TAKE-HOME CLASS QUIZ: DUE WEDNESDAY JANUARY 9: PARAMETRIC STUFF

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

THIS IS A TAKE-HOME CLASS QUIZ, BUT I WILL GIVE YOU ABOUT 5 MINUTES TO REVIEW YOUR ANSWERS IN CLASS AND DISCUSS WITH OTHER STUDENTS.

YOU ARE ALLOWED TO DISCUSS ONLY QUESTIONS THAT BEGIN WITH A (*) OR (**). PLEASE ATTEMPT ALL OTHER QUESTIONS BY YOURSELF. EVEN FOR THE QUESTIONS YOU DISCUSS, PLEASE FINALLY ENTER ONLY THE ANSWER OPTION YOU ARE PERSONALLY MOST CONVINCED ABOUT – DON'T ENGAGE IN GROUPTHINK.

(1) Consider the curve given by the parametric description $x = \cos t$, $y = \sin t$, where t varies over the interval $[a, b]$ with $a < b$. What is a necessary and sufficient condition on a and b for this curve to be the circle $x^2 + y^2 = 1$? *Last time:* 11/24 *correct*
(A) $b - a = \pi$
(B) $b - a > \pi$
(C) $b - a = 2\pi$
(D) $b - a > 2\pi$
(E) $b - a \geq 2\pi$

Your answer: ____________________________

(2) (**) Consider the curve given by the parametric description $x = \arctan t$ and $y = \arctan t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve? *Last time:* 8/24 *correct*
(A) It is the graph of the function arctan
(B) It is the line $y = x$
(C) It is a line segment (without endpoints) that is part of the line $y = x$
(D) It is a half-line (with endpoint) that is part of the line $y = x$
(E) It is a disjoint union of two half-lines that are both part of the line $y = x$

Your answer: ____________________________

(3) (**) Consider the curve given by the parametric description $x = \sin^2 t$ and $y = \cos^2 t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve? *Last time:* 5/24 *correct*
(A) It is the arc of the circle $x^2 + y^2 = 1$ comprising the first quadrant, i.e., when $x \geq 0$ and $y \geq 0$.
(B) It is the entire circle $x^2 + y^2 = 1$
(C) It is the line segment joining the points $(0, 1)$ and $(1, 0)$
(D) It is the line $y = 1 - x$
(E) It is a portion of the parabola $y = x^2$

Your answer: ____________________________

(4) Identify the parametric description which *does not* correspond to the set of points (x, y) satisfying $x^3 = y^5$. *Last time:* 16/24 *correct*

(A) $x = t^3$, $y = t^5$, for $t \in \mathbb{R}$
(B) $x = t^5$, $y = t^3$, for $t \in \mathbb{R}$
(C) $x = t$, $y = t^{3/5}$, for $t \in \mathbb{R}$
(D) $x = t^{5/3}$, $y = t$, for $t \in \mathbb{R}$
(E) All of the above parametric descriptions work

Your answer: ____________________________

(5) (**) Consider the parametric description $x = f(t)$, $y = g(t)$ where t varies over all of $\mathbb{R}$. What is the necessary and sufficient condition for the curve given by this to be the graph of a function, i.e., to satisfy the vertical line test? *Last time:* 10/24 *correct*

(A) For any t_1 and t_2 satisfying $f(t_1) = f(t_2)$, we must have $g(t_1) = g(t_2)$.
(B) For any t_1 and t_2 satisfying $g(t_1) = g(t_2)$, we must have $f(t_1) = f(t_2)$.
(C) Both f and g are one-to-one functions.
(D) For any t_1 and t_2, we must have $f(t_1) = f(t_2)$.
(E) For any t_1 and t_2, we must have $g(t_1) = g(t_2)$.

Your answer: ____________________________

(6) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \circ g)''$? *Last time:* 20/21 *correct*

(A) $(f'' \circ g) \cdot g''$
(B) $(f'' \circ g) \cdot (f' \circ g') \cdot g''$
(C) $(f'' \circ g) \cdot (f' \circ g') \cdot (f \circ g'')$
(D) $(f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$
(E) $(f' \circ g') \cdot (f \circ g) + (f'' \circ g'')$

Your answer: ____________________________

(7) Suppose $x = f(t)$ and $y = g(t)$ where f and g are both twice differentiable functions. What is d^2y/dx^2 in terms of f and g and their derivatives evaluated at t? *Last time:* 20/21 *correct*

(A) $(f'(t)g''(t) - g'(t)f''(t))/(f'(t))^3$
(B) $(f'(t)g''(t) - g'(t)f''(t))/(g'(t))^3$
(C) $(g'(t)f''(t) - f'(t)g''(t))/(f'(t))^3$
(D) $(g'(t)f''(t) - f'(t)g''(t))/(g'(t))^3$
(E) None of the above

Your answer: ____________________________

(8) Which of the following pair of bounds works for the arc length for the portion of the graph of the sine function between $(a, \sin a)$ and $(b, \sin b)$ where $a < b$? *Last time:* 15/21 *correct*

(A) Between $(b - a)/\sqrt{3}$ and $(b - a)/\sqrt{2}$

(B) Between $(b-a)/\sqrt{2}$ and $b-a$
(C) Between $(b-a)$ and $\sqrt{2}(b-a)$
(D) Between $\sqrt{2}(b-a)$ and $\sqrt{3}(b-a)$
(E) Between $\sqrt{3}(b-a)$ and $2(b-a)$

Your answer: ____________________________

(9) (*) Consider the parametric curve $x = e^t$, $y = e^{t^2}$. How does y grow in terms of x as $x \to \infty$? *Last time:* 7/21 *correct*
(A) y grows like a polynomial in x.
(B) y grows faster than any polynomial in x but slower than an exponential function of x.
(C) y grows exponentially in x.
(D) y grows super-exponentially in x but slower than a double exponential in x.
(E) y grows like a double exponential in x.

Your answer: ____________________________

(10) We say that a curve is *algebraic* if it admits a parameterization of the form $x = f(t)$, $y = g(t)$, where f and g are rational functions and t varies over some subset of the real numbers. Which of the following curves is *not* algebraic? *Last time:* 11/21 *correct*
(A) $x = \cos t$, $y = \sin t$, $t \in \mathbb{R}$
(B) $x = \cos t$, $y = \cos(3t)$, $t \in \mathbb{R}$
(C) $x = \cos t$, $y = \cos^2 t$, $t \in \mathbb{R}$
(D) $x = \cos t$, $y = \cos(t^2)$, $t \in \mathbb{R}$
(E) None of the above, i.e., they are all algebraic

Your answer: ____________________________

(11) (**) Suppose $x = f(t)$, $y = g(t)$, $t \in \mathbb{R}$ is a parametric description of a curve Γ and both f and g are continuous on all of $\mathbb{R}$. If both f and g are even, what can we conclude about Γ and its parameterization? *Last time:* 5/21 *correct*
(A) Γ is symmetric about the y-axis
(B) Γ is symmetric about the x-axis
(C) Γ is symmetric about the line $y = x$
(D) Γ has half turn symmetry about the origin
(E) The parameterizations of Γ for $t \leq 0$ and for $t \geq 0$ both cover all of Γ, and in directions mutually reverse to each other.

Your answer: ____________________________

CLASS QUIZ: FRIDAY JANUARY 11: POLAR COORDINATES

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

YOU ARE ALLOWED TO DISCUSS ONLY QUESTIONS THAT BEGIN WITH A (*) OR (**). PLEASE ATTEMPT ALL OTHER QUESTIONS BY YOURSELF. EVEN FOR THE QUESTIONS YOU DISCUSS, PLEASE FINALLY ENTER ONLY THE ANSWER OPTION YOU ARE PERSONALLY MOST CONVINCED ABOUT – DON'T ENGAGE IN GROUPTHINK.

(1) (*) Consider a straight line that does not pass through the pole in a polar coordinate system. The equation of such a line in the polar coordinate system can be expressed as $r = F(\theta)$. What kind of function is F? *Last time:* 8/21 *correct*
(A) $F(\theta)$ is a linear combination of $\sin\theta$ and $\cos\theta$
(B) $F(\theta)$ is the reciprocal of a linear combination of $\sin\theta$ and $\cos\theta$.
(C) $F(\theta)$ is a linear combination of $\tan\theta$ and $\cot\theta$.
(D) $F(\theta)$ is the reciprocal of a linear combination of $\tan\theta$ and $\cot\theta$.
(E) $F(\theta)$ is a linear combination of $\sec\theta$ and $\csc\theta$.

Your answer: ____________________________

(2) Consider the curve $r = \sin^2\theta$. Which of the following symmetries does the curve enjoy? Please see options (D) and (E) before answering. *Last time:* 10/21 *correct*
(A) Mirror symmetry about the polar axis
(B) Mirror symmetry about an axis perpendicular to the polar axis (what would be the y-axis if the polar axis is the x-axis)
(C) Half turn symmetry about the pole
(D) All of the above
(E) None of the above

Your answer: ____________________________

(3) Which of the following is the correct expression for the length of the part of the curve $r = F(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, with $\alpha < \beta$? *Last time:* 14/21 *correct*
(A) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$
(B) $\int_\alpha^\beta |F(\theta) + F'(\theta)|\, d\theta$
(C) $\int_\alpha^\beta |F(\theta) - F'(\theta)|\, d\theta$
(D) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 + 4F(\theta)F'(\theta)}\, d\theta$
(E) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 - 4F(\theta)F'(\theta)}\, d\theta$

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE WEDNESDAY JANUARY 16: THREE DIMENSIONS

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ________________________________

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(1) (*) Consider the subset of $\mathbb{R}^3$ given by the condition $(x^2 + y^2 - 1)(y^2 + z^2 - 1)(x^2 + z^2 - 1) = 0$. What kind of subset is this? *Last time:* 12/24 *correct*
   (A) It is a sphere centered at the origin and of radius 1.
   (B) It is the union of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
   (C) It is the union of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.
   (D) It is the intersection of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
   (E) It is the intersection of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.

   Your answer: ____________________

(2) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that AC and BC have equal length (i.e., C is equidistant from A and B)? *Didn't appear last time*
   (A) Sphere
   (B) Plane
   (C) Circle
   (D) Line
   (E) Two points

   Your answer: ____________________

(3) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right triangle with AB as its hypotenuse? *Last time:* 15/24 *correct*
   (A) Sphere (minus two points)
   (B) Plane
   (C) Circle (minus two points)
   (D) Line
   (E) Square

   Your answer: ____________________

(4) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right isosceles triangle with AB as its hypotenuse? *Didn't appear last time.*
   (A) Sphere
   (B) Plane
   (C) Circle
   (D) Line
   (E) Square

   Your answer: ____________________________

(5) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is equilateral? *Last year:* 23/24 *correct.*
   (A) Sphere
   (B) Plane
   (C) Circle
   (D) Line
   (E) Two points

   Your answer: ____________________________

(6) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that $|AC|/|BC| = \lambda$ for λ a fixed positive real number not equal to 1? *Didn't appear last time.*
   (A) Sphere
   (B) Plane
   (C) Circle
   (D) Line
   (E) Square

   Your answer: ____________________________

(7) Consider the parametric curve in three dimensions given by the coordinate description $t \mapsto (\cos t, \sin t, \cos(2t))$, with $t \in \mathbb{R}$. We can consider the *projections* of this curve onto the xy-plane, yz-plane, and xz-plane, which are basically what we get by dropping perpendiculars from the curve to these planes. What is the correct description of the curves obtained by doing the three projections? *Last time:* 17/24 *correct*
   (A) The projections on the xy-plane and yz-plane are both parts of parabolas, and the projection on the xz-plane is a circle.
   (B) The projections on the xy-plane and yz-plane are both circles, and the projection on the xz-plane is a part of a parabola.
   (C) The projection on the xy-plane is a circle, and the projections on the yz-plane and xz-plane are both parts of parabolas.
   (D) The projection on the xy-plane is a part of a parabola, the projection on the xz-plane and yz-plane are both circles.
   (E) All the three projections are circles.

   Your answer: ____________________________

CLASS QUIZ: FRIDAY JANUARY 18: VECTORS

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

YOU ARE ALLOWED TO DISCUSS ONLY QUESTIONS THAT BEGIN WITH A (*) OR (**). PLEASE ATTEMPT ALL OTHER QUESTIONS BY YOURSELF. EVEN FOR THE QUESTIONS YOU DISCUSS, PLEASE FINALLY ENTER ONLY THE ANSWER OPTION YOU ARE PERSONALLY MOST CONVINCED ABOUT – DON'T ENGAGE IN GROUPTHINK.

(1) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ with the property that the dot product of any two distinct elements of S is zero. What is the maximum possible size of S? *Last time:* 14/23 *correct*
(A) 1
(B) 2
(C) 3
(D) 4
(E) There is no finite bound on the size of S

Your answer: ____________________________

(2) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ such that the cross product of any two distinct elements of S is the zero vector. What is the maximum possible size of S? *Last time:* 17/23 *correct*
(A) 1
(B) 2
(C) 3
(D) 4
(E) There is no finite bound on the size of S

Your answer: ____________________________

(3) (**) Suppose a and b are vectors in $\mathbb{R}^3$. Which of the following is/are true? *Last time:* 6/23 *correct*
(A) If both a and b are nonzero vectors, then $a \times b$ is a nonzero vector.
(B) If $a \times b$ is a nonzero vector, then $a \cdot (a \times b)$ is a nonzero real number.
(C) If $a \times b$ is a nonzero vector, then $a \times (a \times b)$ is a nonzero vector.
(D) All of the above
(E) None of the above

Your answer: ____________________________

(4) (*) Suppose a, b, c, and d are vectors in $\mathbb{R}^3$, with $a \times b \neq 0$ and $c \times d \neq 0$. What does $(a \times b) \times (c \times d) = 0$ mean? *Last time:* 9/23 *correct*

(A) Both the vectors a and b are perpendicular to both the vectors c and d.
(B) a and b are perpendicular to each other and c and d are perpendicular to each other.
(C) a and c are perpendicular to each other and b and d are perpendicular to each other.
(D) The plane spanned by a and b is perpendicular to the plane spanned by c and d.
(E) a, b, c, and d are all coplanar.

Your answer: ____________________________

(5) (**) The *correlation* between two vectors in $\mathbb{R}^n$ is defined as the quotient of the dot product of the vectors by the product of their lengths. Suppose a, b, and c are vectors in $\mathbb{R}^n$ such that the correlation between vectors a and b is a number x and the correlation between b and c is a number y, and suppose x, y are both positive. What is the maximum possible value of the correlation between a and c given this information? *Hint: Geometrically if θ_{ab} is the angle between a and b, θ_{bc} is the angle between b and c, and θ_{ac} is the angle between a and c, then $|\theta_{ab} - \theta_{bc}| \le \theta_{ac} \le \theta_{ab} + \theta_{bc}$. Last time:* 5/23 *correct*
(A) xy
(B) $\max\{1, xy\}$
(C) $\min\{1, xy\}$
(D) $xy + \sqrt{(1-x^2)(1-y^2)}$
(E) $xy - \sqrt{(1-x^2)(1-y^2)}$

Your answer: ____________________________

(6) If the correlation between nonzero vector v and nonzero vector w in $\mathbb{R}^n$ is c, then we say that the *proportion* of vector w *explained by* vector v is c^2. If $v_1, v_2, \dots, v_k$ are all pairwise orthogonal nonzero vectors, and c_i is the correlation between v_i and w, then $c_1^2 + c_2^2 + \cdots + c_k^2 \le 1$, with equality occurring if and only if $k = n$. (This is all a result of the Pythagorean theorem). If $k < n$, then $1 - (c_1^2 + c_2^2 + \cdots + c_k^2)$ is the *unexplained proportion* of w.

Suppose w is the *variation of beauty* vector, v_1 is the *variation of genes* vector, and v_2 is the *variance of make-up* vector. Assume that v_1 and v_2 are orthogonal (i.e., there is no correlation between genes and make-up choice). If the correlation between v_1 and w is 0.6 and the correlation between v_2 and w is 0.3, what proportion of the variation of beauty remains unexplained (i.e., is not explained by either genes or make-up)? *Last time:* 17/23 *correct*
(A) 0.1
(B) 0.19
(C) 0.55
(D) 0.74
(E) 1

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE WEDNESDAY JANUARY 23: VECTORS, 3D, AND PARAMETRIC STUFF – MISCELLANEA

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

THIS IS A TAKE-HOME CLASS QUIZ, BUT I WILL GIVE YOU ABOUT 5 MINUTES TO REVIEW YOUR ANSWERS IN CLASS AND DISCUSS WITH OTHER STUDENTS.

YOU ARE FREE TO DISCUSS *ALL* QUESTIONS, BUT PLEASE FINALLY ENTER ONLY THE ANSWER OPTION YOU ARE PERSONALLY MOST CONVINCED ABOUT – DON'T ENGAGE IN GROUPTHINK.

Yes, you are free to discuss *all* questions for this quiz.

(1) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x,y,z) = 0$, Γ_2 is the set of solutions to $F_2(x,y,z) = 0$, and Γ_3 is the set of solutions to $F_3(x,y,z) = 0$. Which of the following equations gives precisely the set of points that lie in *at least two* of the subsets Γ_1, Γ_2, and Γ_3?
(A) $F_1(x,y,z)F_2(x,y,z)F_3(x,y,z) = 0$
(B) $(F_1(x,y,z))^2 + (F_2(x,y,z))^2 + (F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(D) $(F_1(x,y,z)F_2(x,y,z)) + (F_2(x,y,z)F_3(x,y,z)) + (F_3(x,y,z)F_1(x,y,z)) = 0$
(E) $(F_1(x,y,z)F_2(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 + (F_3(x,y,z)F_1(x,y,z))^2 = 0$

Your answer: ____________________________

(2) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x,y,z) = 0$, Γ_2 is the set of solutions to $F_2(x,y,z) = 0$, and Γ_3 is the set of solutions to $F_3(x,y,z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$?
(A) $F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z) = 0$
(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Your answer: ____________________________

(3) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x,y,z) = 0$, Γ_2 is the set of solutions to $F_2(x,y,z) = 0$, and Γ_3 is the set of solutions to $F_3(x,y,z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cup (\Gamma_2 \cap \Gamma_3)$?
(A) $F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z) = 0$
(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Your answer: ____________________________

(4) Start with two vectors a and b in $\mathbb{R}^3$ such that $a \times b \neq 0$. Consider a sequence of vectors $c_1, c_2, \dots, c_n, \dots$ in $\mathbb{R}^3$ (note: each c_n is a three-dimensional vector) defined as follows: $c_1 = a \times b$ and $c_{n+1} = a \times c_n$ for $n \geq 1$. Which *one* of the following statements is **false** about the c_ns? (5 points)

(A) All the vectors c_n are nonzero vectors.
(B) c_n and c_{n+1} are orthogonal for every n.
(C) c_n and c_{n+2} are parallel for every n.
(D) c_n and a are orthogonal for every n.
(E) c_n and b are orthogonal for every n.

Your answer: ____________________________

(5) As a general rule, what would you expect should be the dimensionality of the set of solutions to m independent and consistent equations in n variables? By solution, we mean here that a n-tuple with coordinates in $\mathbb{R}$ (in other words, a point in $\mathbb{R}^n$) that satisfy all the m equations. Assume $n \geq m \geq 1$.
(A) n
(B) m
(C) $n - 1$
(D) $n - m$
(E) 1

Your answer: ____________________________

(6) As a general rule, what would you expect should be the dimensionality of the set of points in $\mathbb{R}^n$ that satisfy at least one of m independent and consistent equations in n variables? Assume $n \geq m \geq 1$.
(A) n
(B) m
(C) $n - 1$
(D) $n - m$
(E) 1

Your answer: ____________________________

(7) Measuring time t in seconds since the beginning of the year 2013, and stock prices on a 24×7 stock exchange in predetermined units, the stock prices of companies A, B, and C were found to be given by $30 + t/5000000 - \sin(t/10000)$, $16 + 7t/3000000$, and $40 + t/1000000 - 5\sin(t/10000)$. To what extent can we deduce the stock prices of the companies from each other at a given point in time, without knowing what the time is?
(A) The stock price of any of the three companies can be used to deduce the other stock prices.
(B) The stock price of company A can be used to deduce the stock prices of companies B and C, but no other deductions are possible.
(C) The stock price of company A can be used to deduce the stock prices of companies B and C, and the stock price of company C can be used to deduce the stock prices of companies A and B.
(D) The stock price of company B can be used to determine the stock prices of companies A and C, and no other deductions are possible.
(E) The stock price of company B can be used to determine the stock prices of companies A and C, and the stock prices of companies A and C can be used to deduce each other but cannot be used to uniquely deduce the stock price of company B.

Your answer: ____________________________

(8) Lushanna is coaching 30 young athletes for a 100 meter sprint. Every day, at the beginning of the day, she asks the athlete to run 100 meters as fast as they can and notes the time taken. She thus gets a vector with 30 coordinates (measuring the time taken by all the athletes) everyday. Lushanna then plots a graph in thirty-dimensional space that includes all the points for her daily measurements. Each of the following is a sign that Lushanna's young athletes are improving. Which of these signs is **strongest**, in the sense that it would imply all the others?
(A) The norm (length) of the vector every day (after the first) is less than the norm of the vector the previous day.

(B) The sum of the coordinates of the vector every day (after the first) is less than the sum of the coordinates of the vector the previous day.
(C) The minimum of the coordinates of the vector every day (after the first) is less than the minimum of the coordinates of the vector the previous day.
(D) The maximum of the coordinates of the vector every day (after the first) is less than the maximum of the coordinates of the vector the previous day.
(E) Each of the coordinates of the vector every day (after the first) is less than the corresponding coordinate of the vector the previous day.

Your answer: ____________________________

(9) In a closed system (no mass exchanged with the surroundings) a reversible chemical reaction $A+B \to C+D$, and its reverse, are proceeding. There are no other chemicals in the system, and no other reactions are proceeding in the system. A chemist studying the reaction decides to track the masses of A, B, C, and D in the system as a function of time, and plots a parametric curve in four-dimensional space. What can we say about the nature of the curve, ignoring the parametrization (i.e., just looking at the set of points covered)?
(A) It is a part of a straight line.
(B) It is a part of a circle.
(C) It is a part of a parabola.
(D) It is a part of an astroid.
(E) It is a part of a cissoid.

Your answer: ____________________________

Lobbying special: Casa is a lobbyist for a special interest group. There are three politicians P_1, P_2, P_3 competing for a general election. Casa has computed that the probabilities of the politicians winning are p_1 for P_1, p_2 for P_2, and p_3 for P_3, with $p_1, p_2, p_3 \in [0,1]$ and $p_1 + p_2 + p_3 = 1$. Casa estimates a payoff of m_1 money units to her special interest group if P_1 wins, m_2 money units if P_2 wins, and m_3 money units if P_3 wins. (These payoffs may be in terms of passage of favorable laws, repeal of unfavorable laws, or enforcement of laws unfavorable to competitors).

(10) What is the expected payoff to the special interest group that Casa represents?
(A) $m_1 + m_2 + m_3$
(B) $(m_1 + m_2 + m_3)/3$
(C) $(p_1 + p_2 + p_3)(m_1 + m_2 + m_3)$
(D) $p_1 m_1 + p_2 m_2 + p_3 m_3$
(E) $\sqrt{m_1^2 + m_2^2 + m_3^2}$

Your answer: ____________________________

(11) Casa has discovered that the bribe multipliers of the politicians are the positive reals b_1, b_2, and b_3 respectively. In other words, if Casa donates u_i money units to P_i, then the expected payoff from politician P_i winning is now $m_i + b_i u_i$. Consider the vectors $p = \langle p_1, p_2, p_3 \rangle$, $m = \langle m_1, m_2, m_3 \rangle$, $c = \langle p_1 b_1, p_2 b_2, p_3 b_3 \rangle$, and $f = \langle p_1/b_1, p_2/b_2, p_3/b_3 \rangle$ and let $u = \langle u_1, u_2, u_3 \rangle$ be the vector of the bribe quantities Casa gives to the politicians respectively. Assume that bribing politicians does not affect the relative probabilities of winning the election. Which of the following describes Casa's expected payoff from the election, once the bribe is made (if you want to include the cost of bribes, you'd need to subtract $u_1 + u_2 + u_3$ from this answer, but we're not doing that. *Note: Some of the answer options may not make sense from a dimensions/units viewpoint, but the correct answer does make sense.*
(A) $p \cdot (m + u)$
(B) $p \cdot (m + (b \cdot u))$
(C) $p \cdot (m + (f \cdot u))$
(D) $(p \cdot m) + (c \cdot u)$
(E) $p \cdot (f \cdot m + u)$

Your answer: ____________________________

(12) Continuing with the full setup of the preceing question, what is Casa's optimal bribing strategy on a fixed budget of money to be used for bribes?

(A) Donate all the money to the politician with the maximum value of $p_i b_i$, i.e., to the politician corresponding to the largest coordinate of the vector c.

(B) Donate all the money to the politician with the minimum value of $p_i b_i$, i.e., to the politician corresponding to the smallest coordinate of the vector c.

(C) Donate all the money to the politician with the maximum value of p_i/b_i, i.e., to the politician corresponding to the largest coordinate of the vector f.

(D) Donate all the money to the politician with the minimum value of p_i/b_i, i.e., to the politician corresponding to the smallest coordinate of the vector f.

(E) Split the bribery budget between the politicians in the ratio $p_1 b_1 : p_2 b_2 : p_3 b_3$.

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE FRIDAY JANUARY 25: LIMITS, CONTINUITY, DIFFERENTIATION REVIEW

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____

These questions are all related to single variable calculus. Specifically, these are some of the harder questions from material typically covered in Math 151/152. I've given these questions in past quizzes/tests in Math 152 and Math 153 and the scores indicated here are the scores in appearances of these questions in previous quizzes/tests.

PLEASE FEEL FREE TO DISCUSS THESE QUESTIONS WITH OTHERS, BUT YOUR FINAL ANSWERS SHOULD BE THE ANSWERS YOU ARE PERSONALLY CONVINCED ABOUT.

(1) Which of the following statements is **always true**? *Earlier scores:* 2/11*,* 9/16*,* 16/28
- (A) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).
- (B) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).
- (C) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.
- (D) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.
- (E) None of the above.

Your answer: ____________________________

(2) For which of the following specifications is there **no continuous function** satisfying the specifications? *Earlier score:* 7/14*,* 21/28
- (A) Domain $(0, 1)$ and range $(0, 1)$
- (B) Domain $[0, 1]$ and range $(0, 1)$
- (C) Domain $(0, 1)$ and range $[0, 1]$
- (D) Domain $[0, 1]$ and range $[0, 1]$
- (E) None of the above, i.e., we can get a continuous function for each of the specifications.

Your answer: ____________________________

(3) Suppose f is a continuously differentiable function from the open interval $(0, 1)$ to $\mathbb{R}$. Suppose, further, that there are exactly 14 values of c in $(0, 1)$ for which $f(c) = 0$. What can we say is **definitely true** about the number of values of c in the open interval $(0, 1)$ for which $f'(c) = 0$? *Earlier scores:* 7/15*,* 19/28
- (A) It is at least 13 and at most 15.
- (B) It is at least 13, but we cannot put any upper bound on it based on the given information.
- (C) It is at most 15, but we cannot put any lower bound (other than the meaningless bound of 0) based on the given information.
- (D) It is at most 13.
- (E) It is at least 15.

Your answer: ____________________________

(4) Consider the function $f(x) := \begin{cases} x, & 0 \le x \le 1/2 \\ x - (1/7), & 1/2 < x \le 1 \end{cases}$. Define by $f^{[n]}$ the function obtained by iterating f n times, i.e., the function $f \circ f \circ f \circ \cdots \circ f$ where f occurs n times. What is the smallest n for which $f^{[n]} = f^{[n+1]}$? *Earlier scores:* 3/16, 10/28
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
Your answer: ____________________________

(5) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both right continuous on $(0,1)$. Which of the following is *not* guaranteed to be right continuous on $(0,1)$? *Earlier scores:* 3/11, 9/14, 20/28
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be right continuous functions
Your answer: ____________________________

(6) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing function from $\mathbb{R}$ to $\mathbb{R}$? *Earlier scores:* 1/15, 9/16, 18/28
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.
Your answer: ____________________________

(7) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero) *Earlier scores:* 6/16, 10/28
(A) $k - 2$
(B) $k - 1$
(C) k
(D) $k + 1$
(E) There is no bound in terms of k.
Your answer: ____________________________

(8) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy). *Earlier scores:* 3/16, 10/28
(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.
Your answer: ____________________________

(9) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see options (D) and (E) before answering. *Earlier scores:* 4/16, 15/28
(A) The value of F at any two positive numbers.

(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Your answer: ____________________________

(10) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? *Earlier scores:* 0, 10/16, 11/28

(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE FRIDAY FEBRUARY 1: LIMITS

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________

This is based on the concept of limits you have seen in single variable calculus. You can view video playlists of the material here:

http://www.youtube.com/playlist?list=PL8483BCA409563C88

http://www.youtube.com/playlist?list=PLC0bHnWu122lmsGOHv39OSaNwD8MXvlTH

http://www.youtube.com/playlist?list=PLC0bHnWu122lZYgoCzVmWtnMAmUO6RDXC

PLEASE FEEL FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY CONSIDER MOST LIKELY TO BE CORRECT – DO NOT ENGAGE IN GROUPTHINK.

(1) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0, 1)$? If all are examples, please select Option (E).
(A) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x, & 1/2 < x < 1 \end{cases}$
(B) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x, & 1/2 \le x < 1 \end{cases}$
(C) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{cases}$
(D) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \le x < 1 \end{cases}$
(E) All of the above

Your answer: ____________________

(2) Suppose f and g are functions $(0, 1)$ to $(0, 1)$ that are both left continuous on $(0, 1)$. Which of the following is *not* guaranteed to be left continuous on $(0, 1)$? Please see Option (E) before answering.
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Your answer: ____________________

(3) Which of these is the correct interpretation of $\lim_{x \to c} f(x) = L$ in terms of the definition of limit? Please see Option (E) before answering.
(A) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
(B) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
(C) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
(D) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.
(E) None of the above

Your answer: ____________________

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x \to c} f(x)$ exists)? If all, please select Option (E).

(A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.

(B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.

(C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.

(D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.

(E) All of the above.

Your answer: ______________________________

(5) In the usual $\varepsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\varepsilon > 0$, then which of the following is true? Please see Option (E) before answering.

(A) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.

(B) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.

(C) Every larger value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.

(D) Every larger value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.

(E) None of the above statements need always be true.

Your answer: ______________________________

(6) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of εs and δs? Please see Option (E) before answering.

(A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(E) None of the above.

Your answer: ______________________________

(7) Consider the function:

$$f(x) := \begin{cases} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{cases}$$

What is the set of all points at which f is continuous?
(A) $\{0, 1\}$
(B) $\{-1, 1\}$
(C) $\{-1, 0\}$
(D) $\{-1, 0, 1\}$
(E) f is continuous everywhere

Your answer: ____________________________

(8) The graph $y = f(x)$ of a function f defined on all reals has a horizontal asymptote $y = c$ as x approaches $+\infty$. Which of the following is the correct definition of this?
(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(B) For every $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
(C) For every $\varepsilon > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.
(D) For every $\delta > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(E) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - c| < \varepsilon$.

Your answer: ____________________________

(9) Which of the following is the correct definition of $\lim\limits_{x \to c^-} f(x) = -\infty$ (in words: the left hand limit of f at c is $-\infty$)?
(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.
(B) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) > a$.
(C) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) < a$.
(D) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) > a$.
(E) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) < a$.

Your answer: ____________________________

(10) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon - \delta$ definition for the statement "f is differentiable at c"?
(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$.
(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.
(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$
(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$
(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.

Your answer: ____________________________

(11) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon - \delta$ definition for the statement "f is not differentiable at c"?
(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$.

(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.
(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$
(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$
(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.

Your answer: ____________________________

(12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Identify which of these definitions is *not* correct for $\lim_{x \to c} f(x) = L$, where c and L are both finite real numbers. If all are correct, please select Option (E).
(A) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta) \setminus \{c\}$, then $f(x) \in (L - \varepsilon, L + \varepsilon)$.
(B) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c - \delta_1, c + \delta_2) \setminus \{c\}$, then $f(x) \in (L - \varepsilon_1, L + \varepsilon_2)$.
(C) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta) \setminus \{c\}$, then $f(x) \in (L - \varepsilon_1, L + \varepsilon_2)$.
(D) For every $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c - \delta_1, c + \delta_2) \setminus \{c\}$, then $f(x) \in (L - \varepsilon, L + \varepsilon)$.
(E) None of these, i.e., all definitions are correct.

Your answer: ____________________________

(13) In the usual $\varepsilon - \delta$ definition of limit, we find that the value $\delta = 0.2$ for $\varepsilon = 0.7$ for a function f at 0, and the value $\delta = 0.5$ works for $\varepsilon = 1.6$ for a function g at 0. What value of δ *definitely* works for $\varepsilon = 2.3$ for the function $f + g$ at 0?
(A) 0.2
(B) 0.3
(C) 0.5
(D) 0.7
(E) 0.9

Your answer: ____________________________

(14) The sum of limits theorem states that $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ *if* the right side is defined. One of the choices below gives an example where the left side of the equality is defined and finite but the right side makes no sense. Identify the correct choice.
(A) $f(x) := 1/x$, $g(x) := -1/(x + 1)$, $c = 0$.
(B) $f(x) := 1/x$, $g(x) := (x - 1)/x$, $c = 0$.
(C) $f(x) := \arcsin x$, $g(x) := \arccos x$, $c = 1/2$.
(D) $f(x) := 1/x$, $g(x) = x$, $c = 0$.
(E) $f(x) := \tan x$, $g(x) := \cot x$, $c = 0$.

Your answer: ____________________________

CLASS QUIZ: FRIDAY FEBRUARY 1: MULTIVARIABLE FUNCTION BASICS

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

PLEASE DISCUSS *ONLY* THE STARRED OR DOUBLE-STARRED QUESTIONS.

(1) Suppose f is a function of two variables, defined on all of $\mathbb{R}^2$, with the property that $f(x, y) = f(y, x)$ for all real numbers x and y. What does this say about the symmetry of the graph $z = f(x, y)$ of f? *Last time:* 16/21 *correct* – DO NOT DISCUSS.
   (A) It has mirror symmetry about the plane $z = x + y$.
   (B) It has mirror symmetry about the plane $x = y$.
   (C) It has mirror symmetry about the plane $z = x - y$.
   (D) It has half turn symmetry about the line $x = y = z$.
   (E) It has half turn symmetry about the origin.

   Your answer: __________________________

(2) (**) Consider the function $f(x, y) := ax + by$ where a and b are fixed nonzero reals. The level curves for this function are a bunch of parallel lines. What vector are they all parallel to? *Last time:* 5/21 *correct*
   (A) $\langle a, b\rangle$
   (B) $\langle a, -b\rangle$.
   (C) $\langle b, a\rangle$
   (D) $\langle b, -a\rangle$
   (E) $\langle a - b, a + b\rangle$

   Your answer: __________________________

(3) (**) Suppose f is a function of one variable and g is a function of two variables. What is the relationship between the level curves of $f \circ g$ and the level curves of g? *Last time:* 7/21 *correct*
   (A) Each level curve of $f \circ g$ is a union of level curves of g corresponding to the pre-images of the point under f.
   (B) Each level curve of $f \circ g$ is an intersection of level curves of g corresponding to the pre-images of the point under f.
   (C) The level curves of $f \circ g$ are precisely the same as the level curves of g.
   (D) Each level curve of g is a union of level curves of $f \circ g$.
   (E) Each level curve of g is an intersection of level curves of $f \circ g$.

   Your answer: __________________________

(4) Consider the following function f from $\mathbb{R}^2$ to $\mathbb{R}^2$: the function that sends $\langle x, y\rangle$ to $\langle \frac{x+y}{2}, \frac{x-y}{2}\rangle$. What is the image of $\langle x, y\rangle$ under $f \circ f$? *Last time:* 12/21 *correct* – DO NOT DISCUSS.
   (A) $\langle x, y\rangle$
   (B) $\langle 2x, 2y\rangle$
   (C) $\langle x/2, y/2\rangle$

(D) $\langle x + (y/2), y + (x/2) \rangle$
(E) $\langle 2x + y, 2x - y \rangle$

Your answer: ____________________________

(5) (**) Consider the following functions defined on the subset $x > 0$ of the xy-plane: $f(x, y) = x^y$. Consider the surface $z = f(x, y)$. What do the intersections of this surface with planes parallel to the xz-plane and yz-plane look like (ignore the following two special intersections: intersection with the plane $x = 1$ and intersection with the plane $y = 0$, also ignore intersections that turn out to be empty). *Last time:* 5/21 *correct*

(A) Intersections with any plane parallel to the xz or yz plane look like graphs of exponential functions.
(B) Intersections with any plane parallel to the xz or yz plane look like graphs of power functions (only positive inputs allowed).
(C) Intersections with any plane parallel to the xz-plane look like graphs of exponential functions, and intersections with any plane parallel to the yz-plane look like graphs of power functions (only positive inputs allowed).
(D) Intersections with any plane parallel to the yz-plane look like graphs of exponential functions, and intersections with any plane parallel to the xz-plane look like graphs of power functions (only positive inputs allowed).
(E) All the intersections are straight lines.

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE WEDNESDAY FEBRUARY 6: MULTIVARIABLE FUNCTION BASICS CONTINUED

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

PLEASE FEEL FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY CONSIDER MOST LIKELY TO BE CORRECT. DO NOT ENGAGE IN GROUPTHINK.

(1) Suppose F is an additively separable function of two variables x and y that is defined everywhere, i.e., there exist functions f and g of one variable, both defined on all of $\mathbb{R}$, such that $F(x, y) = f(x) + g(y)$ for all $x, y \in \mathbb{R}$.

We call two curves *parallel* if there is a vector by which we can translate all the points in one curve to get precisely the other curve.

Consider the following three statements:

(i) All curves obtained as the intersections of the graph of F with planes parallel to the xy-plane are parallel to each other.

(ii) All curves obtained as the intersections of the graph of F with planes parallel to the xz-plane are parallel to each other.

(iii) All curves obtained as the intersections of the graph of F with planes parallel to the yz-plane are parallel to each other.

Which of the statements (i)-(iii) is/are necessarily true?

(A) All of (i), (ii), and (iii) are true.
(B) Both (i) and (ii) are true but (iii) need not be true.
(C) Both (ii) and (iii) are true but (i) need not be true.
(D) Both (i) and (iii) are true but (ii) need not be true.
(E) (i) is true but (ii) and (iii) need not be true.

Your answer: ________________________

(2) Suppose f is a continuous function of two variables x and y, defined on the entire xy-plane. Suppose further that f is increasing in x for each fixed value of y, and that f is increasing in y for every fixed value of x. Which of the following is the most plausible description of the level curves of f in the xy-plane? *Note: You might wish to take an extremely simple example, e.g., an additively separable function where each of the pieces is the simplest possible increasing function you can think of.*

(A) They are all upward-sloping, i.e., they are of the form $y = g(x)$ with g an increasing function.
(B) They are all downward-sloping, i.e., they are of the form $y = g(x)$ with g a decreasing function.
(C) They look like closed loops (e.g., circles).
(D) They look like graphs of functions with a unique local and absolute minimum (such as the parabola $y = x^2$, though the actual picture may be different).
(E) They look like graphs of functions with a unique local and absolute maximum (such as the parabola $y = -x^2$, though the actual function may be different).

Your answer: ________________________

(3) What do the level curves of the function $f(x, y) := \sin(x + y)$ look like for output value in $[-1, 1]$? Note that all these level curves are being considered as curves in the xy-plane. *Note: This builds upon the idea of Question 3 of the previous quiz.*

(A) Each level curve is a single line.
(B) Each level curve is a union of two intersecting lines.

(C) Each level curve is a union of two distinct parallel lines.
(D) Each level curve is a union of infinitely many concurrent lines (i.e., infinitely many lines, all passing through the same point).
(E) Each level curve is a union of infinitely many distinct parallel lines (i.e., infinitely many lines, all parallel to each other).

Your answer: ____________________________

(4) Suppose f and g are both continuous functions of two variables x and y, both defined on all of $\mathbb{R}^2$, and such that $f(x, y) + g(x, y)$ is a constant C. What is the relation between the level curves of f and the level curves of g, all drawn in the xy-plane?
(A) Every level curve of f is a level curve of g and vice versa, with the same level value for both functions.
(B) Every level curve of f is a level curve of g and vice versa, but the value for which it is a level curve may be different for the two functions.
(C) The level curves of f need not be precisely the same as the level curves of g, but we can go from one set of level curves to the other via a parallel translation.
(D) Each level curve of f can be obtained by reflecting a suitable level curve of g about a suitable line in the xy-plane.
(E) Each level curve of f can be obtained by reflecting a suitable level curves of g about a suitable line in the xy-plane and then performing a suitable translation.

Your answer: ____________________________

CLASS QUIZ: WEDNESDAY FEBRUARY 6: MULTIVARIABLE LIMIT COMPUTATIONS

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ________________________________

PLEASE FEEL FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY CONSIDER MOST LIKELY TO BE CORRECT. DO NOT ENGAGE IN GROUPTHINK.

(1) (**) Consider the function $f(x,y) := x\sin(1/(x^2+y^2))$, defined on all points other than the point $(0,0)$. What is the limit of the function at $(0,0)$? *Last time:* 8/22 *correct*
(A) 0
(B) $1/\sqrt{2}$
(C) 1
(D) The limit is undefined, because the expression becomes unbounded around 0.
(E) The limit is undefined, because the expression is oscillatory around 0.

Your answer: ________________________

(2) The typical $\varepsilon-\delta$ definition of limit in two dimensions makes use of open disks centered at the points on the domain and range side, where the open disk is the interior region bounded by a circle centered at the point. Which other geometric shapes can we use instead of a circle of specified radius centered at the point? Please see Options (D) and (E) before answering and make the most appropriate selection. *Last time:* 12/22 *correct.*
(A) A square of specified side length centered at the point
(B) An equilateral triangle of specified side length centered at the point
(C) A regular hexagon of specified side length centered at the point
(D) Any of the above
(E) None of the above

Your answer: ________________________

(3) Here's a quick recap of the limit definition for a function of a vector variable. We say that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}$ satisfying $0 < |\mathbf{x}-\mathbf{c}| < \delta$, we have $|f(\mathbf{x}) - L| < \varepsilon$. We define $|\mathbf{x}-\mathbf{c}|$ as the Euclidean norm of $\mathbf{x}-\mathbf{c}$ where the Euclidean norm of a vector is the square root of the sum of the squares of its coordinates.

We could replace the Euclidean norm by other measurements. For instance, we could use:
(i) The *sum* of the absolute values of the coordinates of $\mathbf{x}-\mathbf{c}$.
(ii) The *maximum* of the absolute values of the coordinates of $\mathbf{x}-\mathbf{c}$.
(iii) The *minimum* of the absolute values of the coordinates of $\mathbf{x}-\mathbf{c}$.

For any of (i) - (iii), we could replace $|\mathbf{x}-\mathbf{c}|$ in our current definition of limit with that notion. The question is: for which of the replacements will our new notion of limit be the same as the old one? The deeper idea here is that limit depends upon a concept of what it means for two points to be close. So another way of phrasing the question is: which of the notions (i)-(iii) capture the same notion of closeness as the usual Euclidean distance?
(A) All of (i), (ii), and (iii).
(B) (i) and (ii) but not (iii).
(C) (i) and (iii) but not (ii).
(D) Only (i).
(E) None of (i), (ii), or (iii).

Your answer: ____________________________

(4) Suppose f is a function of two variables x, y and is defined on the whole xy-plane. Consider three conditions: (i) f is continuous on the whole xy-plane, (ii) for every fixed value $x = x_0$, the function $y \mapsto f(x_0, y)$ is continuous in y for all $y \in \mathbb{R}$, (iii) for every fixed value $y = y_0$, the function $x \mapsto f(x, y_0)$ is continuous in x for all $x \in \mathbb{R}$, (iv) the function $t \mapsto f(p(t), q(t))$ is continuous for all $t \in \mathbb{R}$ whenever p and q are both constant or linear functions (in other words, the restriction of f to any straight line in $\mathbb{R}^2$ is continuous).

Which of the following correctly describes the implications between (i), (ii), (iii), and (iv)?

(A) (i) implies both (ii) and (iii), and (ii) and (iii) together imply (iv).
(B) (i) implies (iv), and (iv) implies both (ii) and (iii).
(C) (iv) implies (ii) and (iii), and (ii) and (iii) together imply (i).
(D) (iv) implies (i), and (i) implies both (ii) and (iii).
(E) (ii) and (iii) together imply (iv), and (iv) implies (i).

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE MONDAY FEBRUARY 18: PARTIAL DERIVATIVES

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY ENDORSE. DO NOT ENGAGE IN GROUPTHINK.

(1) For this and the next question, consider the function on $\mathbb{R}^2$ given as:

$$f(x,y) := \begin{cases} 1, & x \text{ rational or } y \text{ rational} \\ 0, & x \text{ and } y \text{ both irrational} \end{cases}$$

What can we say about the subset S of $\mathbb{R}^2$ defined as the set of points where f_x is defined?
(A) S is the set of points for which at least one coordinate is rational.
(B) S is the set of points for which both coordinates are rational.
(C) S is the set of points for which the x-coordinate is rational.
(D) S is the set of points for which the y-coordinate is rational.
(E) S is the set of points for which at least one coordinate is irrational.

Your answer: ______________________

(2) With f as in the previous question, what is the subset T of $\mathbb{R}^2$ at which the second-order mixed partial derivative f_{xy} is defined?
(A) T is the empty subset.
(B) T is the set of points for which both coordinates are rational.
(C) T is the set of points for which the x-coordinate is rational.
(D) T is the set of points for which the y-coordinate is rational.
(E) T is the set of points for which both coordinates are irrational.

Your answer: ______________________

(3) For this and the next three questions, consider the function on $\mathbb{R}^2$ given as:

$$g(x,y) := \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

What can we say about the subset U of $\mathbb{R}^2$ defined as the set of points where g_x is defined?
(A) U is the empty subset.
(B) U is the set of points for which both coordinates are rational.
(C) U is the set of points for which the x-coordinate is rational.
(D) U is the set of points for which the y-coordinate is rational.
(E) U is the whole plane $\mathbb{R}^2$.

Your answer: ______________________

(4) With g as in the preceding question, what can we say about the subset V of $\mathbb{R}^2$ defined as the set of points where g_y is defined?
(A) V is the empty subset.
(B) V is the set of points for which both coordinates are rational.
(C) V is the set of points for which the x-coordinate is rational.
(D) V is the set of points for which the y-coordinate is rational.
(E) V is the whole plane $\mathbb{R}^2$.

Your answer: ____________________________

(5) With g as in the preceding question, what can we say about the subset W of $\mathbb{R}^2$ defined as the set of points where g_{xy} is defined?
(A) W is the empty subset.
(B) W is the set of points for which both coordinates are rational.
(C) W is the set of points for which the x-coordinate is rational.
(D) W is the set of points for which the y-coordinate is rational.
(E) W is the whole plane $\mathbb{R}^2$.

Your answer: ____________________________

(6) With g as in the preceding question, what can we say about the subset X of $\mathbb{R}^2$ defined as the set of points where g_{yx} is defined?
(A) X is the empty subset.
(B) X is the set of points for which both coordinates are rational.
(C) X is the set of points for which the x-coordinate is rational.
(D) X is the set of points for which the y-coordinate is rational.
(E) X is the whole plane $\mathbb{R}^2$.

Your answer: ____________________________

(7) For this and the next two questions, consider the function on $\mathbb{R}^2$ given as:

$$h(x, y) := \begin{cases} 1, & x \text{ an integer or } y \text{ an integer} \\ 0, & x \text{ not an integer and } y \text{ not an integer} \end{cases}$$

What can we say about the subset A of $\mathbb{R}^2$ defined as the set of points where h_{xy} is defined?
(A) A is the empty set.
(B) A is the set of points whose x-coordinate is an integer.
(C) A is the set of points whose x-coordinate is not an integer.
(D) A is the set of points whose y-coordinate is an integer.
(E) A is the set of points whose y-coordinate is not an integer.

Your answer: ____________________________

(8) With h as defined in the previous question, what can we say about the subset B of $\mathbb{R}^2$ defined as the set of points where h_x is defined but h_{xy} is not defined?
(A) B is the empty set.
(B) B is the set of points for which both coordinates are integers.
(C) B is the set of points for which both coordinates are non-integers.
(D) B is the set of points for which at least one coordinate is an integer.
(E) B is the set of points for which at least one coordinate is a non-integer.

Your answer: ____________________________

(9) With h as defined in the previous question, what can we say about the subset C of $\mathbb{R}^2$ defined as the set of points where both h_{xy} and h_{yx} are defined?
(A) C is the empty set.
(B) C is the set of points for which both coordinates are integers.
(C) C is the set of points for which both coordinates are non-integers.
(D) C is the set of points for which at least one coordinate is an integer.
(E) C is the set of points for which at least one coordinate is a non-integer.

Your answer: ____________________________

(10) Students training for an examination can spend money either on purchasing textbooks or on private tuitions. A student's expected performance on the examination is a function of the money the student spends on textbooks and on tuition (viewed as separate variables). Two researchers want to consider the question of whether increased expenditure on textbooks leads to improved performance on the examination, and if so, by how much.

One researcher decides to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the expenditure on tuitions*, arguing that in order to determine the effect of changes in textbook expenditures, the other expenditures need to be kept constant.

The other researcher believes that since the student has a limited budget, it would be more realistic to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the total expenditure on both textbook and tuitions.* This is because the student is likely to allocate money away from tuition expenditures in order to spend money on textbooks.

Which of the following best describes what's happening?

(A) Both researchers are effectively computing the same quantity.
(B) The two quantities that the researchers are computing have a simple linear relationship, i.e., their sum or difference is a constant.
(C) The two quantities that the researchers are computing are meaningfully different and there is a relationship between them but that relationship involves other partial derivatives.

Your answer: ____________________________

(11) F is an everywhere twice differentiable function of two variables x and y. Which of the following captures the manner in which the inputs x and y *interact* with each other in the description of F?

(A) The difference $F_x - F_y$
(B) The quotient F_x/F_y.
(C) The product $F_x F_y$.
(D) The product $F_{xx} F_{yy}$.
(E) The mixed partial F_{xy}

Your answer: ____________________________

(12) F is a function of two variables x and y such that both F_x and F_y exist. Which of the following is generically true?

(A) In general, F_x depends only on x (i.e., it is independent of y) and F_y depends only on y. An exception is if F is multiplicatively separable.
(B) In general, F_x depends only on y (i.e., it is independent of x) and F_y depends only on x (i.e., it is independent of y). An exception is if F is multiplicatively separable.
(C) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on y and F_y depends only on x.
(D) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on x and F_y depends only on y.
(E) In general, either both F_x and F_y depend only on x or both F_x and F_y depend only on y.

Your answer: ____________________________

(13) Consider a production function $f(L, K, T)$ of three inputs L (labor expenditure), K (capital expenditure), and T (technology expenditure). Suppose all mixed partials of f with respect to L, K, and T are continuous. Suppose we have the following signs of partial derivatives: $\partial f/\partial L > 0$, $\partial f/\partial K > 0$, $\partial^2 f/(\partial L \partial K) < 0$, and $\partial^3 f/(\partial L \partial K \partial T) > 0$. What does this mean?

(A) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital substitute for each other.
(B) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital substitute for each other, i.e., with more technology investment, labor and capital become more complementary.
(C) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital complement for each other.
(D) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital complement for each other.

(E) Increasing labor or capital decreases production.

Your answer: ____________________________

(14) Analysis of usage of an online social network finds that the total time spent by people on the social network is $P^{1.3}L^{0.5}$ where P is the total number of people on the network and L is a number of processors used at the social network's server facility. Which of these is true?

(A) Increasing returns both on persons and on processors: every new person joining the network increases the average time spent *per person* (and not just the total time), and every new processor added to the server facility increases the average time spent per processor.
(B) Constant returns on persons, increasing returns on processors
(C) Constant returns on persons, decreasing returns on processors
(D) Increasing returns on persons, decreasing returns on processors
(E) Decreasing returns on persons, increasing returns on processors

Your answer: ____________________________

(15) *Not a calculus question, but has deep calculus interpretations – it is basically measuring the derivative of the $1/x$ function with respect to x*: A person travels fifty miles every day by car and the travel distance is fixed. The price of gasoline, which she uses to fuel her car, is also fixed. Which of the following increases in fuel efficiency result in the maximum amount of savings for her?

(A) From 11 to 12 miles per gallon
(B) From 12 to 14 miles per gallon
(C) From 20 to 25 miles per gallon
(D) From 36 to 54 miles per gallon
(E) From 50 to 100 miles per gallon

Your answer: ____________________________

(16) For which of the following production functions $f(L, K)$ of labor and capital is it true that labor and capital can be complementary for some choices of (L, K), and substitutes for others? In other words, for which of these are labor and capital neither globally complements nor globally substitutes? Assume the domain $L > 0, K > 0$.

(A) $L^2 + LK + K^2$
(B) $L^2 - LK + K^2$
(C) $L^3 + L^2K + LK^2 + K^3$
(D) $L^3 + L^2K - LK^2 + K^3$
(E) $L^3 - L^2K - LK^2 + K^3$

Your answer: ____________________________

(17) Consider the following Leontief-like production function $f(L, K) = (\min\{L, K\})^2$. Assume the domain $L > 0$, $K > 0$. What is the nature of returns and complementarity here?

(A) Positive increasing returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(B) Positive constant returns of the smaller of the inputs, positive increasing returns on the larger of the inputs
(C) Zero returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(D) Positive decreasing returns on the smaller of the inputs, zero returns on the larger of the inputs
(E) Positive increasing returns on the smaller of the inputs, zero returns on the larger of the inputs

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE WEDNESDAY FEBRUARY 20: INTEGRATION TECHNIQUES (ONE VARIABLE)

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY ENDORSE. DO NOT ENGAGE IN GROUPTHINK.

(1) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? Please see Option (E) before answering.
   (A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
   (B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
   (C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
   (D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
   (E) None of the above, i.e., they are all necessarily true.

   Your answer: ______________________

(2) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
   (A) $k - 2$
   (B) $k - 1$
   (C) k
   (D) $k + 1$
   (E) There is no bound in terms of k.

   Your answer: ______________________

(3) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).
   (A) $F(1) = F(0)$.

(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Your answer: ____________________________

(4) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see options (D) and (E) before answering.
(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Your answer: ____________________________

(5) Suppose F, G are continuously differentiable functions defined on all of $\mathbb{R}$. Suppose a, b are real numbers with $a < b$. Suppose, further, that $G(x)$ is identically zero everywhere except on the open interval (a, b). Then, what can we say about the relationship between the numbers $P = \int_a^b F(x)G'(x)\,dx$ and $Q = \int_a^b F'(x)G(x)\,dx$?
(A) $P = Q$
(B) $P = -Q$
(C) $PQ = 0$
(D) $P = 1 - Q$
(E) $PQ = 1$

Your answer: ____________________________

(6) Consider the integration $\int p(x)q''(x)\,dx$. Apply integration by parts twice, first taking p as the part to differentiate, and q as the part to integrate, and then again apply integration by parts to avoid a circular trap. What can we conclude?
(A) $\int p(x)q''(x)\,dx = \int p''(x)q(x)\,dx$
(B) $\int p(x)q''(x)\,dx = \int p'(x)q'(x)\,dx - \int p''(x)q(x)\,dx$
(C) $\int p(x)q''(x)\,dx = p'(x)q'(x) - \int p''(x)q(x)\,dx$
(D) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) + \int p''(x)q(x)\,dx$
(E) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) - \int p''(x)q(x)\,dx$

Your answer: ____________________________

(7) Suppose p is a polynomial function. In order to find the indefinite integral for a function of the form $x \mapsto p(x)\exp(x)$, the general strategy, which always works, is to take $p(x)$ as the part to differentiate and $\exp(x)$ as the part to integrate, and keep repeating the process. Which of the following is the best explanation for why this strategy works?
(A) exp can be repeatedly differentiated (staying exp) and polynomials can be repeatedly integrated (giving polynomials all the way).
(B) exp can be repeatedly integrated (staying exp) and polynomials can be repeatedly differentiated, eventually becoming zero.
(C) exp and polynomials can both be repeatedly differentiated.
(D) exp and polynomials can both be repeatedly integrated.
(E) We need to use the recursive version of integration by parts whereby the original integrand reappears after a certain number of applications of integration by parts (i.e., the polynomial equals one of its higher derivatives, up to sign and scaling).

Your answer: ____________________________

(8) Consider the function $x \mapsto \exp(x)\sin x$. This function can be integrated using integration by parts. What can we say about how integration by parts works?

(A) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once to get the answer directly.
(B) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(C) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice to get the answer directly.
(D) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(E) We choose exp as the part to integrate and sin as the part to differentiate, and we apply integration by parts four times to get the answer directly.

Your answer: ______________________________

(9) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ______________________________

(10) Suppose f is a continuous function on $(0, \infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0, \infty)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ______________________________

(11) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.
(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Your answer: ______________________________

(12) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin\sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0, 1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**?
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.

(D) f_2 and f_3.
(E) f_2 and f_4.

Your answer: ______________________________

(13) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?
(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2 e^{-x^2}$
(E) $x \mapsto x^4 e^{-x^2}$

Your answer: ______________________________

(14) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?
(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\dots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Your answer: ______________________________

(15) Which of these functions of x is *not* elementarily integrable?
(A) $x\sqrt{1+x^2}$
(B) $x^2\sqrt{1+x^2}$
(C) $x(1+x^2)^{1/3}$
(D) $x\sqrt{1+x^3}$
(E) $x^2\sqrt{1+x^3}$

Your answer: ______________________________

(16) Consider the function $f(k) := \int_1^2 \frac{dx}{\sqrt{x^2+k}}$. f is defined for $k \in (-1, \infty)$. What can we say about the nature of f within this interval?
(A) f is increasing on the interval $(-1, \infty)$.
(B) f is decreasing on the interval $(-1, \infty)$.
(C) f is increasing on $(-1, 0)$ and decreasing on $(0, \infty)$.
(D) f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.
(E) f is increasing on $(-1, 0)$, decreasing on $(0, 2)$, and increasing again on $(2, \infty)$.

Your answer: ______________________________

(17) For which of these functions of x does the antiderivative necessarily involve *both* arctan *and* ln?
(A) $1/(x+1)$
(B) $1/(x^2+1)$

(C) $x/(x^2+1)$
(D) $x/(x^3+1)$
(E) $x^2/(x^3+1)$

Your answer: ____________________________

(18) Suppose F is a (not known) function defined on $\mathbb{R} \setminus \{-1, 0, 1\}$, differentiable everywhere on its domain, such that $F'(x) = 1/(x^3 - x)$ everywhere on $\mathbb{R} \setminus \{-1, 0, 1\}$. For which of the following sets of points is it true that knowing the value of F at these points **uniquely** determines F?
(A) $\{-\pi, -e, 1/e, 1/\pi\}$
(B) $\{-\pi/2, -\sqrt{3}/2, 11/17, \pi^2/6\}$
(C) $\{-\pi^3/7, -\pi^2/6, \sqrt{13}, 11/2\}$
(D) Knowing F at any of the above determines the value of F uniquely.
(E) None of the above works to uniquely determine the value of F.

Your answer: ____________________________

(19) Suppose F is a continuously differentiable function whose domain contains (a, ∞) for some $a \in \mathbb{R}$, and $F'(x)$ is a rational function $p(x)/q(x)$ on the domain of F. Further, suppose that p and q are nonzero polynomials. Denote by d_p the degree of p and by d_q the degree of q. Which of the following is a **necessary and sufficient condition** to ensure that $\lim_{x\to\infty} F(x)$ is finite?
(A) $d_p - d_q \geq 2$
(B) $d_p - d_q \geq 1$
(C) $d_p = d_q$
(D) $d_q - d_p \geq 1$
(E) $d_q - d_p \geq 2$

Your answer: ____________________________

For the next two questions, build on the observation: For any nonconstant monic polynomial $q(x)$, there exists a finite collection of transcendental functions $f_1, f_2, \dots, f_r$ such that the antiderivative of any rational function $p(x)/q(x)$, on an open interval where it is defined and continuous, can be expressed as $g_0 + f_1 g_1 + f_2 g_2 + \cdots + f_r g_r$ where $g_0, g_1, \dots, g_r$ are rational functions.

(20) For the polynomial $q(x) = 1 + x^2$, what collection of f_is works (all are written as functions of x)?
(A) $\arctan x$ and $\ln|x|$
(B) $\arctan x$ and $\arctan(1 + x^2)$
(C) $\ln|x|$ and $\ln(1 + x^2)$
(D) $\arctan x$ and $\ln(1 + x^2)$
(E) $\ln|x|$ and $\arctan(1 + x^2)$

Your answer: ____________________________

(21) For the polynomial $q(x) := 1 + x^2 + x^4$, what is the size of the smallest collection of f_is that works?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE FRIDAY FEBRUARY 22: MULTI-VARIABLE INTEGRATION

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

YOU ARE ALLOWED TO DISCUSS *ONLY* THE STAR-MARKED QUESTIONS!

The following setup is for the first five questions only.

Suppose F is a function of two real variables, say x and t, so $F(x,t)$ is a real number for x and t restricted to suitable open intervals in the real number. Suppose, further, that F is jointly continuous (whatever that means) in x and t.

Define $f(t) := \int_0^\infty F(x,t)\,dx$. Here, while doing the integration, t is treated as a constant. x, the variable of integration, is being integrated on $[0,\infty)$.

Suppose further that f is defined and continuous for t in $(0,\infty)$.

In the next few questions, you are asked to compute the function f explicitly given the function F, for $t \in (0,\infty)$.

(1) *Do not discuss!* $F(x,t) := e^{-tx}$. Find f. *Last time:* 15/19 *correct*
(A) $f(t) = e^{-t}/t$
(B) $f(t) = e^{t}/t$
(C) $f(t) = 1/t$
(D) $f(t) = -1/t$
(E) $f(t) = -t$

Your answer: ________________________

(2) *Do not discuss!* $F(x,t) := 1/(t^2+x^2)$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/(2t)$
(B) $f(t) = \pi/t$
(C) $f(t) = 2\pi/t$
(D) $f(t) = \pi t$
(E) $f(t) = 2\pi t$

Your answer: ________________________

(3) *Do not discuss!* $F(x,t) := 1/(t^2+x^2)^2$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/t^3$
(B) $f(t) = \pi/(2t^3)$
(C) $f(t) = \pi/(4t^3)$
(D) $f(t) = \pi/(8t^3)$
(E) $f(t) = 3\pi/(8t^3)$

Your answer: ________________________

(4) (*) *You can discuss this!* $F(x,t) = \exp(-(tx)^2)$. Use that $\int_0^\infty \exp(-x^2)\,dx = \sqrt{\pi}/2$. Find f. *Last time:* 7/19 *correct*
(A) $f(t) = t^2\sqrt{\pi}/2$
(B) $f(t) = t\sqrt{\pi}/2$
(C) $f(t) = \sqrt{\pi}/2$
(D) $f(t) = \sqrt{\pi}/(2t)$
(E) $f(t) = \sqrt{\pi}/(2t^2)$

Your answer: ____________________________

(5) (**) *You can discuss this!* (could confuse you if you don't understand it): In the same general setup as above (but with none of these specific Fs), which of the following is a *sufficient* condition for f to be an increasing function of t? *Last time:* 3/19 *correct*
(A) $t \mapsto F(x_0, t)$ is an increasing function of t for every choice of $x_0 \geq 0$.
(B) $x \mapsto F(x, t_0)$ is an increasing function of x for every choice of $t_0 \in (0, \infty)$.
(C) $t \mapsto F(x_0, t)$ is a decreasing function of t for every choice of $x_0 \geq 0$.
(D) $x \mapsto F(x, t_0)$ is a decreasing function of x for every choice of $t_0 \in (0, \infty)$.
(E) None of the above.

Your answer: ____________________________

(end of the setup)

(6) *Do not discuss!* Suppose f is a homogeneous polynomial of degree $d > 0$. Define g as the following function on positive reals: $g(a)$ is the double integral of f on the square $[0, a] \times [0, a]$. Assuming that $g(a)$ is not identically the zero function, which of these best describes the nature of $g(a)$? *Last time:* 12/19 *correct.*
(A) A constant times a^d
(B) A constant times a^{d+1}
(C) A constant times a^{d+2}
(D) A constant times a^{2d+1}
(E) A constant times a^{2d+2}

Your answer: ____________________________

(7) (*) *You can discuss this!* Suppose $g(x, y)$ and $G(x, y)$ are continuous functions of two variables and $G_{xy} = g$. How can the double integral $\int_s^t \int_u^v g(x, y)\, dy\, dx$ be described in terms of the values of G? *Last time:* 8/19 *correct*
(A) $G(v, t) + G(u, s) - G(u, t) - G(v, s)$
(B) $G(v, t) - G(v, s) + G(u, t) - G(u, s)$
(C) $G(t, v) + G(s, u) - G(t, u) - G(s, v)$
(D) $G(t, v) - G(s, v) + G(t, u) - G(s, u)$
(E) $G(t, v) + G(v, t) - G(s, u) - G(u, s)$

Your answer: ____________________________

(8) (**) *You can discuss this!* Suppose f is an elementarily integrable function, but $f(x^k)$ is not elementarily integrable for any integer $k > 1$ (examples are sin, exp, cos). For which of the following types of regions D are we *guaranteed to be able* to compute, in elementary function terms, the double integral $\int_D \int f(x^2)\, dA$ over the region (note that f is just a function of x, but we treat it as a function of two variables)? Please see Option (E) before answering and select that if applicable. *Last time:* 1/19 *correct.*
(A) A rectangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, and (a, b), with $a, b > 0$.
(B) A triangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, with $a, b > 0$.
(C) A triangle with vertices $(0, 0)$, $(0, b)$, (a, b), with $a, b > 0$.
(D) A triangle with vertices $(0, 0)$, $(a, 0)$, (a, b), with $a, b > 0$.
(E) All of the above

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE WEDNESDAY FEBRUARY 27: TAYLOR SERIES AND POWER SERIES

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________

FEEL FREE TO DISCUSS ALL QUESTIONS, BUT ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE. DO NOT ENGAGE IN GROUPTHINK.

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We denote by $C^k(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are at least k times continuously differentiable on all of $\mathbb{R}$. Note that for $k \geq l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k.

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

(1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which $\lim_{x \to c} |f(x)|/|x - c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f? *Two years ago:* 11/26 *correct*
(A) The order of any zero of f must be between 0 and 1.
(B) The order of any zero of f must be between 1 and 2.
(C) The order of any zero of f, if finite, must be a positive integer.
(D) The order of any zero of f must be exactly 1.
(E) The order of any zero of f must be ∞.

Your answer: ______________________

(2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on $\mathbb{R}$, what can we say about the Taylor polynomials about 0? *Two years ago:* 8/26 *correct*
(A) No Taylor polynomial is defined for f.
(B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
(C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
(D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
(E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

Your answer: ______________________

(3) Consider the function $F(x, p) = \sum_{n=1}^\infty x^n/n^p$. For fixed p, this is a power series in x. What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0, \infty)$? *Two years ago:* 4/26 *correct*
(A) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
(B) The interval of convergence is $(-1, 1)$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
(C) The interval of convergence is $[-1, 1)$ for $0 < p \leq 1$ and $[-1, 1]$ for $p > 1$.
(D) The interval of convergence is $(-1, 1]$ for $0 < p < 1$ and $[-1, 1]$ for $p \geq 1$.
(E) The interval of convergence is $(-1, 1)$ for $0 < p \leq 1$ and $[-1, 1)$ for $p > 1$.

Your answer: ______________________

(4) Which of the following functions of x has a power series $\sum_{k=0}^\infty x^{4k}/(4k)!$? *Two years ago:* 9/26 *correct*

(A) $(\sin x + \sinh x)/2$
(B) $(\sin x - \sinh x)/2$
(C) $(\sinh x - \sin x)/2$
(D) $(\cosh x + \cos x)/2$
(E) $(\cosh x - \cos x)/2$

Your answer: ____________________________

(5) What is the sum $\sum_{k=0}^{\infty}(-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$. *Two years ago:* 12/26 *correct*
(A) $\cos x$
(B) $\sin x$
(C) $\cos(x^2)$
(D) $\cosh(x^2)$
(E) $\exp(-x^2)$

Your answer: ____________________________

(6) Define an operator R from the set of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b, $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$?
(A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
(B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
(C) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
(D) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
(E) $R(a+b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Your answer: ____________________________

(7) Which of the following is/are true? *Two years ago:* 5/26 *correct*
(A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on $\mathbb{R}$.
(B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence (which may not be all of $\mathbb{R}$).
(C) If we start with a power series about 0 that converges everywhere in $\mathbb{R}$, then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
(D) All of the above.
(E) None of the above.

Your answer: ____________________________

(8) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \to \infty$? *Two years ago:* 12/26 *correct*
(A) f grows polynomially in x.
(B) f grows faster than any polynomial function but slower than any exponential function of x (i.e., any function of the form $x \mapsto e^{mx}, m > 0$).
(C) f grows like an exponential function of x, i.e., it can be sandwiched between two exponentially growing functions of x.
(D) f grows faster than any exponential function but slower than any doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.
(E) f grows like a doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.

Your answer: ___________________________

(9) Consider the function $f(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$? *Two years ago:* 12/26 *correct*

(A) $\exp(x)$ from below and $\cosh(2x)$ from above.
(B) $\exp(x)$ from below and $\cosh(x^2)$ from above.
(C) $\exp(x/2)$ from below and $\exp(x)$ from above.
(D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.
(E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Your answer: ___________________________

(10) Consider the function $f(x) := \max\{0, x\}$. What can we say about the Taylor series of f centered at various points?

(A) The Taylor series of f centered at any point is the zero series.
(B) The Taylor series of f centered at any point simplifies to x.
(C) The Taylor series of f centered at any point other than zero converges to f globally. However, the Taylor series centered at 0 is not defined.
(D) The Taylor series of f centered at any point is either the zero series or simplifies to x.
(E) The Taylor series of f centered at any point other than the point 0 is either the zero series or simplifies to x. However, the Taylor series is not defined at 0.

Your answer: ___________________________

(11) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0? *Two years ago:* 3/26 *correct*

(A) $f_1(x) := \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$
(B) $f_2(x) := \begin{cases} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(C) $f_3(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(D) $f_4(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$
(E) All of the above.

Your answer: ___________________________

(12) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic? *Two years ago:* 7/26 *correct*

(A) $x \mapsto \ln(1 + x^2)$
(B) $x \mapsto \ln(1 + x)$
(C) $x \mapsto \ln(1 - x)$
(D) $x \mapsto \exp(1 + x)$
(E) $x \mapsto \exp(1 - x)$

Your answer: ___________________________

(13) Suppose f and g are globally analytic functions and g is nowhere zero. Which of the following is *not necessarily* globally analytic?

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) fg, i.e., the function $x \mapsto f(x)g(x)$
(D) f/g, i.e., the function $x \mapsto f(x)/g(x)$
(E) $f \circ g$, i.e., the function $x \mapsto f(g(x))$

Your answer: ___________________________

(14) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic? *Two years ago:* 10/26 *correct*

(A) x
(B) x^2
(C) $x+1$
(D) x^2+1
(E) e^x

Your answer: ______________________________

(15) Consider the rational function $1/\prod_{i=1}^{n}(x-\alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_is, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_is? *Two years ago:* 10/26 *correct*

(A) It is the minimum of the distances from c to the α_is.
(B) It is the second smallest of the distances from c to the α_is.
(C) It is the arithmetic mean of the distances from c to the α_is.
(D) It is the second largest of the distances from c to the α_is.
(E) It is the maximum of the distances from c to the α_is.

Your answer: ______________________________

(16) What is the interval of convergence of the Taylor series for arctan about 0? *Two years ago:* 11/26 *correct*

(A) $(-1,1)$
(B) $[-1,1)$
(C) $(-1,1]$
(D) $[-1,1]$
(E) All of $\mathbb{R}$

Your answer: ______________________________

(17) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} 2^{\sqrt{k}} x^k$? Please keep in mind the square root in the exponent.

(A) 0
(B) $1/2$
(C) $1/\sqrt{2}$
(D) 1
(E) infinite

Your answer: ______________________________

TAKE-HOME CLASS QUIZ: DUE FRIDAY MARCH 1: MAX/MIN VALUES: ONE-VARIABLE RECALL

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER CHOICES THAT YOU PERSONALLY ENDORSE.

(1) Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?
(A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
(B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
(C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
(D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
(E) None of the above is true in general.

Your answer: ____________________________

(2) Suppose f is a continuous function defined on an open interval (a, b) and c is a point in (a, b). Which of the following implications is **true**?
(A) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a, b)$ such that f is non-increasing on $(c-\delta, c)$ and non-decreasing on $(c, c+\delta)$.
(B) If there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a, b)$ such that f is non-increasing on $(c-\delta, c)$ and non-decreasing on $(c, c+\delta)$, then c is a point of local minimum for f.
(C) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a, b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$.
(D) If there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a, b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$, then c is a point of local minimum for f.
(E) All of the above are true.

Your answer: ____________________________

(3) Consider all the rectangles with perimeter equal to a fixed length $p > 0$. Which of the following **is true** for the unique rectangle which is a square, compared to the other rectangles?
(A) It has the largest area and the largest length of diagonal.
(B) It has the largest area and the smallest length of diagonal.
(C) It has the smallest area and the largest length of diagonal.
(D) It has the smallest area and the smallest length of diagonal.
(E) None of the above.

Your answer: ____________________________

PLEASE TURN OVER FOR REMAINING QUESTIONS

(4) Suppose the total perimeter of a square and an equilateral triangle is L. (We can choose to allocate all of L to the square, in which case the equilateral triangle has side zero, and we can choose to allocate all of L to the equilateral triangle, in which case the square has side zero). Which of the following statements **is true** for the sum of the areas of the square and the equilateral triangle? (The area of an equilateral triangle is $\sqrt{3}/4$ times the square of the length of its side).

(A) The sum is minimum when all of L is allocated to the square.
(B) The sum is maximum when all of L is allocated to the square.
(C) The sum is minimum when all of L is allocated to the equilateral triangle.
(D) The sum is maximum when all of L is allocated to the equilateral triangle.
(E) None of the above.

Your answer: ____________________________

(5) Suppose x and y are positive numbers such as $x + y = 12$. For **what values** of x and y is x^2y maximum?

(A) $x = 3$, $y = 9$
(B) $x = 4$, $y = 8$
(C) $x = 6$, $y = 6$
(D) $x = 8$, $y = 4$
(E) $x = 9$, $y = 3$

Your answer: ____________________________

(6) Consider the function $p(x) := x^2 + bx + c$, with x restricted to integer inputs. Suppose b and c are integers. The minimum value of p is attained either at a single integer or at two consecutive integers. Which of the following is a **sufficient condition** for the minimum to occur at two consecutive integers?

(A) b is odd
(B) b is even
(C) c is odd
(D) c is even
(E) None of these conditions is sufficient.

Your answer: ____________________________

TAKE-HOME CLASS QUIZ: DUE MONDAY MARCH 11: MAX-MIN VALUES: TWO-VARIABLE VERSION

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY ENDORSE. PLEASE DO *NOT* ENGAGE IN GROUPTHINK.

(1) Suppose $F(x,y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are differentiable functions of one variable, defined for all real numbers. What can we say about the critical points of F in its domain $\mathbb{R}^2$?
(A) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *or* y_0 is a critical point for g.
(B) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *and* y_0 is a critical point for g.
(C) F has a critical point at (x_0, y_0) iff $x_0 + y_0$ is a critical point for $f + g$, i.e., the function $x \mapsto f(x) + g(x)$.
(D) F has a critical point at (x_0, y_0) iff $x_0 y_0$ is a critical point for fg, i.e., the function $x \mapsto f(x)g(x)$.
(E) None of the above.

Your answer: ___________________________

(2) Suppose $F(x,y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both differentiable functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following is true?
(A) F has a critical point at (x_0, y_0) if and only if x_0 is a critical point for f and y_0 is a critical point for g.
(B) If x_0 is a critical point for f and y_0 is a critical point for g, then (x_0, y_0) is a critical point for F. However, the converse is not necessarily true, i.e., (x_0, y_0) may be a critical point for F even without x_0 being a critical point for f and y_0 being a critical point for g.
(C) If (x_0, y_0) is a critical point for F, then x_0 must be a critical point for f and y_0 must be a critical point for g. However, the converse is not necessarily true.
(D) (x_0, y_0) is a critical point for F if and only if *at least* one of these is true: x_0 is a critical point for f and y_0 is a critical point for g.
(E) None of the above.

Your answer: ___________________________

(3) Consider a homogeneous polynomial $ax^2 + bxy + cy^2$ of degree two in two variables x and y. Assume that at least one of the numbers a, b, and c is nonzero. What can we say about the local extreme values of this polynomial on $\mathbb{R}^2$?
(A) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.
(B) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.

(C) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained only at the origin.
(D) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.
(E) If $b^2 - 4ac = 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.

Your answer: ____________________________

A subset of $\mathbb{R}^n$ is termed *convex* if the line segment joining any two points in the subset is completely within the subset. A function f of two variables defined on a closed convex domain is termed *quasiconvex* if given any two points P and Q in the domain, the maximum of f restricted to the line segment joining P and Q is attained at one (possibly both) of the endpoints P or Q.

There are many examples of quasiconvex functions, including linear functions (which are quasiconvex but not strictly quasiconvex) and all convex functions.

(4) What can we say about the maximum of a continuous quasiconvex function defined on the circular disk $x^2 + y^2 \leq 1$?
(A) It must be attained at the center of the disk, i.e., the origin $(0, 0)$.
(B) It must be attained somewhere in the interior of the disk, but we cannot be more specific with the given information.
(C) It must be attained somewhere on the boundary circle $x^2 + y^2 = 1$. However, we cannot be more specific than that with the given information.
(D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
(E) It could be attained at any point. We cannot be specific at all.

Your answer: ____________________________

(5) What can we say about the maximum of a continuous quasiconvex function defined on the square region $|x| + |y| \leq 1$? This is the region bounded by the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
(A) It must be attained at the center of the square, i.e., the origin $(0, 0)$.
(B) It must be attained somewhere in the interior of the square, but we cannot be more specific with the given information.
(C) It must be attained somewhere on the boundary square $|x| + |y| \leq 1$. However, we cannot be more specific than that with the given information.
(D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
(E) It could be attained at any point. We cannot be specific at all.

Your answer: ____________________________

(6) Suppose $F(x, y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are continuous functions of one variable, defined for all real numbers. Which of the following statements about local extrema of F is **false**?
(A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
(B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).

(C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
(D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).
(E) None of the above, i.e., they are all true.

Your answer: ____________________________

(7) Suppose $F(x, y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both continuous functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following statements about local extrema is **true**?
(A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
(B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).
(C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
(D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).
(E) None of the above, i.e., they are all false.

Your answer: ____________________________