EQUATIONS OF LINES AND PLANES

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 12.5.

What students should definitely get: Parametric equation of line given in point-direction and two-point form, symmetric equations of line, degenerate cases where direction vector has one or more coordinate zero, intersecting lines, equation of plane, angle between planes, line of intersection of planes, distance of point and plane.

What students should hopefully get: How the equation setup relates to the general setup for curves and surfaces. Understanding of the degenerate cases. Role of parameter restrictions in defining a line segment. Deeper understanding of relationship of direction vector and direction cosines.

Executive summary

0.1. Direction cosines.

(1) For a nonzero vector v, there are two unit vectors parallel to v, namely $v/|v|$ and $-v/|v|$.
(2) The direction cosines of v are the coordinates of $v/|v|$. if $v/|v| = \langle \ell, m, n \rangle$, then the direction cosines are ℓ, m, and n. We have the relation $\ell^2 + m^2 + n^2 = 1$. Further, if α, β, and γ are the angles made by v with the positive x, y, and z axes, then $\ell = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$.

0.2. Lines. Words ...

(1) A line in $\mathbb{R}^3$ has dimension 1 and codimension 2. A parametric description of a line thus requires 1 parameter. A top-down equational description requires two equations.
(2) Given a point with radial vector $\mathbf{r}_0$ and a direction vector $\mathbf{v}$ along a line, the parametric description of the line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, this is more explicitly described as $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$.
(3) Given two points with radial vectors $\mathbf{r_0}$ and $\mathbf{r_1}$, we obtain a vector equation for the line as $\mathbf{r}(t) = t\mathbf{r_1} + (1-t)\mathbf{r_0}$. If we restrict t to the interval $[0, 1]$, then we get the line segment joining the points with these radial vectors.
(4) If the line is not parallel to any of the coordinate planes, this parametric description can be converted to symmetric equations by eliminating the parameter t. With the above notation, we get:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is actually *two* equations rolled into one.
(5) If $c = 0$ and $ab \neq 0$, the line is parallel to the xy-plane, and we get the equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad z = z_0$$

Similarly for the other cases where precisely one coordinate is zero.
(6) If $a = b = 0$ and $c \neq 0$, the line is parallel to the z-axis, and we get the equations:

$$x = x_0, \qquad y = y_0$$

Actions ...

(1) To intersect two lines both given parametrically: Choose different letters for parameters, equate coordinates, solve 3 equations in 2 variables. *Note: Expected dimension of solution space is* $2 - 3 = -1$.

(2) To intersect a line given parametrically and a line given by equations: Plug in the coordinates as functions of parameters into both equations, solve. Solve 2 equations in 1 variable. *Note: Expected dimension of solution space is* $1 - 2 = -1$.
(3) To intersect two lines given by equations: Combine equations, solve 4 equations in 3 variables. *Note: Expected dimension of solution space is* $3 - 4 = -1$.

0.3. **Planes.** Words ...

(1) Vector equation of a plane (for the radial vector) is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$ where $\mathbf{n}$ is a normal vector to the plane and $\mathbf{r_0}$ is the radial vector of any fixed point in the plane. This can be rewritten as $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$. Using $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, we get the corresponding scalar equation $ax + by + cz = ax_0 + by_0 + cz_0$. Set $d = -(ax_0 + by_0 + cz_0)$ and we get $ax + by + cz + d = 0$.
(2) The "direction" or "parallel family" of a plane is determined by its normal vector. The angle between planes is the angle between their normal vectors. Two planes are parallel if their normal vecors are parallel. And so on.

Actions ...

(1) Given three non-collinear points, we find the equation of the unique plane containing them as follows: first we find a normal vector by taking the cross product of two of the difference vectors. Then we use any of the three points to calculate the dot product with the normal vector in the above vector equation or the corresponding scalar equation.
    Note that if the points are collinear, there is no unique plane through them – any plane containing their line is a plane containing them.
(2) We can compute the angle of intersection of two planes by computing the angle of intersection of their normal vectors.
(3) The line of intersection of two planes that are not parallel can be computed by simply taking the equations for *both* planes. This, however, is not a standard form for a line in $\mathbb{R}^3$. To find a standard form, either find two points by inspection and join them, or find one point by inspection and another point by taking the cross product of the normal vectors to the plane.
(4) To intersect a plane and a line, plug in parametric expressions for the coordinates arising from the line into the equation of the plane. We get one equation in the one parameter variable. In general, this is expected to have a unique solution for the parameter. Plug in the value of the parameter into the parametric expressions for the line and get the coordinates of the point of intersection.
(5) For a point with coordinates (x_1, y_1, z_1) and a plane $ax + by + cz + d = 0$, the distance of the point from the plane is given by $|ax_1 + by_1 + cz_1 + d|/\sqrt{a^2 + b^2 + c^2}$.

1. Lines and planes

1.1. **Lines: dimension and codimension.** A line in $\mathbb{R}^n$ has dimension one and codimension $n - 1$. In particular, a line in Euclidean space $\mathbb{R}^3$ has dimension 1 and codimension $3 - 1 = 2$. In particular, based on what we know of dimension and codimension, we expect that:

- In a top-down or relational description, we should need *two* independent equations to define a line.
- In a bottom-up or parametric description, we should need *one* parameter to define a line.

1.2. **Planes: dimension and codimension.** A plane in $\mathbb{R}^3$ is 2-dimensional, and it has codimension $3 - 2 = 1$. In particular, based on what we know of dimension and codimension, we expect that:

- In a top-down or relational description, we should need *one* equation to define a plane.
- In a bottom-up or parametric description, we should need *two* parameters to define a plane. *This gets into the realm of functions of two variables, so we will defer the actual 2-parameter description of planes for now.*

1.3. **Intersection theory.** We have the following basic intersection facts:

Intersect	Generic case	Special case 1	Special case 2	Special case 3
Plane, plane	Line	Empty (parallel planes)	Plane (equal planes)	
Plane, line	Point	Empty (line parallel to, not on plane)	Line (line on plane)	
Line, line	Empty (skew lines)	Point (intersecting lines)	Empty (parallel lines)	Line (equal lines)

The *generic case* here represents the case that is most likely, i.e., the case that would arise if the things being intersected were chosen randomly. There are mathematical ways of making this precise, but these are beyond the current scope.

In particular, it is worth pointing out that the generic case is exactly as intersection theory predicts. Let's consider the three generic cases:

- *Generic intersection of plane and plane*: A plane has codimension 1, so the intersection of two planes (generically) has codimension $1+1=2$. We know that a line has codimension 2, so this makes sense.
- *Generic intersection of plane and line*: A plane has codimension 1 and a line has codimension 2, so the intersection of a plane and a line (generically) has codimension $1+2=3$, so it is zero-dimensional. A point is zero-dimensional.
- *Generic intersection of line and line*: A line has codimension 2, so the intersection of two lines (generically) has codimension $2+2=4$, so it has dimension $3-4=-1$. Negative dimension indicates that the intersection is generically empty.

After we study the intersection theory in detail for lines and planes, we will be in a position to acquire a better understanding of the *general principles* of intersection theory. Specifically, we will acquire a better grasp of the *non-generic* cases where the intersections don't work out as they generically do.

2. Equations of lines

2.1. **The point-direction form.**

The general principle behind this is the same as it is with the *point-slope form*. Basically, to describe a line, it suffices to specify a point on the line, and the *direction* of the line.

The *direction* is specified by specifying any vector parallel to the line. Specifically, given a line with points A and B on it, the direction of the line is given by taking the vector AB. Note that any two vectors that are scalar multiples of each other (i.e., parallel to each other) specify the same direction.

Suppose $\mathbf{r_0}$ is the radial vector for one point on the line, and $\mathbf{v}$ is any nonzero vector along the line. Then the radial vector (i.e., vector from the origin to a point) for points on the line can be defined by the parametric equation:

$$\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}$$

where t varies over the real numbers. For each value of t, we get a radial vector for some point on the line, and every point on the line is covered this way.

Suppose $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$. Then $\mathbf{r_0} + t\mathbf{v}$ is the vector:

$$\langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

The corresponding parametric description of a curve is:

$$\{(x_0 + ta, y_0 + tb, z_0 + tc) : t \in \mathbb{R}\}$$

Note that the *choice* of parametric description depends on the choice of basepoint in the line and the choice of vector (which can be varied up to scalar multiples).

By the way, here is some terminology (which we overlooked earlier). The *direction cosines* for a particular direction are defined as the coordinates of the *unit vector* in that direction. The direction cosines of a particular direction are denoted ℓ, m, and n. For instance, if a direction vector is $\langle 1, 2, 3 \rangle$, then the corresponding unit vector is $\langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle$, so the direction cosines are $\ell = 1/\sqrt{14}$, $m = 2/\sqrt{14}$, and $n = 3/\sqrt{14}$.

The direction cosines are also the cosines of the angles made by the vectors with the x-axis, y-axis, and z-axis. They satisfy the relation:

$$\ell^2 + m^2 + n^2 = 1$$

2.2. **The two-point form.**

Suppose $\mathbf{r_0}$ and $\mathbf{r_1}$ are the radial vectors of two points on a line. Then, we can get a line in the point-direction form by setting $\mathbf{v} = \mathbf{r_1} - \mathbf{r_0}$. We thus get the form:

$$\mathbf{r}(t) = \mathbf{r_0} + t(\mathbf{r_1} - \mathbf{r_0})$$

This simplifies to:

$$\mathbf{r}(t) = t\mathbf{r_1} + (1-t)\mathbf{r_0}$$

As t varies over all of $\mathbb{R}$, this gives the whole line. When $t = 0$, we get the point with radial vector $\mathbf{r_0}$ and when $t = 1$, we get the point with radial vector $\mathbf{r_1}$. If we allow only $0 \le t \le 1$, we get the *line segment* joining the two points.

2.3. **Top-down description: symmmetric equations.** To obtain the symmetric equations, we start with the parametric equations and then eliminate the parameter. In other words, with the parametric description:

$$\{(x_0 + ta, y_0 + tb, z_0 + tc) : t \in \mathbb{R}\}$$

We note that:

$$x = x_0 + ta, \qquad \Longrightarrow \qquad t = \frac{x - x_0}{a}$$

Similarly, we get $t = (y - y_0)/b$ and $t = (z - z_0)/c$. Eliminating t, we get:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note that while this looks like a single long equation, it is actually *two* equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

and

$$\frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is in keeping with what we expect/hope – that to describe a 1-dimensional subset in 3-dimensional space, we need $3 - 1 = 2$ equations.

Intuitively, what these equations are saying is that the coordinate changes are in the ratio $a : b : c$.

2.4. **Exceptional case of lines parallel to one of the coordinate planes.** The symmetric equations formulation breaks down if one of the coordinates of the direction vector $\langle a, b, c \rangle$ is zero. In this case, the line is parallel to one of the three coordinate planes, with the third coordinate being unchanged (e.g., if $c = 0$, then the line is parallel to the xy-plane, because its z-coordinate is unchanged).

They break down even more when two coordinates of the direction vector are zero, which means that the line is parallel to one of the axes.

In this case, the symmetric equations given above do not work, and we instead do the following.

- If only one coordinate of the direction vector is zero: If $c = 0$ and $a, b \ne 0$, then we get the two equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad z = z_0$$

  Similarly for the other cases.
- If two coordinates are zero: If, say $a = b = 0$, then we get the two equations:

$$x = x_0, \qquad y = y_0$$

  z does not appear in the equations because it can vary freely. This line is parallel to the z-axis.

2.5. **Pairs of lines: questions about intersection.** As we noted earlier, lines in $\mathbb{R}^3$ have codimension 2, so the intersection of two lines is expected to be empty. There are qualitatively four possibilities:

(1) The lines are skew lines: This is the most "independent" case possible. Here, the equations describing the two lines are as independent of each other as possible and the two lines thus do not lie in the same plane. They do not intersect.
(2) The lines are intersecting lines in the same plane: This is a somewhat less independent case. Here, there is a plane (not necessarily containing the origin) that contains both lines, and the lines are not parallel, so they intersect at a point.
(3) The lines are parallel lines in the same plane: Here, the equations for the line are inconsistent in a specific way, so they lie in the same plane but are parallel. They do not intersect. *Although the conclusion about intersection is the same both for pairs of parallel lines and for pairs of skew lines, the reasons behind this conclusion are different.*
(4) The two lines are actually the same line: In this case, their intersection is the same line. This is the most dependent case possible.

We now examine how to find the intersection of two lines. The approach is simply a special case of finding the intersection of two curves. Since the equations are all linear, we can actually devise specific procedures to solve the equations.

- *Both lines are given parametrically*: In this case, we first make sure we have different letters for the parameters for each line. Then we equate coordinate-wise and solve the system of 3 linear equations in 2 variables (the parameter variables for the two lines). Note that the number of equations is more than the number of variables – unsurprising since the generic case is one of skew lines.
  After finding solutions for the two parameters, plug back to find the points.
- *One line is given parametrically in terms of t, the other using symmetric equations*: We substitute the parametric expressions into the values of x, y, and z in the symmetric equations and solve the system of two equations in the one (parameter) variable t. After finding solutions for t, plug back to find the points.
- *Both lines are given by symmetric equations*: We solve all the four symmetric equations.

3. Planes

3.1. **Vector description in terms of dot product.** For a given plane in $\mathbb{R}^3$, it either already passes through the origin, or there is a unique plane parallel to it that passes through the origin. We say that two planes are *parallel* if they either coincide or they do not intersect – equivalently, if for every line in one plane, there is a line in the other plane parallel to it.

A family of parallel planes can be thought of as sharing a direction. But how do we specify the direction of a plane, which is a two-dimensional object? The idea is to look at the *complement*, or the *codimension*, of the plane. Specifically, we look at the direction that is *orthogonal* to the plane.

There is a unique direction vector (up to scalar multiples) orthogonal to a family of parallel planes. Further, the dot product of this vector with the radial vector in any fixed plane in the family is a constant, and this constant differs for each plane in the family. This allows us to give equations for planes as follows.

Let $\mathbf{n}$ be a normal vector (orthogonal vector) to a plane and let $\mathbf{r_0}$ be the radial vector for a fixed point in the plane. Then, if $\mathbf{r}$ is the radial vector for an arbitrary point in the plane, we have:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

Rearranging, we get:

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

Note that the right side is an actual real number.

If $\mathbf{n} = \langle a, b, c\rangle$ and $\mathbf{r_0} = \langle x_0, y_0, z_0\rangle$, we get the scalar equation:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

If we define $d = -(ax_0 + by_0 + cz_0)$, we can rewrite the above as:

$$ax + by + cz + d = 0$$

Conversely, any equation of the above sort, where at least one of the numbers a, b, and c is nonzero, gives a plane.

3.2. **Plane parallel to the coordinate axes and planes.** We say that a plane and a line are parallel if either the line lies on the plane or they do not intersect at all.

If $a = 0$, the plane is parallel to the x-axis. If $b = 0$, the plane is parallel to the y-axis. If $c = 0$, the plane is parallel to the z-axis.

If $a = b = 0$, the plane is parallel to the xy-plane. If $b = c = 0$, the plane is parallel to the yz-plane. If $a = c = 0$, the plane is parallel to the xz-plane.

3.3. **Finding the equation of a plane given three points.** To specify a plane, we need to provide at least three points on the plane. Given these three points, we can find the equation of the plane as follows:

- We first take two difference vectors and take their cross product to find a normal vector to the plane: If the points are P, Q, and R, we take the difference vectors PQ and PR and compute their cross product.
- We now use the vector equation, and hence from that the scalar equation, taking any of of the three points P, Q, or R as the basepoint.

Note that if the three points given are *collinear*, then they do not define a unique plane. Rather, any plane through the line joining these three points works. It is no surprise that the above procedure fails at the stage where we need to take cross product, because the cross product turns out to be the zero vector.

3.4. **Intersecting two planes: line of intersection.** Given two planes, the typical case is that they intersect in a line. If we have scalar equations for both planes, then the intersection line can be described by taking the two equations together.

Unfortunately, this pair of two equations together, while it does define a line, is not directly one of the *standard* descriptions of a line.

There are many ways of obtaining the line in standard form. One of these is as follows: first, find normal vectors to the planes. For instance, if the equations for the planes are:

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}$$

Then the normal vectors to these planes are $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$. A direction vector along the line of intersection must be perpendicular to *both* these normal vectors, hence, it must be in the line of the *cross product*. Hence, we take the cross product $\langle a_1, b_1, c_1 \rangle \times \langle a_2, b_2, c_2 \rangle$.

Now that we've found the direction vector along the intersection of these planes, we need to find just one point along the intersection and we can then use the point-direction form. One way of finding a point is to set $z = 0$ in both equations and solve the system for x and y (this is assuming that neither is parallel to the xy-plane; otherwise choose some other coordinate).

Note that if the planes are parallel or coincide, then their normal vectors are parallel and thus the cross product of the normal vectors becomes zero. Conversely, the cross product becoming zero means the planes are parallel, so there is a good reason for the line of intersection to not make sense.

3.5. **Intersecting two planes: angle of intersection.** The *angle of intersection* between two planes is the angle of intersection between their normal vectors. As for the line of intersection, we can extract the normal vector from the scalar equation of the planes. To compute the angle of intersection, we use the formula as arc cosine of the quotient of the dot product by the product of the lengths.

3.6. **Intersecting a plane and a line.** Given a plane and a line, we can intersect them as follows: If the plane is given by a scalar equation and the line is given parametrically using a parameter t, then to compute the intersection, we plug in all coordinates as functions of the parameter into the scalar equation for the plane, and solve one equation in the one variable t. After finding the solution t, we plug this into the parametric equation of the line to find the coordinates of the point of intersection.

There are three possibilities:

- The typical case is that we have a linear equation in one variable, and it has a unique solution. In other words, the plane and line intersect at a point.
- Another case is that the equation simplifies to something nonsensical, such as $0 = 1$. In this case, there is no intersection. Geometrically, this means the line is parallel to but not on the plane.
- The final case is that the equation simplifies to a tautology, such as $0 = 0$. In this case, all real t give solutions. Geometrically, this means that the line is on the plane.

3.7. **Distance of a point from a plane.** We will not have much occasion to use this formula, but we note it briefly nonetheless. Given a point with coordinates (x_1, y_1, z_1) and a plane $ax + by + cz + d = 0$, the distance from the point to the plane is given by the formula:

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$