POLAR COORDINATES

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 10.3.

What students should definitely get: The information needed to set up a polar coordinate system, how to go between a physical point and its polar coordinates, conversion back and forth between Cartesian and polar coordinates. Thinking of polar coordinate descriptions as parametric descriptions when viewed in Cartesian coordinates. Using this to compute the slope of the tangent line.

What students should hopefully get: The dimensionality of the plane is 2, so 2 parameters are needed to describe a point in any decent coordinate system. What happens when we fix one coordinate in a Cartesian or polar coordinate system. Why spirals are easy to describe in polar coordinates. How various symmetries in implicit and functional descriptions correspond to geometric symmetries.

Executive summary

Words ...

(1) *Specifying a polar coordinate system*: To specify a polar coordinate system, we need to choose a point (called the *origin* or *pole*), a half-line starting at the point (called the *polar axis* or *reference line*) and an orientation of the plane (chosen counter-clockwise in the usual depictions).
(2) *Finding the polar coordinates of a point and vice versa*: The radial coordinate r is the distance between the point and the pole. The angular coordinate θ is the angle (measured in the counter-clockwise direction) from the polar axis to the line segment from the pole to the point. Note that θ is uniquely defined up to addition of multiples of 2π, and it becomes truly unique if we restrict it to a half-open half-closed interval of length 2π. *The exception is the pole itself, for which θ is undefined* in the sense that any value of θ could be chosen.
(3) *Converting between Cartesian and polar coordinates*: If we take the polar axis as the positive x-axis and the axis at an angle of $+\pi/2$ from it as the positive y-axis, we get a Cartesian coordinate system. The point defined by polar coordinates (r, θ) has Cartesian coordinates $(r\cos\theta, r\sin\theta)$. Conversely, given a point with Cartesian coordinates (x, y) the corresponding polar coordinates are $r = \sqrt{x^2 + y^2}$ and θ is the unique angle (up to addition of multiples of 2π) such that $x = r\cos\theta$, $y = r\sin\theta$.

Actions ...

(1) A functional description of the form $r = F(\theta)$ gives rise to a parametric description in Cartesian coordinates: $x = F(\theta)\cos\theta$ and $y = F(\theta)\sin\theta$. We can do the usual things (like find slopes of tangent lines) using this parametric description. Note that here, θ is typically allowed to vary over all of $\mathbb{R}$ rather than simply being restricted to an interval of length 2π. The slope of the tangnet line in Cartesian terms is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d(F(\theta)\sin\theta)/d\theta}{d(F(\theta)\cos\theta)/d\theta} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$$

(2) An implicit (relational) description in Cartesian coordinates can be converted to a description in polar coordinates by replacing x by $r\cos\theta$ and y by $r\sin\theta$.
(3) An implicit (relational) description in polar coordinates can sometimes be converted to a description in Cartesian coordinates, but with some ambiguity. General idea: replace r by $\sqrt{x^2 + y^2}$, $\cos\theta$ by $x/\sqrt{x^2 + y^2}$, and $\sin\theta$ by $y/\sqrt{x^2 + y^2}$.

1. Review of Cartesian coordinates

1.1. Descartes' achievement. We're so used to Cartesian coordinates that we don't often give them much thought. But the idea of using Cartesian coordinates to describe a plane, or to describe three-dimensional space, was a fundamental breakthrough when it did occur. The idea here being that something as geometric as a plane or space could be represented purely by a tuple of real numbers.

There are some aspects of the *rectangular Cartesian coordinate system* that are worth disaggregating:

(1) The *number* of parameters used is equal to the *dimensionality* of the system being studied. The concept of dimensionality as the *number of free parameters* or *number of degrees of freedom* is probably not something totally new to you.
(2) In order to actually specify a Cartesian coordinate system, we need to choose a pair of orthogonal lines, an ordering of these lines, and a direction to be labeled positive within each line.
(3) Once we choose an origin and an ordered pair of orthogonal directed lines through it, we can use Cartesian coordinates to proceed from an ordered pair of real numbers to a point in the plane, and back. The two procedures are reverses of each other.
(4) Different choices of origin and different choices of direction for the pair of perpendicular lines give different choices of Cartesian coordinate systems. Moving from one to the other *geometrically* corresponds to translations, rotations, and reflections. Moving from one to the other *algebraically* corresponds to some specific algebraic operations on the coordinates.

The most important of these ideas is (1). In some sense, the dimensionality of the plane – namely 2, is far more fundamental than the specific choice of coordinate system used. We will soon construct a new kind of coordinate system called a *polar coordinate system.* This looks very different from a Cartesian coordinate system, but the number of real parameters needed to describe a point remains 2.

1.2. **Fixing one coordinate in the Cartesian system.** Let's consider another aspect of the Cartesian coordinate system. A point in the Cartesian coordinate system is given by a pair of coordinates (x, y). What happens if we fix one coordinate and let the other vary over $\mathbb{R}$. Specifically:

- If we fix a value of x to x_0 and let y vary over $\mathbb{R}$, we get a vertical line given by $x = x_0$. For different choices of x_0, we get parallel lines. Overall, we get a family of parallel lines.
- If we fix a value of y to y_0 and let x vary over $\mathbb{R}$, we get a horizontal line given by $y = y_0$. Overall, we get a family of parallel lines.

2. Polar coordinates

2.1. **The key definitions.** The key idea behind polar coordinates is to specify the *distance* from a specified origin and the *direction* of the line joining the point to the origin.

To create a polar coordinate system, we need the following pieces of data:

- A point selected as the *pole* or *origin* for the coordinate system.
- A half-line (ray) with the point at its endpoint. We will call this the *polar axis* or *reference line.*
- An orientation (counter-clockwise) on the plane.

Every point has two coordinates:

- The *radial coordinate*, denoted r, which is the distance from the origin to that point.
- The *angular coordinate*, denoted θ, which is the angle made between the reference line and the line segment joining the origin to that point, measured counter-clockwise.

Some important notes:

- The radial coordinate is a nonnegative real number.
- The angular coordinate for the pole is not defined. In fact, *any* value of θ could be used for the pole and it would serve to describe the pole. This is a kind of degeneracy or singularity.
- For any other point, the angular coordinate is unique up to multiples of 2π. To make it truly unique, we usually adopt the convention that the angle θ must satisfy $0 \leq \theta < 2\pi$.

The role of coordinates in a polar coordinate system is asymmetric. The r-coordinate is a length coordinate, and the θ-coordinate is a dimensionless angle coordinate. This contrasts with the Cartesian coordinate system where both coordinates play a symmetric role as lengths.

2.2. **What happens if we fix one coordinate?** We note that:

- If we fix a given value of r but allow θ to vary freely, we get a circle centered at the origin. The exceptional case $r = 0$ gives us the single point namely the origin. Thus, we get a family of concentric circles centered at the origin.
- If we fix a given value of θ but allow r to vary freely, we get a ray (half-line) starting at the origin.

2.3. **Polar and Cartesian coordinates: conversion.** Every polar coordinate system has a corresponding Cartesian coordinate system, and vice versa. For a Cartesian coordinate system, we convert to a polar coordinate system by selecting the same origin, taking the reference line as the positive x-axis, and choosing the orientation as counter-clockwise, from the positive x-axis to the positive y-axis.

With this back-and-forth, here are the conversion rules:

- The Cartesian coordinates (x, y) gives $r = \sqrt{x^2 + y^2}$ and θ is the angle $0 \le \theta < 2\pi$ such that $r \cos\theta = x$ and $r \sin\theta = y$.
- The polar coordinates (r, θ) gives the Cartesian coordinates $x = r\cos\theta$ and $y = r \sin\theta$.

With these conversion rules, we can derive formulas involving polar coordinates from the corresponding formulas involving Cartesian coordinates.

2.4. **Polar coordinates with negative radial coordinate.** A slight variation on the polar coordinate theme is the case where we consider polar coordinates with *negative* radial coordinate value. Here r is the negative of the distance from the pole to the point, and θ is the usual θ shifted by π, so it is the angle made from the polar axis to the half-line in the *opposite* direction to that joining the pole to the point.

3. Descriptions of curves in polar coordinates

3.1. **r as a function of θ.** When we give this kind of description, we *usually* allow θ to vary over all real numbers, rather than just restrict it to an interval of length 2π. Typically, the curves for which these descriptions work well are *spirals* starting out at the origin and spiraling outward. For instance, the curve with equation $r = e^\theta$ is a spiral.

One way of thinking of these curves in terms of the *usual Cartesian coordinate system* is in parametric terms – θ is a parameter. If we denote $r = F(\theta)$, then the two coordinate functions x and y are given by $x = F(\theta)\cos\theta$ and $y = F(\theta)\sin\theta$.

3.2. **Case of negative r values.** In some cases, the expression $r = F(\theta)$ gives rise to negative r-values for some values of θ. In this case, we interpret these the way we discussed the interpretation of negative r-values.

3.3. **Slope of tangent line in Cartesian terms.** We can determine the slope of the tangent line as follows:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d(F(\theta)\sin\theta)/d\theta}{d(F(\theta)\cos\theta)/d\theta} = \frac{F'(\theta)\sin\theta + F(\theta)\cos\theta}{F'(\theta)\cos\theta - F(\theta)\sin\theta}$$

Using this, we can determine the equation of the tangent line in a Cartesian coordinate system.

3.4. **Relational description.** In addition to functional descriptions of the form $r = F(\theta)$, we could more generally have implicit (relational) descriptions between r and θ, i.e., descriptions of the form $H(r, \theta) = 0$ where H is a function of two variables. These are more general than functional descriptions.

Note that it is generally possible to convert a relational description in a Cartesian coordinate system to a relational description in a polar coordinate system. Starting with a relational description $G(x, y) = 0$ in a Cartesian coordinate system we get the corresponding polar relation by setting $x = r\cos\theta$ and $y = r\sin\theta$. For instance, the parabola $y = x^2$, in polar coordinates, becomes:

$$r \sin\theta = r^2 \cos^2\theta$$

which simplifies to:

$$r(\sin\theta - r\cos^2\theta) = 0$$

It turns out that the r solution is subsumed in the other, so we get:

$$\sin\theta = r\cos^2\theta$$

Similarly if we have the equation:

$$xy = \sin(x^2 + y^2)$$

Then, in polar coordinates, this becomes:

$$r^2\cos\theta\sin\theta = \sin(r^2\cos^2\theta + r^2\sin^2\theta)$$

This simplifies to:

$$r^2\cos\theta\sin\theta = \sin(r^2)$$

Going the other way around, the rule is to plug in $r = \sqrt{x^2+y^2}$, $\cos\theta = x/\sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$. Note that directly plugging θ in reverse is tricky, because there is no single shorthand expression for θ – the expression depends on the signs of x and y and gets a little messy.

3.5. **Symmetries for polar equations.** Here are two symmetries of note:

- *Mirror symmetry*: A functional or relational description is symmetric about the polar axis (the reference line) if it satisfies the condition that the condition is satisfied by replacing θ with $-\theta$. More generally, if replacing θ by $2\alpha - \theta$ preserves the condition, then there is symmetry about the line $\theta = \alpha$. Thus, for instance, if replacing θ by $\pi - \theta$ preserves the condition, the curve has mirror symmetry about the y-axis.
- *Half turn symmetry*: If replacing θ by $\theta + \pi$ preserves the condition, then the curve has half turn symmetry about the pole (origin). Similarly, if replacing r by $-r$ preserves the condition, then the curve has half turn symmetry about the pole (origin).