MAXIMUM AND MINIMUM VALUES: EXAMPLES

MATH 195, SECTION 59 (VIPUL NAIK)

What students should hopefully get: The description of critical points, local extreme values, and absolute extreme values for additively separable functions and (the more complicated version for) multiplicatively separable functions. The special nature of extreme values for quasiconvex and strictly quasiconvex functions and the notion of extreme points. The nature of extreme values for linear, quadratic, and homogeneous polynomials. The use of Lagrange multipliers to find extrema on the boundary.

Executive summary

(1) *Additively separable, critical points*: For an additively separable function $F(x,y) := f(x) + g(y)$, the critical points of F are the points whose x-coordinate gives a critical point for f and y-coordinate gives a critical point for g.
(2) *Additively separable, local extreme values*: The local maxima occur at points whose x-coordinate gives a local maximum for f and y-coordinates gives a local maximum for g. Similarly for local minima. If one coordinate gives a local maximum and the other coordinate gives a local minimum, we get a saddle point.
(3) *Additively separable, absolute extreme values*: If the domain is a rectangular region, rectangular strip, or the whole plane, then the absolute maximum occurs at the point for which each coordinate gives the absolute maximum for that coordinate, and analogously for absolute minimum. This does *not* work for non-rectangular regions in general.
(4) *Multiplicatively separable, critical points*: For a multiplicatively separable function $F(x,y) := f(x)g(y)$ with f, g, differentiable, there are four kinds of critical points (x_0, y_0): (1) $f'(x_0) = g'(y_0) = 0$, (2) $f(x_0) = f'(x_0) = 0$, (3) $g(y_0) = g'(y_0) = 0$, (4) $f(x_0) = g(y_0) = 0$.
(5) *Multiplicatively separable, local extreme values*: At a critical point of Type (1), the nature of local extreme value for F depends on the signs of f and g *and* on the nature of local extreme values for each. See the table. Critical points of Type (4) alone do not give local extreme values. The situation with critical points of Types (2) and (3) is more ambiguous and too complicated for discussion.
(6) *Multiplicatively separable, absolute extreme values*: Often, these don't exist, if one function takes arbitrarily large magnitude values and the other one takes nonzero values (details based on sign). If both functions are everywhere positive, and we are on a rectangular region, then the absolute maximum/minimum for the product occur at points whose coordinates give respective absolute maximum/minimum for f and g. (See notes)
(7) For a continuous quasiconvex function on a convex domain, the maximum must occur at one of the extreme points, in particular on the boundary. If the function is strictly quasiconvex, the maximum can occur only at a boundary point.
(8) For a continuous quasiconvex function on a convex domain, the minimum must occur on a convex subset. If the function is strictly quasiconvex, it must occur at a unique point.
(9) Linear functions are quasiconvex but not strictly so. The negative of a linear function is also quasiconvex. The maximum and minimum for linear functions on convex domains must occur at extreme points.
(10) To find maxima/minima on the boundary, we can use the method of Lagrange multipliers.

See also: tables, discussion for linear, quadratic, and homogeneous functions (hard to summarize).

1. Additively separable functions

1.1. **Partial derivatives, critical points and Hessian.** Consider a function of the form $F(x,y) := f(x) + g(y)$, i.e., F is additively separable. Then, $F_x(x,y) = f'(x)$, $F_y(x,y) = g'(y)$, $F_{xx}(x,y) = f''(x)$, $F_{yy}(x,y) = g''(y)$, and $F_{xy}(x,y) = 0$.

Note that in this case, the Hessian (the determinant used to determine the nature of extreme value at a critical point) is simply the product $F_{xx}F_{yy} = f''(x)g''(y)$.

We have the following:

> The critical points for F are precisely the points whose x-coordinate gives a critical point for f and y-coordinate gives a critical point for g. In other words, (x_0, y_0) in the domain of F gives a critical point if and only if x_0 gives a critical point for f and y_0 gives a critical point for g.

1.2. **Local extreme values.** We can say the following about local extreme values:

- The points of local maximum for F are precisely the points whose x-coordinate gives a local maximum for f and whose y-coordinate gives a local maximum for g. In other words, (x_0, y_0) in the domain of F gives a local maximum if and only if x_0 gives a local maximum for f and y_0 gives a local maximum for g.
- The points of local minimum for F are precisely the points whose x-coordinate gives a local minimum for f and whose y-coordinate gives a local minimum for g. In other words, (x_0, y_0) in the domain of F gives a local minimum if and only if x_0 gives a local minimum for f and y_0 gives a local minimum for g.
- In particular, a critical point gives a saddle point if any of these conditions hold: it is not a point of local extremum in one of the variables (e.g., a point of inflection type), *or* it is a point of local extrema of opposite kinds in the two variables. For instance, for the function $(x-1)^2 - (y-2)^2$, the point $(1,2)$ is a point of local minimum for the first coordinate but local maximum for the second, so we get a saddle point overall.
- For an additively separable function, the second derivative test simply boils down to checking whether $f''(x)$ and $g''(y)$ have the same sign. This is because the *interaction term* arising as a *mixed partial* is *absent*.

1.3. **Absolute extreme values.** Continuing notation from above, we note that:

- If the domain of F is rectangular (or the whole plane or a rectangular infinite strip) then the absolute maximum value for F occurs at a point whose x-coordinate maximizes f and whose y-coordinate maximizes g.
- If the domain of F is rectangular (or the whole plane or a rectangular infinite strip) then the absolute minimum value for F occurs at a point whose x-coordinate minimizes f and whose y-coordinate minimizes g.

These results don't hold for non-rectangular domains because we cannot carry out separate analysis of the variables. For instance, consider the function $x + y$ on the circular disk $x^2 + y^2 \leq 1$. The maximum for x occurs at $x = 1$, and the maximum for y occurs at $y = 1$. However, the point $(1,1)$ lies outside the domain of the function.

We will deal with non-rectangular regions in more detail a little later.

1.4. **Examples.** Consider the function:

$$F(x,y) := x^2 - 3x + \sin^2 y$$

This is the sum of the functions $f(x) := x^2 - 3x$ and $g(y) := \sin^2 y$. f attains its local and absolute minimum at $x_0 = 3/2$ with value $-9/4$, and it has no local or absolute maximum. g attains its local and absolute minima at multiples of π with value 0, and its local and absolute maxima at odd multiples of $\pi/2$, with value 1.

The upshot is that:

- F attains its local and absolute minima at points of the form $(3/2, n\pi)$, n an integer. This is because f is minimum on the x-coordinate and g is minimum on the y-coordinate.

- F has saddle points at $(3/2, n\pi + \pi/2)$. This is because f is minimum on the x-coordinate and g is maximum on the y-coordinate.

1.5. **Key observation: cases where second derivative test doesn't work.** Consider the additively separable function:

$$F(x,y) = (x-1)^3 - (y-2)^2$$

The function has a unique critical point for the point $(1,2)$ in the domain. If we didn't notice additive separability, and directly tried to compute the Hessian, we'd get 0, indicating that the second derivative test is inconclusive.

We note that this is the sum of the functions $f(x) := (x-1)^3$ and $g(y) := -(y-2)^2$. Since we are now dealing with functions of *one* variable, we have methods other than the second derivative test (for instance, the first derivative test or higher derivative tests) to find out whether a given critical point gives a local extreme value. In this case we figure that $x = 1$ gives a point of inflection and *not* a local extreme value for f, whereas $y = 2$ gives a local maximum for g. Thus, overall, we conclude that $(1,2)$ gives a saddle point.

In other words, for additively separable functions, we can go beyond the second derivative test using our knowledge of functions of one variable, despite our ignorance of analogous results for functions of two variables.

2. Multiplicatively separable functions

2.1. **Partial derivatives, critical points, and Hessian.** Consider a function of the form $F(x,y) := f(x)g(y)$, i.e., F is multiplicatively separable. Then, $F_x(x,y) = f'(x)g(y)$, $F_y(x,y) = f(x)g'(y)$, $F_{xx}(x,y) = f''(x)g(y)$, $F_{xy}(x,y) = f'(x)g'(y)$, and $F_{yy}(x,y) = f(x)g''(y)$.

The Hessian determinant (used for the second derivative test) at a point (x_0, y_0) thus becomes:

$$f(x_0)g(y_0)f''(x_0)g''(y_0) - [f'(x_0)g'(y_0)]^2$$

We first try to figure out the necessary and sufficient conditions for a point (x_0, y_0) to be a critical point for F. This happens iff $f'(x_0)g(y_0) = 0$ and $f(x_0)g'(y_0) = 0$. This could occur for four different reasons. We provide each reason along with an interpretation.

1. $f'(x_0) = g'(y_0) = 0$: This means that x_0 is a critical point for f and y_0 is a critical point for g.
2. $f(x_0) = f'(x_0) = 0$: Note that in this case the partial with respect to y is 0 at the point, not because of y_0, but because of x_0. What's happening is that on the line $x = x_0$, the function is identically zero, so changes in g do not matter.
3. $g(y_0) = g'(y_0) = 0$: Note that in this case the partial with respect to x is 0 at the point, not because of x_0, but because of y_0. What's happening is that on the line $y = y_0$, the function is identically zero, so changes in f do not matter.
4. $f(x_0) = 0$ and $g(y_0) = 0$: In this case, the function is identically zero along both the vertical and the horizontal line containing (x_0, y_0).

Note that any critical point that is of Type (4) above but not any of the preceding types must *fail* the second derivative test. For a critical point of Type (2) or (3) above, the second derivative test is inconclusive because we get 0 (more is discussed in the next subsection). For a critical point of Type (1), the second derivative test is most useful. Note that for such a critical point, the Hessian determinant simply becomes $f(x_0)g(y_0)f''(x_0)g''(y_0)$, so its sign depends not only on the signs of the second derivatives of f and g but *also* on the signs of the functions f and g themselves.

2.2. **Local extreme values: Type 1 case.** The following table gives conclusions for the nature of local extreme values of $F(x,y) = f(x)g(y)$ at (x_0, y_0) if x_0 gives a local extreme value for f and y_0 gives a local extreme value for g.

$f(x_0)$ sign	$g(y_0)$ sign	$f(x_0)$ (local max/min)	$g(y_0)$ (local max/min)	$F(x_0, y_0)$ (local max/min/saddle)
positive	positive	local max	local max	local max
positive	positive	local max	local min	saddle point
positive	positive	local min	local max	saddle point
positive	positive	local min	local min	local min
positive	negative	local max	local max	saddle point
positive	negative	local max	local min	local min
positive	negative	local min	local max	local max
positive	negative	local min	local min	saddle point
negative	positive	local max	local max	saddle point
negative	positive	local max	local min	local max
negative	positive	local min	local max	local min
negative	positive	local min	local min	saddle point
negative	negative	local max	local max	local min
negative	negative	local max	local min	saddle point
negative	negative	local min	local max	saddle point
negative	negative	local min	local min	local max

Note that the conclusion about F depends not merely on whether f and g have local max/min but also on the sign of the local max/min for f. The *saddle point cases* arise when f and g are pulling (multiplicatively) in opposite directions. Here, the function is a local maximum along one of the x- and y-directions and a local minimum along the other.

In cases where the second derivative test is conclusive for both f and g as functions of one variable, the above observations can be cross-checked by looking at the sign of the Hessian, which is $f(x_0)g(y_0)f''(x_0)g''(y_0)$, and of $F_{xx} = g(y_0)f''(x_0)$ and $F_{yy} = f(x_0)g''(y_0)$.

We do two of the sixteen examples for illustration:

- If f has a positive local maximum at x_0 and g has a negative local maximum at y_0, then F has a saddle point: We get $f(x_0) > 0$, $g(y_0) < 0$, $f''(x_0) < 0$, $g''(y_0) < 0$. So, the Hessian is negative (product of one positive and three negatives). Thus, by the second derivative test, F has a saddle point.
- If f and g both have negative local minima, then F has a local maximum. Here's how we see this: we get $f(x_0) < 0$, $g(y_0) < 0$, $f''(x_0) > 0$ (local minimum), $g''(y_0) > 0$ (local minimum), so multiplying all the signs, we see that the Hessian is positive. Thus, the function does attain a local extreme value. Next, we look at the sign of $F_{xx}(x_0, y_0)$, which is $g(y_0)f''(x_0)$. This is negative, since it is the product of a negative and a positive number. Thus, F has a local minimum.

It's important to keep in mind that the statements in the table *are more general and apply even when the second derivative tests are inconclusive.* We'll be looking at some examples shortly.

2.3. **Critical points of Types 2 and 3.** We now turn to the situation $F(x, y) = f(x)g(y)$ where there are points x_0 satisfying $f(x_0) = f'(x_0) = 0$. In this case, the second derivative test is inconclusive because the Hessian determinant takes the value 0.

Let's try to examine what's happening near the point. On the line $x = x_0$, the function is constant at 0. That explains why $F_y(x_0, y_0) = 0$ – the function is not changing along the y-direction because it's the product of $f(x_0)$, which is zero, and a changing number. On the line $y = y_0$, the function has derivative zero because $f'(x_0) = 0$, so that is why $F_x(x_0, y_0) = 0$.

Now, the first condition we need to obtain a local minimum is that the function be a local minimum under slight perturbations in the x-direction. So, we would like that the function $x \mapsto f(x)g(y_0)$ have a local minimum at x_0. If $g(y_0) > 0$, this is equivalent to wanting f to have a local minimum at x_0. If $g(y_0) < 0$, this is equivalent to wanting f to have a local maximum at x_0. In fact, as long as $g(y_0) \neq 0$, these are necessary and sufficient conditions to impose. Let's make this explicit in a table:

$g(y_0)$ sign	$f(x_0)$ (local max/min) at point with $f(x_0) = f'(x_0) = 0$	$F(x_0, y_0)$ (local max/min)
positive	local max	local max
negative	local max	local min
positive	local min	local min
negative	local min	local max

2.4. **Absolute extreme values.** Here are some results on the *non-existence* of absolute extreme values:

- If f takes a positive value anywhere on its domain, and g takes arbitrarily large positive values, then F takes arbitrarily large positive values, and hence has no absolute maximum.
- If f takes a negative value anywhere on its domain, and g takes arbitrarily large positive values, then F takes arbitrarily large magnitude negative values, and hence has no absolute minimum.
- If f takes a positive value anywhere on its domain, and g takes arbitrarily large magntiude negative values, then F takes arbitrarily large magnitude negative values, and hence has no absolute minimum.
- If f takes a negative value anywhere on its domain, and g takes arbitrarily large magnitude negative values, then F takes arbitrarily large positive values, and hence has no absolute maximum.

To each of the above, an analogous statement holds if we interchange the roles of f and g.

On the other hand, the following is true: if both f and g are everywhere positive, and the domain is a rectangular region, then the absolute minimum for F occurs at the point whose x-coordinate gives the absolute minimum for f and whose y-coordinate gives the absolute minimum for g.

2.5. **Examples (Type 1 critical points only).** Consider the function:

$$F(x, y) := (x^2 - x + 2)(3 + \cos y)$$

F is multiplicatively separable and can be written as $f(x)g(y)$ where $f(x) = x^2 - x + 2$ and $g(y) = 3 + \cos y$.

f has a unique critical point with a local and absolute *minimum* at $x = 1/2$, and the value of the minimum is $7/4$.

As for g, it attains its local and absolute maxima at multiples of 2π (with value 4) and its local and absolute minima at odd multiples of π (with value 2).

Note that there are no critical points of types (2), (3), and (4). This is because f is never 0 and further, g and g' are never simultaneously 0.

We can see from this that:

- F has local and absolute minimum attained at points of the form $(1/2, (2n + 1)\pi)$ with value $7/2$. This is because f has a local and absolute *positive* minimum at the point $1/2$ and g has a local and absolute *positive* minimum at the point $(2n + 1)\pi$.
- F has saddle points at points of the form $(1/2, 2n\pi)$ with value 7. This is because f has a local minimum at $1/2$ and g has a local maximum at $2n\pi$.
- F has no absolute maximum. To see this, note that f is unbounded from above and g takes values in $[2, 4]$.

Consider a very similar example:

$$F(x, y) := (x^2 - x + 2) \cos y$$

This is similar to the previous example except that the $3 + \cos y$ is replaced by $\cos y$. We take $f(x) = x^2 - x + 2$ and $g(y) = \cos y$. f has a unique local and absolute minimum at $x_0 = 1/2$ with value $7/4$. g has local and absolute maxima at even multiples of π with value 1, and local and absolute minima at odd multiples of π with value -1.

From this, we conclude that:

- F has *no* local maxima or minima. To see this, note that both the $(1/2, 2n\pi)$ and the $(1/2, (2n+1)\pi)$ cases give saddle points. For $(1/2, 2n\pi)$, we get minimum and maximum, all positive, which gives saddle points. For $(1/2, (2n+1)\pi)$, we get minimum and minimum, positive and negative respectively, which again gives saddle points.
- There are no absolute maxima and minima either. To see this, note that f is unbounded from above, and g takes values in $[-1, 1]$, so F can take arbitrarily large positive and negative values.

2.6. **Example (Type 1 and Type 4 critical points).** Consider the function:

$$F(x,y) := (x-2)(x-4)(y+1)(y-5)$$

This is the product of the function $f(x) := (x-2)(x-4)$ and $g(y) := (y+1)(y-5)$.

f has a unique critical point at $x_0 = 3$ and g has a unique critical point at $y_0 = 2$. So F has a unique critical point of Type 1 (with $f'(x_0) = g'(y_0) = 0$), namely the point $(3,2)$ in the domain. Since both f and g have *negative* local *minima* at the point, we conclude that F has a *positive* local *maximum* at the point.

However, there are also other kinds of critical points, specifically Type 4 critical points where $f(x_0) = g(y_0) = 0$. There are in fact four such critical points: $(2,-1)$, $(2,5)$, $(4,-1)$, and $(4,5)$.

As already mentioned earlier, the critical points that are only Type 4 critical points *cannot* must give saddle points, so we obtain saddle points at all these four points. The upshot:

- There is a unique positive local maximum at $(3,2)$.
- There are four saddle points: $(2,-1)$, $(2,5)$, $(4,-1)$, and $(4,5)$.
- There is *no* absolute maximum or minimum. To see this, note that f takes both a positive and a negative value, and g takes arbitrarily large positive values, so we can arrange the product to be a positive or a negative number of arbitrarily large magnitude.

2.7. **We don't need no second derivative test.** Consider a function such as:

$$F(x,y) := ((x-1)^4 - 1)((y+3)^6 - 2)$$

If we directly tried to use the second derivative test on this at the unique critical point $(1,-3)$, we'd get 0, i.e., the inconclusive case. However, thinking of the function as multiplicatively separable allows us to do a little better.

We consider F as a product of the function $f(x) := (x-1)^4 - 1$ and $g(y) := (y+3)^6 - 2$. Using the first derivative test (or the higher derivative tests) we can conclude that f has its unique local and absolute minimum with value -1 at $x_0 = 1$. g has its unique local and absolute minimum with value -2 at $y_0 = -3$. Consulting the table on various combinations, we note that F has a local *maximum* at $(1,-3)$ with value 2. However, F has no absolute maxima or minima because f takes both positive and negative values and g takes arbitrarily large positive values.

Incidentally, this also gives an example of a function with a unique local maximum and no local minimum but where the local maximum is not an absolute maximum.

The saddle points arising as Type 4 critical points in this case are $(0, \pm 2^{1/6} - 3)$ and $(2, \pm 2^{1/6} - 3)$.

3. Polynomials

3.1. **Linear polynomials.** A linear polynomial is a polynomial of the form:

$$f(x,y) := ax + by + c$$

where at least one of the values a and b is nonzero. For a linear polynomial, there are no critical points and hence no local extreme values. We may still have boundary extreme values, discussed later when we talk of maximization on closed bounded subsets.

3.2. **Homogeneous quadratic case.** We begin by looking at a homogeneous quadratic polynomial:

$$f(x,y) := ax^2 + bxy + cy^2$$

where at least one of the coefficients a, b, and c is nonzero.

First, we calculate the partials:

$$\begin{aligned} f_x(x,y) &= 2ax+by \\ f_y(x,y) &= bx+2cy \\ f_{xy}(x,y) &= b \\ f_{xx}(x,y) &= 2a \\ f_{yy}(x,y) &= 2c \end{aligned}$$

The Hessian determinant in this case becomes the constant $4ac-b^2$, which is the *negative* of the *discriminant* of the quadratic polynomial.

If $4ac - b^2 \neq 0$, then the equations $2ax + by = 0$ and $bx + 2cy = 0$ are independent linear equations, so their solution set is the unique point $(0,0)$, so this is the only critical point. We note that:

- If $4ac - b^2 > 0$, i.e., the discriminant is negative, then this is a unique local extreme value for the function with value 0 attained at the origin $(0,0)$. Whether it is a maximum or a minimum depends on whether a and c are positive or negative. If $a > 0$, then the local extreme value of 0 at the origin is the unique minimum for the function. This also turns out to be the absolute maximum/minimum.
- If $4ac - b^2 < 0$, i.e., the discriminant is positive, then the function has no local extreme values. In fact, in this case, there are two lines through the origin on which the function takes the value 0. The two lines divide the plane into four regions. In two of these regions, the function can take negative values of arbitrarily large magnitude. In the other two regions, the function can take positive values of arbitrarily large magnitude.

Finally, if $b^2 = 4ac$, then the function attains its extreme value of 0 along a single line through the origin, obtained as the line $2ax + by = 0$. This is a minimum or maximum again depending on the sign of a.

These cases are also summarized in the table:

$4ac-b^2$ sign	b^2-4ac sign	Extreme value	Points where it is attained	Nature (max or min?)
>0	<0	0	$(0,0)$	min if $a>0$, max if $a<0$
<0	>0	–	–	–
$=0$	$=0$	0	all points on the line $2ax+by=0$	min if $a>0$, max if $a<0$

What's happening here will become clearer with our general analysis of homogeneous functions.

3.3. Homogeneous polynomials and functions.

Suppose $F(x,y)$ is a homogeneous function of degree d. Then, we know by definition that:

$$F(ax,ay) = a^d F(x,y)$$

We note the following:

- For any line through the origin, either F is identically zero along the line, or the gradient of F at any point on the line other than the origin is a nonzero vector.
- In particular, this means that the only possibility for a local extreme value is 0, and this must be attained either only at the origin or at a union of lines through the origin.
- To determine what lines through the origin work, rewrite $F(x,y) = x^d g(m)$ where $m = y/x$. Now, find the values of m for which this function of one variable attains a local extreme value of 0. The lines $y = mx$ for these values of m are the relevant ones. The y-axis needs to be checked separately.

  To check whether the origin works, check whether g has a uniform sign (excluding points where it is zero).

In the light of this, the discussion of quadratics makes extra sense. Rewriting the quadratic $ax^2+bxy+cy^2$, we get $x^2(a+bm+cm^2)$. The function is $g(m) = cm^2 + bm + a$. If the discriminant $b^2 - 4ac$ is less than 0, that means the quadratic has uniform sign, so the origin is a local extreme value, but there are no lines on which the function is zero.

If the discriminant is positive, then the quadratic g does not have uniform sign, and 0 is not a local extreme value, so there are no local extreme values for the function F of two variables.

If the discriminant is zero, then the quadratic g has a uniform sign except at an isolated point where it attains an extreme value of 0, so that corresponds to a line on which F attains its local extreme value of 0.

3.4. **Non-homogeneous quadratics: general discussion and specific example.** We now turn to the situation of a non-homogeneous quadratic polynomial. This is of the general form:

$$f(x, y) := ax^2 + bxy + cy^2 + px + qy + l$$

where at least one of the a, b, and c is nonzero.

Note that the value of l does not affect the points of local extrema, though it affects their values. The partial derivatives are:

$$\begin{aligned} f_x(x, y) &= 2ax + by + p \\ f_y(x, y) &= bx + 2cy + q \\ f_{xx}(x, y) &= 2a \\ f_{xy}(x, y) &= b \\ f_{yy}(x, y) &= 2c \end{aligned}$$

The Hessian determinant at any point is thus $4ac - b^2$. In other words, it is a *constant independent of the point* and is the negative of the discriminant of the homogeneous quadratic part.

Thus, we note the following cases:

- If $4ac - b^2 > 0$, or $b^2 - 4ac < 0$, then we get a unique critical point solving the simultaneous system of linear equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$ and this gives a local extreme value. It is a local minimum if $a > 0$ and a local maximum if $a < 0$.
- If $4ac - b^2 < 0$ or $b^2 - 4ac > 0$, then we get a unique critical point solving the simultaneous system of linear equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$, but this gives a saddle point and not a local extremum.
- If $4ac - b^2 = 0$, then, depending on the values p and q, we either have *no* critical points or *a line's worth* of critical points. In the latter case, the line's worth of critical points gives local extrema.

4. Boundary issues

4.1. **The concept of a quasiconvex function.** Suppose f is a function defined on a domain D in $\mathbb{R}^n$. Suppose the domain D is a convex subset of $\mathbb{R}^n$, i.e., given any two points in D, the line segment joining them lies completely in D.

We say that f is a *quasiconvex function* if given any two points P and Q in D, the maximum value of f on the line segment joining P and Q is attained at one (or possibly both) of the endpoints P and Q. In other words, the value in the interior of a line segment is less than or equal to the value at one (or possibly both) of the endpoints.

We say that f is *strictly quasiconvex* if the maximum can occur *only* at endpoints, i.e., it is not possible for the maximum value to also be attained at an interior point.

We note that:

- All linear functions are quasiconvex but not strictly quasiconvex.
- All convex functions are quasiconvex and all strictly convex functions are strictly quasiconvex. We're not going to go into the meaning of convex and strictly convex here (the definition is a non-calculus definition and is fairly simple but will take us a little off track). For functions of one variable, strictly convex simply means *concave up* if the function is continuously differentiable. For functions of two variables, if the Hessian determinant is strictly positive everywhere (except possibly at isolated points where it is zero) and the second pure partials are positive everywhere, the function is strictly convex.
- Quadratic functions of two variables with negative discriminant of the homogeneous part (i.e., positive Hessian determinant) and with positive coefficients on the square terms are strictly convex and hence strictly quasiconvex.

There are two reasons quasiconvex functions are significant:

- The *maximum* of a continuous quasiconvex function on a closed bounded convex domain *must* be attained somewhere on its boundary. In fact, we can go further and note that the maximum must be attained at an *extreme point* of the domain: a point not in the interior of any line segment within the domain.

For a strictly quasiconvex function, the maximum *cannot* be attained at any point other than an extreme point. For a function that's quasiconvex but not strictly so, the maximum may also be attained at other points.

- The *minimum* of a continuous quasiconvex function on a closed bounded convex domain is attained either at a unique point or on a convex subset of the convex domain, i.e., if the minimum occurs at two distinct points in the domain, it also occurs at all points in the line segment joining them.

  For a strictly quasiconvex function, the minimum *must* be attained at a *unique* point.
- For a continuous function that is the negative of a quasiconvex function, the same observations as above hold but with the roles of maximum and minimum interchanged.

We now consider some examples of maximization for functions that are strictly quasiconvex.

4.2. **Linear examples.** Note that *for linear functions*, both the function and its negative are quasiconvex, so both the maximum and the minimum of a linear function on a closed bounded convex domain must occur at extreme points.

Consider a function $f(x, y) = 2x - 3y$ on the square region $[-1, 1] \times [-1, 1]$.

Since f is a linear function, it is quasiconvex (though not strictly so). This means that the maximum value, if it occurs, must occur at one of the extreme points. The extreme points of a square are its four vertices, i.e., the vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$. We simply need to evaluate f at all these points and see which is the largest. We have $f(-1, -1) = 1$, $f(-1, 1) = -5$, $f(1, 1) = -1$, $f(1, -1) = 5$. The largest occurs at $(1, -1)$ with value 5, so this is the maximum.

In this case, the minimum occurs at $(-1, 1)$ with value -5.

Now consider another example:

$$f(x, y) := 3x + 4y$$

on the circular disk $x^2 + y^2 \leq 1$.

Here, the maximum and minimum both occur on the boundary circle $x^2 + y^2 = 1$. However, all points on the boundary circle are extreme points, so the minimum may be attained at any of them – we cannot rule any point offhand. The problem thus reduces to maximizing a function on the circle. There are many ways of doing this, including Lagrange multipliers (which we'll see shortly) but one approach is to put $x = \cos\theta$, $y = \sin\theta$, and thus convert the problem to a maximization/minimization in one variable of the function $g(\theta) = 3\cos\theta + 4\sin\theta$.

4.3. **Strictly convex quadratic example.**

$$f(x, y) := 2x^2 - 2xy + y^2 - x + 3$$

Suppose we want to calculate the maximum and minimum values of f on the square region $[0, 1] \times [0, 1]$. We compute the first and second partials:

$$\begin{aligned} f_x(x, y) &= 4x - 2y - 1 \\ f_y(x, y) &= 2y - 2x \\ f_{xx}(x, y) &= 4 \\ f_{yy}(x, y) &= 2 \\ f_{xy}(x, y) &= -2 \end{aligned}$$

We note that the Hessian determinant is 4 *everywhere* and $f_{xx} = 4$ is positive everywhere. So this is an example of a strictly convex, and hence strictly quasiconvex, quadratic function. As also seen from the discussion of quadratics, we should get a unique critical point that gives a local extreme value. We first find the critical point by solving:

$$
\begin{aligned}
4x - 2y - 1 &= 0 \\
2y - 2x &= 0
\end{aligned}
$$

Solving, we get $x = 1/2$, $y = 1/2$. This gives the point $(1/2, 1/2)$, which lies within the domain. Evaluating the function at this point gives $f(1/2, 1/2) = 11/4$. This is the unique local minimum of the function on the whole plane and hence on the domain. In this case, it also turns out to be the absolute minimum. *Note: In general, there are examples of continuous functions having a unique local minimum that is not an absolute minimum. But those examples don't include strictly quasiconvex functions.*

Since the function is strictly convex and hence strictly quasiconvex, the maximum must occur at one of the four corner points. It remains to evaluate f at the four boundary points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. We get $f(0, 0) = 3$, $f(1, 0) = 4$, $f(1, 1) = 3$, $f(0, 1) = 4$. Of these, we see that the maximum occurs at the points $(1, 0)$ and $(0, 1)$, with a value of 4.

5. Combining max-min and Lagrange

For quasiconvex functions, we already noted that the maximum must occur somewhere on the boundary of the domain. For strictly quasiconvex functions, it can occur *only* at boundary points.

Even for functions that aren't quasiconvex, maximization over a certain domain requires us to find not just local maxima/minima in the interior but also boundary maxima/minima and then compare them all. Lagrange multipliers can offer a method to do so.

Let's return to our earlier example:

$$f(x, y) := 3x + 4y$$

on the disk $x^2 + y^2 \leq 1$.

Since this is a linear function, both the function and its negative are quasiconvex, and since the domain of consideration is a convex domain, the maximum and minimum must both be attained at the boundary circle $x^2 + y^2 = 1$.

We can now use the method of Lagrange multipliers to carry out the maximization and minimization relative to the boundary. Here, $g(x, y) = x^2 + y^2$, so $\nabla g(x, y) = \langle 2x, 2y \rangle$. Also, $\nabla f(x, y) = \langle 3, 4 \rangle$. We get:

$$\langle 3, 4 \rangle = \lambda \langle 2x, 2y \rangle, \qquad x^2 + y^2 = 1$$

Simplifying, we get:

$$
\begin{aligned}
3 &= 2\lambda x \\
4 &= 2\lambda y \\
x^2 + y^2 &= 1
\end{aligned}
$$

Plug in $x = 3/(2\lambda)$ and $y = 2/\lambda$ into the third equation and we get:

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$$

Simplifying, we get:

$$\lambda = \pm 5/2$$

Plugging back, we obtain that the two candidate critical points are $\langle 3/5, 4/5 \rangle$ and $\langle -3/5, -4/5 \rangle$. In this case, it is easy to see from inspection that $\langle 3/5, 4/5 \rangle$ is a point of local maximum with value 5 and $\langle -3/5, -4/5 \rangle$ is a point of local minimum with value -5.