2) Finding the angle between vectors: This is done using the dot product. The angle between vectors v and w is $\arccos((v \cdot w)/|v||w|)$.
(3) Finding the area of a triangle or a parallelogram: We first find two adjacent sides as vectors both with the same starting vertex (by taking the differences of coordinates of endpoints). For the parallelogram, we take the length of the cross product of these two vectors. For the triangle, we take *half* the length.
(4) Finding the volume of a parallelopiped: We find three sides as vectors, all with the same starting vertex. Then we take the *absolute value of* the scalar triple product of these sides.
(5) Finding a vector orthogonal to two given vectors: Simply take the cross product if they are linearly independent. Otherwise, just pick anything that dots with one of them to zero.
(6) Testing orthogonality: We check whether the dot product is zero.
(7) Testing coplanarity of points: We take one point as the basepoint, compute difference vectors to it from the other three points. We then take the scalar triple product of these three vectors. If we get zero, then the four points are coplanar, otherwise they are not.

7. Vector-valued functions

7.1. Vector-valued functions, limits, and continuity.

(1) *Not for review discussion*: A vector-valued function is a function from $\mathbb{R}$, or a subset of $\mathbb{R}$, to a vector space $\mathbb{R}^n$. It comprises n scalar functions, one for each of the coordinates. For instance, given scalar functions $f_1, f_2, \dots, f_n$, we can construct a vector-valued function $f = \langle f_1, f_2, \dots, f_n \rangle$ defined by $t \mapsto \langle f_1(t), f_2(t), \dots, f_n(t) \rangle$.
(2) *Not for review discussion*: A vector-valued function in n dimensions corresponds to a parametric description of a curve in $\mathbb{R}^n$ whose points are just the heads of the corresponding vectors. The vector-valued function from the previous observation has corresponding curve $\{(f_1(t), f_2(t), \dots, f_n(t)) : t \in D\}$ where D is the appropriate domain.
(3) To add two vector-valued functions in n dimensions, we add them coordinate-wise, where the corresponding scalar functions are added pointwise as usual. This sum is also a vector-valued function in n dimensions.
(4) We can multiply a scalar function and a vector-valued function to get a new vector-valued function. At each point in the domain, this is just multiplication of the corresponding scalar number and the corresponding vector.
(5) We can take the dot product of two vector-valued functions in n dimensions. The dot product is a scalar-valued function. At each point in the domain, it is obtained by taking the dot product of the corresponding vector values.
(6) For $n = 3$, we can take the cross product of two vector-valued functions and get a vector-valued function. This cross product is taken pointwise.

(7) To calculate the limit of a vector-valued function at a point, we calculate the limit separately for each coordinate. We use this idea to define the *limit*, *left hand limit*, and *right hand limit* at any point in the domain.
(8) Limit theorems: Limit of sum is sum of limits, constant scalars pull out of limits, limit of scalar-vector product is product of scalar limit and vector limit, limit of dot product is dot product of limits, limit of cross product (case $n = 3$) is cross product of limits.
(9) A vector-valued function is *continuous* at a point in its domain if each coordinate function is continuous, or equivalently, if the limit equals the value. We say it is continuous on its interval if it is continuous at every point in the interior of the interval and has one-sided continuity at one of the endpoints.
(10) Continuity theorems: Sum of continuous vector-valued functions is continuous, product of continuous scalar function and continuous vector-valued function is continuous, dot product of continuous vector-valued functions is continuous, cross product (case $n = 3$) of continuous vector-valued functions is continuous.
(11) There is no n-dimensional analogue of the intermediate value theorem, multiple things fail.

Actions ...

(1) If no domain is specified, the domain of a vector-valued function is the intersection of the domains of all the constituent scalar functions.

7.2. **Top-down and bottom-up descriptions.** Words ...

(1) A top-down description of a subset of $\mathbb{R}^n$ is in terms of a system of equations and inequality constraints. Each equation (equality constraint) is expected to reduce the dimension by 1 (we start from n) whereas inequality constraints usually have no effect on the dimension. So if there are k independent equality constraints describing a subset of $\mathbb{R}^n$, we expect the subset to have dimension $n - k$.
(2) A bottom-up description is a parametric description with possibly more than one parameter. The number of parameters needed is the dimension of the subset. The parametric descriptions we have seen so far are 1-parameter descriptions and hence they describe curves – 1-dimensional subsets.
(3) The codimension of a m-dimensional subset is $n - m$.
(4) When intersecting, codimensions are expected to add. If the total codimension we get after adding is greater than the dimension of the space, the intersection is expected to be empty.
(5) In $\mathbb{R}^3$, curves are one-dimensional, surfaces are two-dimensional. Thus, curves are not expected to intersect each other, but curves and surfaces are expected to intersect at finite collections of points (in general).

Actions ...

(1) Strategy for finding intersection of subsets in $\mathbb{R}^n$ (specifically, curves and surfaces in $\mathbb{R}^3$) given with top-down descriptions: Take all the equations together and solve simultaneously.
(2) Strategy for finding intersection of curve given parametrically and curve or surface given by top-down description: Plug in the functions of the parameter for the coordinates in the top-down description.
(3) Strategy for finding intersection of curves given parametrically: Choose different letters for parameter values, and then equate coordinate by coordinate. We get a bunch of equations in two variables (the two parameter values).
(4) Strategy for finding collision of curves given parametrically: Just equate coordinates, using the same letter for parameter values. Get a bunch of equations all in one variable.

7.3. **Differentiation, tangent vectors, integration.**

(1) The derivative of a n-dimensional vector-valued function is again a n-dimensional vector-valued function. It can be defined by differentiating each coordinate with respect to the parameter, or by using a difference quotient expression. These definitions are equivalent.
(2) This derivative operation satisfies the sum rule, pulling out constant scalars, and product rules for scalar-vector multiplication, dot product, and cross product (case $n = 3$).

(3) As a free vector, the tangent vector at $t = t_0$ to a parametric description of a curve is just the derivative vector for the corresponding vector-valued function. As a localized vector, it starts off at the corresponding point in $\mathbb{R}^n$.
(4) The tangent vector for a curve with parametric description depends on the choice of parameterization. The *unit tangent vector* does not, apart from the issue of direction (forward or backward). The unit tangent vector is a unit vector (i.e., length 1 vector) in the direction of the tangent vector. It is unique for a given curve (independent of parameterization) up to forward-backward issues.
(5) To perform definite or indefinite integration of a vector-valued function, we perform the integration coordinate-wise.

REVIEW SHEET FOR MIDTERM 1: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.

1. Formula summary

1.1. Parametric.
Set $x = f(t)$, $y = g(t)$, parametric curve in $\mathbb{R}^2$.

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- $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$.
- $\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$
- Arc length: $\int \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

1.2. Polar.
Set $r = F(\theta)$, polar equation of a curve.

- $y = F(\theta)\sin\theta$ and $x = F(\theta)\cos\theta$.
- $dy/d\theta = F'(\theta)\sin\theta + F(\theta)\cos\theta$ and $dx/d\theta = F'(\theta)\cos\theta - F(\theta)\sin\theta$.
- $\frac{dy}{dx} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$
- Arc length: $\int \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$

1.3. Three-dimensional geometry.

- Distance formula between (x_1, y_1, z_1) and (x_2, y_2, z_2): $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- Sphere with center having coordinates (h, k, l) and radius r is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

1.4. Vectors.

- Vector dot product: $\langle v_1, v_2, \dots, v_n\rangle \cdot \langle w_1, w_2, \dots, w_n\rangle = v_1w_1 + v_2w_2 + \dots + v_nw_n$.
- Length of vector $\langle v_1, v_2, \dots, v_n\rangle$ is $\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.
- Unit vector in the direction of a vector v is $v/|v|$. Unit vector in opposite direction but along same line (so parallel) is $-v/|v|$.
- Vector cross product: $\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$.
- For nonzero vectors v and w in three dimensions, we have $|v \times w| = |v||w|\sin\theta$ where θ is the angle between v and w.
- Scalar triple product is $a \cdot (b \times c)$.
- Angle between nonzero vectors v and w is $\arccos\left(\frac{v \cdot w}{|v||w|}\right)$.
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- Vector projection of b onto a is $((a \cdot b)/|a|^2)a$.
- Area of triangle with vertices P, Q and R is $(1/2)|PQ \times PR|$. Need to: (i) compute difference vectors, (ii) take cross product, (iii) compute length of the cross product, (iv) divide by 2.
- Area of parallelogram with vertices P, Q, R, S is $|PQ \times PR|$ or $|PQ \times PS|$ (same number). Steps (i)-(iii) of above.
- Volume of parallelepiped is *absolute value of* scalar triple product of vectors for adjacent triple of edges.

2. Quickly: what you should know from one-variable calculus

You need to be able to do the following from one-variable calculus and before:

(1) Finding domains of functions
(2) Basic algebraic manipulation and trigonometric identities

(3) Graphing: Know equation of circle centered at origin, graph linear functions, sine, cosine.
(4) Differentiation and integration: Everything you saw in one-variable calculus. However, for this midterm, you will get only simple integrations that rely on the very basic formulas and not, for instance, those that use integration by parts.

3. Parametric stuff

Error-spotting exercises ...

(1) Consider the parametric curve given by $x = \sin^3 t$, $y = t^3$. We want to calculate dy/dx at $t = 0$. We note that $dy/dt = 3t^2$, and at $t = 0$, this takes the value 0. Thus:

$$\frac{dy}{dx}|_{t=0} = \frac{dy/dt}{dx/dt}|_{t=0} = \frac{3t^2}{dx/dt}|_{t=0} = \frac{0}{dx/dt} = 0$$

(2) Consider the curve $x = (\cos t)^{2/3}$ and $y = (\sin t)^{2/3}$, $t \in \mathbb{R}$. This curve is described by the relation $x^3 + y^3 = 1$.
(3) Consider the curve given by $x = e^t$, $y = e^{t^2}$, $t \in \mathbb{R}$. Then, the graph of this function is the part of the parabola $y = x^2$ for $x \geq 0$.
(4) Consider the curve given parametrically by $x = \cos(t^2)$, $y = \sin(t^2)$. To calculate the length of the arc of this curve from $t = 0$ to $t = 5$, we calculate:

$$\int_0^5 \sqrt{(\cos(t^2))^2 + (\sin(t^2))^2}\, dt = \int_0^5 \sqrt{\cos^2(t^2) + \sin^2(t^2)}\, dt = \int_0^5 dt = 5$$

4. Polar coordinates

Error-spotting exercises ...

(1) Consider the parametric description $x = \cos^2\theta$, $y = \sin^2\theta$. To convert to a polar description, we set $x = r\cos\theta$, $y = r\sin\theta$, so we get $r\cos\theta = \cos^2\theta$ and $r\sin\theta = \sin^2\theta$. Simplifying, we get either $r = \cos\theta = \sin\theta$ or $r = \cos\theta$, $\sin\theta = 0$, or $r = \sin\theta$, $\cos\theta = 0$.

5. Three-dimensional geometry

Error-spotting exercises ...

(1) Suppose A and B are points in $\mathbb{R}^3$. Suppose λ is a fixed positive real number. Then, the set of points C such that $|AC|/|BC| = \lambda$ is a plane whose intersection with the line segment AB divides it into the ratio $\lambda : 1$. The case $\lambda = 1$ is a case in point: in this case, the plane is the perpendicular bisector of AB.

6. Introduction to vectors and relation with geometry

6.1. n-dimensional generality.
Error-spotting exercises ...

(1) The product of the vectors $\langle 1, 2, 3\rangle$ and $\langle 3, 4, 5\rangle$ is the vector $\langle 3, 8, 15\rangle$.
(2) If a is a scalar and $v = \langle v_1, v_2, \ldots, v_n\rangle$ is a vector, the length of $av = \langle av_1, av_2, \ldots, av_n\rangle$ is a times the length of v.
(3) The dot product of the three vectors $\langle 1, 2, 3\rangle$, $\langle 4, 5, 6\rangle$, and $\langle 7, 8, 9\rangle$ is $\langle 28, 80, 162\rangle$.

6.2. Three-dimensional geometry.
Error-spotting exercises ...

(1) The cross product of the vectors $\langle 2, 3, 0\rangle$ and $\langle 4, 5, 0\rangle$ is $\langle (2)(5) - (4)(3), (3)(0) - (0)(5), (0)(4) - (0)(2)\rangle$ which simplifies to $\langle -2, 0, 0\rangle$.
(2) We can compute the angle between vectors v and w by using the formula $\arcsin(|v \times w|/(|v||w|))$.
(3) Because the dot product of two vectors a and b is symmetric in a and b, the scalar projection of a on b is the same as the scalar projection of b on a.
(4) To check whether three points are coplanar, we take the scalar triple product of the vectors giving their coordinates and check if the scalar triple product is zero.

7. Vector-valued functions

7.1. Vector-valued functions, limits, and continuity. Error-spotting exercises ...

(1) Consider the vector-valued function $\langle 1/t, 1/(t-1), 1/(t+1)\rangle$. The domain is all real numbers, because at every real number, at least one of the coordinates is defined.
(2) Consider the vector-valued functions $\langle t, 1, t\rangle$ and $\langle t, -2t^2, t\rangle$. The dot product of these vector-valued functions is identically the 0 function. Thus, the corresponding parametric curves for these functions are orthogonal curves, i.e., they intersect at right angles.

7.2. Top-down and bottom-up descriptions. Error-spotting exercises ...

(1) If S_1 and S_2 are two surfaces in $\mathbb{R}^3$ given as the solutions to $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$ respectively, then $S_1 \cap S_2$ is given by the equation $F_1(x, y, z) + F_2(x, y, z) = 0$ and $S_1 \cup S_2$ is given by the equation $F_1(x, y, z)F_2(x, y, z) = 0$.
(2) The intersection of finitely many two-dimensional subsets of $\mathbb{R}^3$ is generically expected to be one-dimensional. For instance, the intersection of two planes (each two-dimensional) is expected to be a line (one-dimensional).
(3) Surfaces in $\mathbb{R}^3$ have dimension 2 and codimension 1. So, the intersection of two surfaces should have codimension $1 + 1 = 2$ and dimension $3 - 2 = 1$, hence should be a curve. This means that the intersection of any surface with itself should be a curve. In other words, every surface should be a curve.
(4) The intersection of the surfaces $x^2 + y^2 + z^2 = 1$ and $x^4 + y^4 + z^4 = 1/2$ is the surface $(x^2 + y^2 + z^2 - 1)(x^4 + y^4 + z^4 - (1/2)) = 0$.
(5) $x^2 + y^2 = 1$ defines a circle in the xy-plane in $\mathbb{R}^3$ centered at the origin and with radius 1. Hence, the solution set in $\mathbb{R}^3$ to $(x^2 + y^2 - 1)(y^2 + z^2 - 1)(z^2 + x^2 - 1) = 0$ is the union of the three circles in the xy-plane, yz-plane, and xz-plane, with center at the origin and radius 1.

7.3. Differentiation, tangent vectors, integration. Error-spotting exercises ...

(1) The indefinite integral of the vector-valued function $t \mapsto \langle 2t, 3t^2, 4t^3\rangle$ is $t \mapsto \langle t^2 + C, t^3 + C, t^4 + C\rangle$.
(2) Suppose f and g are vector-valued functions. Then:

$$\int (f(t) \cdot g(t))\, dt = f(t) \cdot \left(\int g(t)\, dt\right) + \left(\int f(t)\, dt\right) \cdot g(t)$$

REVIEW SHEET FOR MIDTERM 2: BASIC

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.

The document does not include material that was part of the midterm 1 syllabus. Very little of that material will appear directly in midterm 2; however, you should have reasonable familiarity with the material.

We will not be going over this sheet, but rather, we'll be going over the advanced review sheet in the session. Please review this sheet on your own time.

1. Formula summary

1.1. Formula formulas.

(1) Unit vectors parallel to a nonzero vector v are $v/|v|$ and $-v/|v|$.
(2) Coordinates of the unit vector are the direction cosines. If $v/|v| = \langle \ell, m, n \rangle$, these are the direction cosines. If $\alpha, \beta, \gamma \in [0, \pi]$ are such that $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$, then α, β, γ are the direction angles.
(3) Parametric equation of line in $\mathbb{R}^3$: $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, $\mathbf{r_0}$ is the radial vector for a point in the line, $\mathbf{v}$ is the difference vector between two points in the line. In scalar terms, $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, where $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. (See also two-point form parametric equation).
(4) Symmetric equation of line in $\mathbb{R}^3$ not parallel to any coordinate plane (i.e., $abc \neq 0$ case):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

with same notation as for parametric equation. (See also cases of parallel to coordinate plane).
(5) Equation of plane: vector equation $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$ where $\mathbf{n}$ is a nonzero normal vector, $\mathbf{r_0}$ is a fixed point in the plane. If $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{r} = \langle x, y, z \rangle$, we get:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(6) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that f is differentiable at the point* (x_0, y_0) is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

1.2. Artistic formulas.

(1) Partial differentiation, multiplicatively separable – differentiate each piece in the corresponding variable the corresponding number of times.
(2) Partial differentiation, additively separable – pure partials, just care about function of that variable, mixed partials are zero.
(3) Integration along rectangular region, multiplicatively separable – product of integrals for function of each variable.
(4) Integration along non-rectangular region, multiplicatively separable – outer variable function can be pulled to outer integral.

2. Equations of lines and planes

2.1. Direction cosines.

(1) For a nonzero vector v, there are two unit vectors parallel to v, namely $v/|v|$ and $-v/|v|$.
(2) The direction cosines of v are the coordinates of $v/|v|$. if $v/|v| = \langle \ell, m, n \rangle$, then the direction cosines are ℓ, m, and n. We have the relation $\ell^2 + m^2 + n^2 = 1$. Further, if α, β, and γ are the angles made by v with the positive x, y, and z axes, then $\ell = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$.

2.2. Lines. Words ...

(1) A line in $\mathbb{R}^3$ has dimension 1 and codimension 2. A parametric description of a line thus requires 1 parameter. A top-down equational description requires two equations.
(2) Given a point with radial vector $\mathbf{r}_0$ and a direction vector $\mathbf{v}$ along a line, the parametric description of the line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, this is more explicitly described as $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$.
(3) Given two points with radial vectors $\mathbf{r_0}$ and $\mathbf{r_1}$, we obtain a vector equation for the line as $\mathbf{r}(t) = t\mathbf{r_1} + (1-t)\mathbf{r_0}$. If we restrict t to the interval $[0, 1]$, then we get the line segment joining the points with these radial vectors.
(4) If the line is not parallel to any of the coordinate planes, this parametric description can be converted to symmetric equations by eliminating the parameter t. With the above notation, we get:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is actually *two* equations rolled into one.
(5) If $c = 0$ and $ab \neq 0$, the line is parallel to the xy-plane, and we get the equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad z = z_0$$

Similarly for the other cases where precisely one coordinate is zero.
(6) If $a = b = 0$ and $c \neq 0$, the line is parallel to the z-axis, and we get the equations:

$$x = x_0, \qquad y = y_0$$

Actions ...

(1) To intersect two lines both given parametrically: Choose different letters for parameters, equate coordinates, solve 3 equations in 2 variables. *Note: Expected dimension of solution space is* $2 - 3 = -1$.
(2) To intersect a line given parametrically and a line given by equations: Plug in the coordinates as functions of parameters into both equations, solve. Solve 2 equations in 1 variable. *Note: Expected dimension of solution space is* $1 - 2 = -1$.
(3) To intersect two lines given by equations: Combine equations, solve 4 equations in 3 variables. *Note: Expected dimension of solution space is* $3 - 4 = -1$.

2.3. Planes. Words ...

(1) Vector equation of a plane (for the radial vector) is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ where $\mathbf{n}$ is a normal vector to the plane and $\mathbf{r_0}$ is the radial vector of any fixed point in the plane. This can be rewritten as $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$. Using $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, we get the corresponding scalar equation $ax + by + cz = ax_0 + by_0 + cz_0$. Set $d = -(ax_0 + by_0 + cz_0)$ and we get $ax + by + cz + d = 0$.
(2) The "direction" or "parallel family" of a plane is determined by its normal vector. The angle between planes is the angle between their normal vectors. Two planes are parallel if their normal vecors are parallel. And so on.

Actions ...

(1) Given three non-collinear points, we find the equation of the unique plane containing them as follows: first we find a normal vector by taking the cross product of two of the difference vectors. Then we use any of the three points to calculate the dot product with the normal vector in the above vector equation or the corresponding scalar equation.

Note that if the points are collinear, there is no unique plane through them – any plane containing their line is a plane containing them.

(2) We can compute the angle of intersection of two planes by computing the angle of intersection of their normal vectors.
(3) The line of intersection of two planes that are not parallel can be computed by simply taking the equations for *both* planes. This, however, is not a standard form for a line in $\mathbb{R}^3$. To find a standard form, either find two points by inspection and join them, or find one point by inspection and another point by taking the cross product of the normal vectors to the plane.
(4) To intersect a plane and a line, plug in parametric expressions for the coordinates arising from the line into the equation of the plane. We get one equation in the one parameter variable. In general, this is expected to have a unique solution for the parameter. Plug in the value of the parameter into the parametric expressions for the line and get the coordinates of the point of intersection.
(5) For a point with coordinates (x_1, y_1, z_1) and a plane $ax + by + cz + d = 0$, the distance of the point from the plane is given by $|ax_1 + by_1 + cz_1 + d|/\sqrt{a^2 + b^2 + c^2}$.

3. Functions of several variables

3.1. Introduction. Words ...

(1) A function of n variables is a function on a subset of $\mathbb{R}^n$. We can think of it in three ways: as a function with n real inputs, as a function with input a point in (a subset of) $\mathbb{R}^n$, and as a function with n-dimensional vector inputs. We often write the inputs with numerical subscripts, so a function f of n inputs is written as $f(x_1, x_2, \dots, x_n)$.
(2) In the case $n = 2$, we often write the inputs as x, y so we write $f(x, y)$. This may be concretely described as an expression in terms of x and y.
(3) The graph of a function $f(x, y)$ of the two variables x and y is the surface $z = f(x, y)$. The xy-plane plays the role of the independent variable plane and the z-axis is the dependent variable axis. Any such graph satisfies the "vertical" line test where vertical means parallel to the z-axis.
(4) The level curves of a function $f(x, y)$ are curves satisfying $f(x, y) = z_0$ for some fixed z_0. These are curves in the xy-plane.
(5) The level surfaces of a function $f(x, y, z)$ of three variables are the surfaces satisfying $f(x, y, z) = c$ for some fixed c.
(6) Domain convention: If nothing else is specified, the domain of a function in n variables given by an expression is defined as the largest subset of $\mathbb{R}^n$ on which that expression makes sense.
(7) We can also define vector-valued functions of many variables, e.g., a function from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$.
(8) We can do various pointwise combination operations on functions of many variables, similar to what we do for functions of one variable (both the scalar and vector cases).
(9) To compose functions, we need that the number of outputs of the inner/right function equals the number of inputs of the outer/left function.

Actions ...

(1) To find the domain, we first apply the usual conditions on denominators, things under square roots, and inputs to logarithms and inverse trigonometric functions. For functions of two variables, each such condition usually gives a region of $\mathbb{R}^2$ bounded by a line or curve.
(2) After getting a bunch of conditions that need to be satisfied, we try to find the common solution set for all of these. This involves intersecting the regions in $\mathbb{R}^2$ obtained previously.

3.2. Limits and continuity. Words ...

(1) Conceptual definition of limit $\lim_{x \to c} f(x) = L$: For any neighborhood of L, however small, there exists a neighborhood of c such that for all $x \neq c$ in that neighborhood of c, $f(x)$ is in the original neighborhood of L.
(2) Other conceptual definition of limit $\lim_{x \to c} f(x) = L$: For any open ball centered at L, however small, there exists an open ball centered at c such that for all $x \neq c$ in that open ball, $f(x)$ lies in the original open ball centered at L.

(3) $\epsilon - \delta$ definition of limit $\lim_{x \to c} f(x) = L$: For any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x = \langle x_1, x_2, \dots, x_n \rangle$ satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. The definition is the same for vector inputs and vector outputs, but we interpret subtraction as vector subtraction and we interpret $|\cdot|$ as length/norm of a vector rather than absolute value if dealing with vectors instead of scalars.
(4) On the range/image side, it is possible to break down continuity into continuity of each component, i.e., a vector-valued function is continuous if each component scalar function is continuous. This cannot be done on the domain side.
(5) We can use the above definition of limit to define a notion of continuity. The usual limit theorems and continuity theorems apply.
(6) The above definition of continuity, when applied to functions of many variables, is termed *joint continuity*. For a jointly continuous function, the restriction to any continuous curve is continuous with respect to the parameterization.
(7) We can define a function of many variables to be a continuous in a particular variable if it is continuous in that variable when we fix the values of all other variables. A function continuous in each of its variables is termed *separately continuous*. Any jointly continuous function is separately continuous, but the converse is not necessarily true.
(8) Geometrically, separate continuity means continuity along directions parallel to the coordinate axes.
(9) For homogeneous functions, we can talk of the order of a zero at the origin by converting to radial/polar coordinates and then seeing the order of the zero in terms of r.

Actions ...

(1) Polynomials and sin and cos are continuous, and things obtained by composing/combining these are continuous. Rational functions are continuous wherever the denominator does not blow up. The usual *plug in to find the limit* rule, as well as the usual list of indeterminate forms, applies.
(2) Unlike the case of functions of one variable, the strategy of canceling common factors is not sufficient to calculate all limits for rational functions. When this fails, and we need to compute a limit at the origin, try doing a polar coordinates substitution, i.e., $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. Now try to find the limit as $r \to 0$. If you get an answer independent of θ in a strong sense, then that's the limit. This method works best for homogeneous functions.
(3) For limit computations, we can use the usual chaining and stripping techniques developed for functions of one variable.

3.3. **Partial derivatives.** Words ...

(1) The partial derivative of a function of many variables with respect to any one variable is the derivative with respect to that variable, keeping others constant. It can be written as a limit of a difference quotient, using variable letter subscript (such as $f_x(x, y)$), numerical subscript based on input position (such as $f_2(x_1, x_2, x_3)$), Leibniz notation (such as $\partial/\partial x$).
(2) In the separate continuity-joint continuity paradigm, partial derivatives correspond to the "separate" side. The corresponding "joint" side notion requires linear algebra and we will therefore defer it.
(3) The expression for the partial derivative of a function of many variables with respect to any one of them involves all the variables, not just the one being differentiated against (the exception is additively separable functions). In particular, the *value* of the partial derivative (as a number) depends on the values of all the inputs.
(4) The procedure for partial derivatives differs from the procedure used for implicit differentiation: in partial derivatives, we assume that the other variable is independent and constant, while in implicit differentiation, we treat the other variable as an unknown (implicit) function of the variable.
(5) We can combine partial derivatives and implicit differentiation, for instance, $G(x, y, z) = 0$ may be a description of z as an implicit function of x and y, and we can compute $\partial z/\partial x$ by implicit differentiation, differentiate G, treat z as an implicit function of x and treat y as a constant.
(6) By iterating partial differentiation, we can define higher order partial derivatives. For instance f_{xx} is the derivative of f_x with respect to x. For a function of two variables x and y, we have four second order partials: f_{xx}, f_{yy}, f_{xy} and f_{yx}.
(7) Clairaut's theorem states that if f is defined in an open disk surrounding a point, and both mixed partials f_{xy} and f_{yx} are jointly continuous in the open disk, then $f_{xy} = f_{yx}$ at the point.

(8) We can take higher order partial derivatives. By iterated application of Clairaut's theorem, we can conclude that under suitable continuity assumptions, the mixed partials having the same number of differentiations with respect to each variable are equal in value.
(9) We can consider a partial differential equation for functions of many variables. This is an equation involving the function and its partial derivatives (first or higher order) all at one point. A solution is a function of many variables that, when plugged in, satisfies the partial differential equation.
(10) Unlike the case of ordinary differential equations, the solution spaces to partial differential equations are huge, usually infinite-dimensional, and there is often no neat description of the general solution.

Pictures ...

(1) The partial derivatives can be interpreted as slopes of tangent lines to graphs of functions of the one variable being differentiated with respect to, once we fix the value of the other variable.

Actions ...

(1) To compute the first partials, differentiate with respect to the relevant variable, treating other variables as constants.
(2) Implicit differentiation for first partial of implicit function of two variables, e.g., z as a function of x and y given via $G(x, y, z) = 0$.
(3) In cases where differentiation formulas do not apply directly, use the limit of difference quotient idea.
(4) To calculate partial derivative at a point, it may be helpful to first fix the values of the other coordinates and then differentiate the function of one variable rather than trying to compute the general expression for the derivative using partial differentiation and then plugging in values. On the other hand, it might not.
(5) Two cases of particular note for computing partial derivatives are the cases of additively and multiplicatively separable functions.
(6) To find whether a function satisfies a partial differential equation, plug it in and check. Don't try to find a general solution to the partial differential equation.

Econ-speak ...

(1) Partial derivatives = marginal analysis. Positive = increasing, negative = decreasing
(2) Second partial derivatives = nature of returns to scale. Positive = increasing returns (concave up), zero = constant returns (linear), negative = decreasing returns (concave down)
(3) Mixed partial derivatives = interaction analysis; positive mixed partial derivative means complementary, negative mixed partial derivative means substitution
(4) The signs of the first partials are invariant under monotone transformations, not true for signs of second partials, pure or mixed.
(5) Examples of quantity demanded, production functions.
(6) Cobb-Douglas production functions (see section of lecture notes and corresponding discussion in the book)

3.4. **Tangent planes and linear approximations.** Words ...

(1) For a d-dimensional subset of $\mathbb{R}^n$, it (occasionally) makes sense to talk of the tangent space and the normal space at a point. The tangent space is a linear/affine d-dimensional space and the normal space is a linear/affine $(n - d)$-dimensional space. Both pass through the point and are mutually orthogonal.
(2) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that f is differentiable at the point* (x_0, y_0) is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

(3) It may be the case that a function f of two variables is not differentiable at a point in its domain but the partial derivatives exist. In this case, although the *above formula makes sense as a formula*, the plane it gives *is not the tangent plane* – in fact, *no tangent plane exists*. Similarly, *no linearization exists*, and the linear function given by the above formula *is not a close approximation to the function near the point*.

3.5. **Chain rule.** Words ...

(1) The general formulation of chain rule: consider a function with m inputs and n outputs, and another function with n inputs and p outputs. Composing these, we get a function with m inputs and p outputs. The m original inputs are termed *independent variables*, the n in-between things are termed *intermediate variables*, and the p final outputs are termed *dependent variables*.

For a given triple of independent variable t, intermediate variable x, and dependent variable u, the partial derivative of u with respect to t via x is defined as:

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

The chain rule says that the partial derivative of u with respect to t is the sum, over all intermediate variables, over the partial derivatives via each intermediate variable.

(2) The $1 \to 2 \to 1$ and $2 \to 2 \to 1$ versions (see the lecture notes or the book).

(3) There is also a tree interpretation of this, where we make pathways based on the directions/paths of dependence. This is discussed in the book, not the lecture notes.

(4) The product rule for scalar functions can be proved using the chain rule. Other variants of the product rule can be proved using generalized formulations of the chain rule, which are beyond our current scope.

(5) Implicit differentiation can be understood in terms of the chain rule and partial derivatives.

4. Double and iterated integrals

Words ...

(1) The double integral of a function f of two variables, over a domain D in $\mathbb{R}^2$, is denoted $\int\int_D f(x,y)\,dA$ and measures an infinite analogue of the sum of f-values at all points in D.

(2) Fubini's theorem for rectangles states that if F is a function of two variables on a rectangle $R = [a,b] \times [p,q]$, such that F is continuous except possibly at the union of finitely many smooth curves, then the integral equals either of these iterated integrals:

$$\int\int_R F(x,y)\,dA = \int_a^b \int_p^q F(x,y)\,dy\,dx = \int_p^q \int_a^b F(x,y)\,dx\,dy$$

(3) For a function f defined on a closed connected bounded domain D with a smooth boundary, we can make sense of $\int\int_D f(x,y)\,dA$ as being $\int\int_R F(x,y)\,dA$ where R is a rectangular region containing D and F is a function that equals f on D and is 0 on the rest of R.

(4) Suppose D is a Type I region, i.e., its intersection with every vertical line is either empty or a point or a line segment. Then, we can describe D as $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(5) Suppose D is a Type II region, i.e., its intersection with every horizontal line is either empty or a point or a line segment. Then, we can describe D as $p \le y \le q$, $g_1(y) \le x \le g_2(y)$, where g_1 and g_2 are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_p^q \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

(6) The double integral of $f + g$ over D is the sum of the double integral of f over D and the double integral of g over D. Similarly, scalars can be pulled out of double integrals.

(7) The integral of the function 1 over a domain is the area of the domain.
(8) If $f(x, y) \geq 0$ on a domain D, the integral of f over D is also ≥ 0.
(9) If $f(x, y) \geq g(x, y)$ on a domain D, the integral of f over D is $\geq$ the integral of g over D.
(10) If $m \leq f(x, y) \leq M$ over a domain D, then $\int\int_D f(x, y)\, dA$ is betweem mA and MA where A is the area of D.
(11) If $f(x, y)$ is odd in x and the domain of integration is symmetric about the y-axis, the integral is zero. If $f(x, y)$ is odd in y and the domain is symmetric about the x-axis, the integral is zero.

Actions ...

(1) To compute a double integral, compute it as an iterated integral. For a rectangle, we can choose either order of integration, as long as the integration is feasible. For other types of regions, we need to first determine whether the region is Type I or Type II, and break it up into pieces of those types.
(2) For a multiplicatively separable function over a rectangular region (or for a sum of such multiplicatively separable functions), things are particularly easy.
(3) Sometimes, an integral cannot be computed using a particular order of integration – we might get stuck on the inner or the outer stage. However, it may be computable using the other order of integration.
(4) We can often use symmetry-based techniques to argue that certain parts of the integrand integrate to zero.
(5) Even in cases where the integral cannot be computed, we can bound it between limits using maximum or minimum values of function and/or using bigger or smaller regions on which the integral can be computed.

REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.

The document does not include material that was part of the midterm 1 syllabus. Very little of that material will appear directly in midterm 2; however, you should have reasonable familiarity with the material.

1. Formula summary

1.1. Formula formulas.

(1) Unit vectors parallel to a nonzero vector v are $v/|v|$ and $-v/|v|$.

(2) Coordinates of the unit vector are the direction cosines. If $v/|v| = \langle \ell, m, n \rangle$, these are the direction cosines. If $\alpha, \beta, \gamma \in [0, \pi]$ are such that $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$, then α, β, γ are the direction angles.

(3) Parametric equation of line in $\mathbb{R}^3$: $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, $\mathbf{r_0}$ is the radial vector for a point in the line, $\mathbf{v}$ is the difference vector between two points in the line. In scalar terms, $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, where $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. (See also two-point form parametric equation).

(4) Symmetric equation of line in $\mathbb{R}^3$ not parallel to any coordinate plane (i.e., $abc \neq 0$ case):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

with same notation as for parametric equation. (See also cases of parallel to coordinate plane).

(5) Equation of plane: vector equation $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$ where $\mathbf{n}$ is a nonzero normal vector, $\mathbf{r_0}$ is a fixed point in the plane. If $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{r} = \langle x, y, z \rangle$, we get:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(6) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that f is differentiable at the point* (x_0, y_0) is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

1.2. Artistic formulas.

(1) Partial differentiation, multiplicatively separable – differentiate each piece in the corresponding variable the corresponding number of times.

(2) Partial differentiation, additively separable – pure partials, just care about function of that variable, mixed partials are zero.

(3) Integration along rectangular region, multiplicatively separable – product of integrals for function of each variable.

(4) Integration along non-rectangular region, multiplicatively separable – outer variable function can be pulled to outer integral.

2. Equations of lines and planes

2.1. Direction cosines. Error-spotting exercises ...

(1) If α, β, γ are the direction angles of the vector $\langle a, b, c\rangle$ then the direction angles of the vector $\langle -a, b, c\rangle$ are $-\alpha, \beta, \gamma$.

2.2. Lines. Error-spotting exercises ...

(1) *Counting issues*: They say that to describe a line in $\mathbb{R}^3$, we need $3 - 1 = 2$ equations in a top down description. However, the symmetric equation of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

is a *single* equation that describes the line.

(2) *Unparalleled lines*: By definition, if two lines do not intersect, they are parallel. Thus, the x-axis is parallel to the line $x = 1 + u$, $y = 2 + u$, $z = 3 + u$.

(3) *And and/or or*: Consider the planes $x + y + z = 0$ and $2x + 3y + 4z = 0$. Their intersection is a line given by the equation $(x + y + z)(2x + 3y + 4z) = 0$.

2.3. Planes. No error-spotting exercises.

3. Functions of several variables

3.1. Introduction. Error-spotting exercises ...

(1) *One-point curves*: Consider the function $f(x, y) := (x - 1)^2 + (y + 1)^2 - 3$. The level "curve" for the value -3 is the single point $(1, -1)$. This is a point, not a curve at all. So, the claim that level curves are one-dimensional is wrong, and the term "curve" itself is a misnomer.

(2) *Count issues again*: Consider the function $f(x, y) := x^2 - y^2$. The level "curve" for the value 1 is a union of two curves, one on the positive x-axis side and the other on the negative x-axis side. The level curve thus isn't a curve at all, it is a union of multiple curves.

(3) *A new unparalleled level*: Consider the function $f(x, y, z) := ax + by + cz$ of three variables. The level curves of this function are the lines parallel to the vector $\langle a, b, c\rangle$.

3.2. Limits and continuity. Error-spotting exercises ...

(1) *Zero ain't infinity*: Consider the limit $\lim_{(x,y)\to(0,0)}(x^4 + y^4)/(x^2 + y^2)^2$. We see that the numerator and denominator are both homogeneous polynomials of degree four, and so the limit of the quotient is the quotient of the leading coefficients, which are both 1. So the limit of the quotient is 1. We can verify this by noting that the limit for approach along the x-axis as well as the y-axis are both equal to 1.

(2) *Curvophobia* or *straightonormativity*: To verify that the limit of a function at the origin equals a particular value, we need to compute the limit along the x-axis, along the y-axis, and along the line $y = mx$ for m fixed but arbitrary. If all the three answers are a constant independent of m, then that is the limit.

3.3. Partial derivatives. Error-spotting exercises ...

(1) *Once it's fixed, it stays fixed*: Here is a simple logical explanation as to why, for any function f of two variables x and y, the second-order mixed partial derivative f_{xy} must be zero. Recall that f_x is the first-order partial derivative of x holding y constant. In other words, we fix the value of y and are allowed to vary only x, and measure the rate of change of f subject to that restriciton.

The second-order mixed partial derivative $f_{xy} = (f_x)_y$ is obtained by taking the first-order partial f_x and figuring out how it changes with respect to y holding x constant. But note from the preceding paragraph that y needs to be held constant in order to make sense of f_x. Thus, for computing f_{xy}, *both* x and y need to be held constant. Since both coordinates are being held constant, there is no scope for f to change, hence f_{xy} is zero.

(2) *Mixed up partials*: To differentiate a multiplicatively separable function, we differentiate the function of x with respect to x the required number of times and the function of y with respect to y the required number of times, and then multiply. Thus, if $f(x, y) := \sin(x^2 \sin y)$, we get $f_{xy}(x, y) = \cos(2x \cos y)$.

(3) *Slaving for joy*: My happiness is proportional to the logarithm of my income; every time my income doubles, my happiness goes up 0.3 units. I have observed that my income obeys increasing returns to effort, and empirically I find that my total income is proportional to the $(4/3)^{th}$ power of the number of hours I work. Therefore, my happiness also obeys increasing returns to effort.

(4) *Futility personified*: Consider a production function $f(L, K) = (\min\{L, K\})^2$. We know that if $L > K$, then $f_L(L, K) = 0$. This means that reducing the value of L has no impact on the output. But if that's true, then L can be reduced to 0, and output would be unaffected. Similarly, K can be reduced to zero, and output would be unaffected. But that's nonsense.

(5) *Mixed up partials – something doesn't add up*: Suppose $F(x, y) := f(x) + g(y)$. Then $F_{xy}(x, y) = f'(x) + g'(y)$.

(6) *Mixed up partials – shut up and multiply*: Suppose $F(x, y) := f(x)g(y)$. We know that the mixed partial $F_{xy}(x, y) = f'(x)g'(y)$. But this is in contradiction with the product rule, which states that the derivative of the product is *not* the product of the derivatives. Shouldn't the answer be $f'(x)g(y) + f(x)g'(y)$?

(7) *Quid est quod custodire cupis constans*: Let f be a function of two variables. Define $g(x, y) := f(x, x + y)$. Then, clearly, $g(2, 3) = f(2, 5)$. Hence also, we have $g_1(2, 3) = f_1(2, 5)$ (where the subscript $_1$ denotes partial differentiation with respect to the first input keeping the second input constant).

(8) *Value depends only on the variable you differentiate with respect to*: A manager wants to figure out the marginal product of labor. He has an expression for the production function in terms of labor and capital. In order to calculate the marginal product of labor, he simply needs to know the current labor expenditure to plug into the formula. Information on the current capital expenditures is redundant.

3.4. **Tangent planes and linear approximations.** Error-spotting exercises...

(1) *The rational elite and the irrational hoi polloi are on different planes*: Consider the function:

$$f(x, y) := \begin{cases} 1, & x \text{ rational or } y \text{ rational} \\ 0, & x \text{ and } y \text{ both irrational} \end{cases}$$

Suppose x_0, y_0 are rational numbers, so (x_0, y_0) is a point both of whose coordinates are rational. Then, we have $f(x_0, y_0) = 1$ and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Thus, we get that the tangent plane to the graph of f through the point $(x_0, y_0, f(x_0, y_0))$ is:

$$z = 1 + 0(x - x_0) + 0(x - x_0)$$

So we get that the equation is:

$$z = 1$$

(2) *So near, yet so far, or, missing the forest for the trees, or, going off on tangents*: The tangent line to $(0, 0)$ for the curve $y = \sin x$ in the xy-plane is the $y = x$ line. This is therefore the best straight line approximation to the curve. Thus, for instance, a reasonable approximation for sin(1000) is 1000.

3.5. **Chain rule.** Error-spotting exercises ...

(1) *x, tx, it's all the same*: Suppose $f(x, y)$ is a function of two variables. Then, we have:

$$f_x(tx, ty) = \frac{\partial}{\partial x}[f(tx, ty)]$$

Note: The underlying issue here affected some people's attempts at advanced HW 6 question 5.

(2) *Functions are born free, yet everywhere they are in chains*: Suppose f and g are functions of one variable. Then, we know that:

$$(f \circ g)'(t) = f'(g(t))g'(t)$$

by the chain rule. Differentiating both sides with respect to t again, and using the product rule, we get:

$$(f \circ g)''(t) = \frac{d}{dt}\left[f'(g(t))g'(t) + f'(g(t))g''(t)\right] = f''(g(t))g'(t) + f'(g(t))g''(t)$$

(3) *On the other hand*: Suppose $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$. Then, we have:

$$\frac{\partial f_x}{\partial t} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t}$$

4. Double and iterated integrals

Error-spotting exercises ...

(1) *Fundamental theorem of miscalculus*: Suppose we are integrating a continuous function $g(x, y)$ of two variables over a rectangular region $[a, b] \times [p, q]$. Then, if $G_{xy} = g$, the value of the integral is $G(b, q) - G(a, p)$. This is just like the fundamental theorem of calculus.

(2) *Separation of abscissa and ordinate*: Suppose $F(x, y) := f(x)g(y)$. We want to integrate F on the region $0 \le x \le 5$, $0 \le y \le x^2$. Since F is multiplicatively separable, we don't need to compute this as an iterated integral, and instead, we can compute it as a product:

$$\left(\int_0^5 f(x)\,dx\right)\left(\int_0^{x^2} g(y)\,dy\right)$$

(3) *Dissolving the bonds of addition*: Suppose $F(x, y) := f(x) + g(y)$, and we need to integrate F on $[a, b] \times [p, q]$. The integral is:

$$\int_a^b f(x)\,dx + \int_p^q g(y)\,dy$$

(4) *Argument from personal incredulity*: The double integral for a function F on a domain D exists only if D is a Type I or Type II region.

(5) *Another argument from personal incredulity*: e^{-x^2} is not an integrable function of one variable, i.e., it does not have an antiderivative.

(6) *Straightonormativity yet again*: If $F(x, y) = f(x)g(y)$ and we have antiderivatives available for f and g, we can use these to successfully integrate F over any closed bounded convex region.

(7) *O mirror to my soul, don't be orthogonal!*: If f is a function and D is a closed convex region centered at the origin symmetric about the x-axis, such that f is odd in x for each fixed value of y, then the integral of f over D is zero.

(8) *Positivity bias yet again, or tunnel vision*: The integral:

$$\int_2^3 \frac{dx}{x^2 + y}$$

gives us:

$$\left[\frac{1}{\sqrt{y}} \arctan\left(\frac{x}{\sqrt{y}}\right)\right]_{x=2}^{x=3}$$

This simplifies to:

$$\frac{1}{\sqrt{y}}\left[\arctan\left(\frac{3}{\sqrt{y}}\right) - \arctan\left(\frac{2}{\sqrt{y}}\right)\right]$$

(9) Consider the following integral on the region $D = [0, a] \times [0, a]$ for the function $f(x, y) := g[(\max\{x, y\})^2]$. We get:

$$\int\int_D f(x, y)\,dA = \max\{\int_0^a g(x^2)\,dx, \int_0^a g(y^2)\,dy\}$$

Since both integrals are the same, this becomes:

$$\int\int_D f(x,y)\,dA = \int_0^a g(x^2)\,dx$$

If G is an antiderivative for g, this becomes:

$$[G(x^2)]_0^a$$

This simplifies to $G(a^2) - G(0)$.

REVIEW SHEET FOR FINAL: BASIC

MATH 195, SECTION 59 (VIPUL NAIK)

The document does not include material that was part of the midterm 1 and midterm 2 review sessions. Please also bring copies of these review sheets to the review session on Monday.

You are expected to review this on your own time. We will concentrate on the *advanced* review sheet during problem session.

1. Directional derivatives and gradient vectors

Words ...

(1) The directional derivative of a scalar function f of two variables along a *unit* vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ at a point (x_0, y_0) is defined as the following limit of difference quotient, if the limit exists:

$$\lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

(2) The directional derivative of a differentiable scalar function f of two variables along a *unit* vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ at a point (x_0, y_0) is $D_{\mathbf{u}}(f) = af_x(x_0, y_0) + bf_y(x_0, y_0)$.

(3) The gradient vector for a *differentiable* scalar function f of two variables at a point (x_0, y_0) is $\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$.

(4) The directional derivative of f is the dot product of the gradient vector of ∇f and the unit vector $\mathbf{u}$.

(5) Suppose ∇f is nonzero. Then, if $\mathbf{u}$ makes an angle θ with ∇f, then $D_{\mathbf{u}}(f)$ is $|\nabla_f| \cos \theta$. The directional derivative is maximum in the direction of the gradient vector, zero in directions orthogonal to the gradient vector, and minimum in the direction opposite to the gradient vector.

(6) The level curves are orthogonal to the gradient vector.

(7) Similar formulas for gradient vector and directional derivative work in three dimensions.

(8) The level surfaces are orthogonal to the gradient vector for a function of three variables.

(9) For a surface given by $F(x, y, z) = 0$, if (x_0, y_0, z_0) is a point on the surface, and $F_x(x_0, y_0, z_0)$, $F_y(x_0, y_0, z_0)$, and $F_z(x_0, y_0, z_0)$ all exist and are nonzero, then the normal line is given by:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

The tangent plane is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

2. Max-min values

Words ...

(1) For a directional local minimum, the directional derivative (in the outward direction from the point) is greater than or equal to zero. For a directional local maximum, the directional derivative (in the outward direction from the point) is less than or equal to zero.

Note that even for *strict* directional local maximum or minimum, the possibility of the directional derivative being zero cannot be ruled out.

(2) If a point is a point of directional local minimum from two opposite directions (i.e., it is a local minimum along a line through the point, from both directions on the line) then the directional derivative along the line, if it exists, must equal zero.

(3) If a function of two variables is differentiable at a point of local minimum or local maximum, then the directional derivative of the function is zero at the point in every direction. Equivalently, the gradient vector of the function at the point is the zero vector. Equivalently, both the first partial derivatives at the point are zero.

Points where the gradient vector is zero are termed *critical points*.

(4) If the directional derivatives along some directions are positive and the directional derivatives along other directions are negative, the point is likely to be a *saddle point*. A saddle point is a point for which the tangent plane to the surface that's the graph of the function slides through the graph, i.e., it is not completely on one side.

(5) For a function f of two variables with continuous second partials, and a critical point (a, b) in the domain (so $f_x(a, b) = f_y(a, b) = 0$) we compute the Hessian determinant:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, the function has a local *minimum* at the point (a, b). If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, the function has a local *maximum* at the point (a, b). If $D(a, b) < 0$, we get a saddle point at the point. If $D(a, b) = 0$, the situation is inconclusive, i.e., the test is indecisive.

(6) For a closed bounded subset of $\mathbb{R}^n$ (and specifically $\mathbb{R}^2$) any continuous function with domain that subset attains its absolute maximum and minimum values. These values are attained either at interior points (in which case they are local extreme values and must be attained at critical points) or at boundary points.

(7) *Relation with level curves*: *Typically*, local extreme values correspond to isolated single point level curves. However, this is not always the case, and there are some counterexamples. To be more precise, any *isolated* or *strict* local extreme value corresponds to a (locally) single point level curve.

Actions ...

(1) Strategy for finding local extreme values: First, find all the critical points by solving $f_x(a, b) = 0$ and $f_y(a, b) = 0$ as a pair of simultaneous equations. Next, use the second derivative test for each critical point, and if feasible, try to figure out if this is a point of local maximum, or local minimum, or a saddle point.

(2) To find absolute extreme values of a function on a closed bounded subset of $\mathbb{R}^2$, first find critical points, then find critical points for a parameterization of the boundary, and then compute values at all of these and see which is largest and smallest. *If the list of critical points is finite, and we need to find absolute maximum and minimum, it is not necessary to do the second derivative test to figure out which points give local maximum, local minimum, or neither, we just need to evaluate at all points and find the maximum/minimum.*

(3) When the domain of the function is bounded but not closed, we must consider the possibility of extreme values occurring as we approach boundary points not in the domain. If the domain is not bounded, we must consider directions of approach to infinity.

3. Lagrange multipliers

Words ...

(1) Two of the reasons why the derivative of a function may be zero: the function is constant around the point, or the function has a local extreme value at the point.

Version for many variables: two of the reasons why the gradient vector of a function of many variables may be zero: the function is constant around the point, or the function has a local extreme value at the point.

Version for function restricted to a subset smooth around a point: two of the reasons why the gradient vector may be *orthogonal* to the subset at the point: the function is constant on the subset around the point, or the function has a local extreme value (relative to the subset) at the point.

(2) For a function f defined on a subset smooth around a point (i.e., with a well defined tangent and normal space), if f has a local extreme value at the point when restricted to the subset, then ∇f lives in the normal direction to the subset (this includes the possibility of it being zero).

(3) For a codimension one subset of $\mathbb{R}^n$ defined by a condition $g(x_1, x_2, \dots, x_n) = k$, if a point $(a_1, a_2, \dots, a_n)$ gives a local extreme value for a function f of n variables, and if ∇g is well defined and nonzero at the point, then there exists a real number λ such that $\nabla f(a_1, a_2, \dots, a_n) = \lambda \nabla g(a_1, a_2, \dots, a_n)$. Note that λ may be zero.

(4) Suppose a codimension r subset of $\mathbb{R}^n$ is given by r independent constraints $g_1(x_1, x_2, \dots, x_n) = k_1$, $g_2(x_1, x_2, \dots, x_n) = k_2$, and so on till $g_r(x_1, x_2, \dots, x_n) = k_r$. Suppose ∇g_i are nonzero for all i at a point $(a_1, a_2, \dots, a_n)$ of local extreme value for a function f relative to this subset. Suppose further that all the ∇g_i are linearly independent. Then $\nabla f(a_1, a_2, \dots, a_n)$ is a linear combination of the vectors $\nabla g_1(a_1, a_2, \dots, a_n)$, $\nabla g_2(a_1, a_2, \dots, a_n)$, $\dots$, $\nabla g_r(a_1, a_2, \dots, a_n)$. In other words, there exist real numbers λ_1, λ_2, $\dots$, λ_r such that:

$$\nabla f(a_1, a_2, \dots, a_n) = \lambda_1 \nabla g_1(a_1, a_2, \dots, a_n) + \lambda_2 \nabla g_2(a_1, a_2, \dots, a_n) + \dots + \lambda_r \nabla g_r(a_1, a_2, \dots, a_n)$$

(5) The Lagrange condition may be violated at points of local extremum where ∇g is zero, or more generally, where the ∇g_i fail to be linearly independent. This may occur either because the tangent and normal space are not well defined or because the functions fail to capture it well.

Actions ...

(1) Suppose we want to maximize and minimize f on the set $g(x_1, x_2, \dots, x_n) = k$. Assume $\nabla g(x_1, x_2, \dots, x_n)$ is defined everywhere on the set and never zero. Suppose ∇f is also defined. Then, all local maxima and local minima are attained at points where $\nabla f = \lambda \nabla g$ for some real number λ. To find these, we solve the system of $n + 1$ equations in the $n + 1$ variables x_1, x_2, $\dots$, x_n, namely the n scalar equations from the Lagrange condition and the equation $g(x_1, x_2, \dots, x_n) = k$.

    To find the actual extreme values, once we've collected all candidate points from the above procedure, we evaluate the function at all these and find the largest and smallest value to find the absolute maximum and minimum.

(2) If there are points in the domain where ∇g takes the value 0, these may also be candidates for local extreme values, and the function should additionally be evaluated at these as well to find the absolute maximum and minimum.

(3) A similar procedure works for a subset given by r constraints. In this case, we have the equation:

$$\nabla f(a_1, a_2, \dots, a_n) = \lambda_1 \nabla g_1(a_1, a_2, \dots, a_n) + \lambda_2 \nabla g_2(a_1, a_2, \dots, a_n) + \dots + \lambda_r \nabla g_r(a_1, a_2, \dots, a_n)$$

    as well as the r equations $g_1(x_1, x_2, \dots, x_n) = k_1$, $g_2(x_1, x_2, \dots, x_n) = k_2$, and so on. In total, we have $n + r$ equations in $n + r$ variables: the x_1, x_2, $\dots$, x_n and the λ_1, λ_2, $\dots$, λ_r.

4. Max-min values: examples

(1) *Additively separable, critical points*: For an additively separable function $F(x, y) := f(x) + g(y)$, the critical points of F are the points whose x-coordinate gives a critical point for f and y-coordinate gives a critical point for g.

(2) *Additively separable, local extreme values*: The local maxima occur at points whose x-coordinate gives a local maximum for f and y-coordinates gives a local maximum for g. Similarly for local minima. If one coordinate gives a local maximum and the other coordinate gives a local minimum, we get a saddle point.

(3) *Additively separable, absolute extreme values*: If the domain is a rectangular region, rectangular strip, or the whole plane, then the absolute maximum occurs at the point for which each coordinate gives the absolute maximum for that coordinate, and analogously for absolute minimum. This does *not* work for non-rectangular regions in general.

(4) *Multiplicatively separable, critical points*: For a multiplicatively separable function $F(x, y) := f(x)g(y)$ with f, g, differentiable, there are four kinds of critical points (x_0, y_0): (1) $f'(x_0) = g'(y_0) = 0$, (2) $f(x_0) = f'(x_0) = 0$, (3) $g(y_0) = g'(y_0) = 0$, (4) $f(x_0) = g(y_0) = 0$.

(5) *Multiplicatively separable, local extreme values*: At a critical point of Type (1), the nature of local extreme value for F depends on the signs of f and g *and* on the nature of local extreme values for

each. See the table. Critical points of Type (4) alone do not give local extreme values. The situation with critical points of Types (2) and (3) is more ambiguous and too complicated for discussion.

(6) *Multiplicatively separable, absolute extreme values*: Often, these don't exist, if one function takes arbitrarily large magnitude values and the other one takes nonzero values (details based on sign). If both functions are everywhere positive, and we are on a rectangular region, then the absolute maximum/minimum for the product occur at points whose coordinates give respective absolute maximum/minimum for f and g. (See notes)

(7) For a continuous quasiconvex function on a convex domain, the maximum must occur at one of the extreme points, in particular on the boundary. If the function is strictly quasiconvex, the maximum can occur only at a boundary point.

(8) For a continuous quasiconvex function on a convex domain, the minimum must occur on a convex subset. If the function is strictly quasiconvex, it must occur at a unique point.

(9) Linear functions are quasiconvex but not strictly so. The negative of a linear function is also quasiconvex. The maximum and minimum for linear functions on convex domains must occur at extreme points.

(10) To find maxima/minima on the boundary, we can use the method of Lagrange multipliers.

See also: tables, discussion for linear, quadratic, and homogeneous functions (hard to summarize). Below is a copy of the table for the multiplicatively separable case.

The setup here is that we have a function $F(x, y) := f(x)g(y)$ and a point (x_0, y_0) in the domain such x_0 is a critical point for f and y_0 is a critical point for g. Visit the lecture notes for more detailed context.

$f(x_0)$ sign	$g(y_0)$ sign	$f(x_0)$ (local max/min)	$g(y_0)$ (local max/min)	$F(x_0, y_0)$ (local max/min/saddle)
positive	positive	local max	local max	local max
positive	positive	local max	local min	saddle point
positive	positive	local min	local max	saddle point
positive	positive	local min	local min	local min
positive	negative	local max	local max	saddle point
positive	negative	local max	local min	local min
positive	negative	local min	local max	local max
positive	negative	local min	local min	saddle point
negative	positive	local max	local max	saddle point
negative	positive	local max	local min	local max
negative	positive	local min	local max	local min
negative	positive	local min	local min	saddle point
negative	negative	local max	local max	local min
negative	negative	local max	local min	saddle point
negative	negative	local min	local max	saddle point
negative	negative	local min	local min	local max

REVIEW SHEET FOR FINAL: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to all review sessions.

1. Directional derivatives and gradient vectors

Error-spotting exercises ...

(1) *Partials don't tell the whole story*: Consider the function $f(x, y) := (xy)^{1/5}$. We note that f takes the value 0 identically both on the x-axis and the y-axis, thus, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Hence, the gradient of f at $(0, 0)$ is the zero vector.

(2) *Directional derivatives don't tell the whole story either*: Let

$$f(x,y) := \begin{cases} 0 & \text{if } y \le 0 \text{ or } y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

We note that on any line approaching $(0, 0)$, f becomes constant at 0 close enough to $(0, 0)$. Hence, the directional derivative of f in every direction is 0. Thus, the gradient vector of f is 0.

(3) *Orthogonal to nothing*: Consider the function $f(x, y) := \sin(xy)$ at the point $(\pi, 1/2)$. At this point, we have $f_x(x, y) = y\cos(xy) = (1/2)\cos(\pi/2) = 0$. Thus, the gradient of f is in the y-direction, so the tangent line to the level curve of f for this function is parallel to the x-axis.

(4) *Zero gradient, level curve not smooth?*: Consider the function $f(x, y) := (x-y)^3$. At the point $(1, 1)$, both $f_x(x, y)$ and $f_y(x, y)$ take the value 0, so the gradient vector is 0. Thus, the level curve of f passing through the point $(1, 1)$ does not have a well defined normal direction at $(1, 1)$.

(5) *Misquare*: The maximum magnitude of directional derivative for a function f with a nonzero gradient at a point occurs in the direction of the gradient vector ∇f, and its value is $\nabla f \cdot \nabla f = |\nabla f|^2$.

(6) *False addition*: The directional derivative along the direction of the vector $a + b$ is the sum of the directional derivatives along the direction of a and the direction of b.

2. Max-min values

Error-spotting exercises ...

(1) *Separate versus joint*: Suppose F is a function of two variables denoted x and y, and (x_0, y_0) is a point in the interior of the domain of F. If F has a local maximum at (x_0, y_0) with respect to both the x- and the y-directions, then F must have a local maximum.

(2) *Saddled with wrong ideas*: Suppose F is a function of two variables denoted x and y, and (x_0, y_0) is a point in the interior of the domain of F. If F has a saddle point at (x_0, y_0), then that means it must have a local maximum from one of the x- and y-directions and a local minimum from the other.

(3) *Hessian as second derivative*: The second derivative test for a function f of two variables says the following: define the Hessian determinant $D(a, b)$ at a point as $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. If $D(a, b) > 0$, this means that f has a local minimum at (a, b). If $D(a, b) < 0$, this means that f has a local maximum at (a, b). If $D(a, b) = 0$, the second derivative test is inconclusive.

3. Lagrange multipliers

Error-spotting exercises ...

(1) *Local maximum, minimum*: To determine whether a point on a level curve of g satisfying the Lagrange condition on f (i.e., $\nabla f = \lambda \nabla g$) gives a local maximum or a local minimum for f, we simply need to check whether $\lambda > 0$ or $\lambda < 0$. If $\lambda > 0$, we have a local minimum, and if $\lambda < 0$, we have a local maximum.

(2) *Hessian confusion*: Consider a function f of two variables. Let D denote the Hessian determinant. To maximize f along the constraint curve $g(x, y) = k$, we first find points on the constraint curve where $\nabla f = \lambda \nabla g$ for some suitable choice of λ, i.e., points satisfying the Lagrange condition. At any such point, if $D < 0$, then we have neither a local maximum nor a local minimum with respect to the curve. If $D > 0$ and $f_{xx} > 0$, then we have a local minimum with respect to the curve. If $D > 0$ and $f_{xx} < 0$, then we have a local maximum with respect to the curve.

4. Max-min values: examples

Error-spotting exercises ...

(1) *Absolute maximum folly, thinking in the box*: Suppose $F(x, y) := f(x) + g(y)$ and we want to maximize F over the domain $|x| + |y| \le 1$. We note that in the domain $|x| + |y| \le 1$, we have the constraints $-1 \le x \le 1$ and $-1 \le y \le 1$. Thus, to find the absolute maximum for F, we do the following: maximize f on the interval $[-1, 1]$ (say at x_0 with value a), maximize g on the interval $[-1, 1]$ (say at y_0 with value b), and then take the combined point (x_0, y_0) and get value $a + b$.

(2) *Critical missed types*: Suppose $F(x, y) := f(x)g(y)$. Then, (x_0, y_0) gives a critical point for F if and only if x_0 gives a critical point for f and y_0 gives a critical point for g.

(3) *Ignoring the signs of a pessimistic world*: Suppose $F(x, y) := f(x)g(y)$. If f attains a local maximum value at x_0 and g attains a local maximum value at y_0, then F attains a local maximum value at (x_0, y_0).

(4) *Maximum, minimum*: Suppose f is a continuous quasiconvex function defined on the set $|x|+|y| \le 1$. We know by the definition of quasiconvex that f must attain both its absolute maximum and its absolute minimum at one of its extreme points, i.e., at one of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

(5) *Pointy circles*: Suppose f is a strictly convex function defined on the circular disk $x^2 + y^2 \le 1$. Then, f can attain its absolute maximum only at one of the four extreme points: $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.