DOUBLE INTEGRALS AND ITERATED INTEGRALS

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 15.2, 15.3. *Note: We are omitting the question types from the book that require three-dimensional visualization, i.e., those that require sketching figures in three dimensions to compute volumes.*

What students should definitely get: The procedure for computing double integrals over rectangles using iterated integrals, the procedure for computing double integrals over other regions using iterated integrals, the idea of Fubini's theorem and its use in interchanging the order of integration. Use of symmetry and inequality-based bounding/estimation techniques.

What students should hopefully get: Relation between single and double integrals, dealing with piecewise cases, breaking up domain into smaller pieces when direct integration over entire domain is infeasible.

Note: The lecture notes contain only a few examples. For more examples, please refer to worked examples in Sections 15.2 and 15.3.

Executive summary

Words ...

(1) The double integral of a function f of two variables, over a domain D in $\mathbb{R}^2$, is denoted $\int\int_D f(x,y)\,dA$ and measures an infinite analogue of the sum of f-values at all points in D.

(2) Fubini's theorem for rectangles states that if F is a function of two variables on a rectangle $R = [a,b] \times [p,q]$, such that F is continuous except possibly at the union of finitely many smooth curves, then the integral equals either of these iterated integrals:

$$\int\int_R F(x,y)\,dA = \int_a^b \int_p^q F(x,y)\,dy\,dx = \int_p^q \int_a^b F(x,y)\,dx\,dy$$

(3) For a function f defined on a closed connected bounded domain D with a smooth boundary, we can make sense of $\int\int_D f(x,y)\,dA$ as being $\int\int_R F(x,y)\,dA$ where R is a rectangular region containing D and F is a function that equals f on D and is 0 on the rest of R.

(4) Suppose D is a Type I region, i.e., its intersection with every vertical line is either empty or a point or a line segment. Then, we can describe D as $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(5) Suppose D is a Type II region, i.e., its intersection with every horizontal line is either empty or a point or a line segment. Then, we can describe D as $p \le y \le q$, $g_1(y) \le x \le g_2(y)$, where g_1 and g_2 are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_p^q \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

(6) The double integral of $f+g$ over D is the sum of the double integral of f over D and the double integral of g over D. Similarly, scalars can be pulled out of double integrals.

(7) The integral of the function 1 over a domain is the area of the domain.

(8) If $f(x,y) \ge 0$ on a domain D, the integral of f over D is also ≥ 0.

(9) If $f(x,y) \ge g(x,y)$ on a domain D, the integral of f over D is $\ge$ the integral of g over D.

(10) If $m \leq f(x,y) \leq M$ over a domain D, then $\int\int_D f(x,y)\,dA$ is betweem mA and MA where A is the area of D.
(11) If $f(x,y)$ is odd in x and the domain of integration is symmetric about the y-axis, the integral is zero. If $f(x,y)$ is odd in y and the domain is symmetric about the x-axis, the integral is zero.

Actions ...

(1) To compute a double integral, compute it as an iterated integral. For a rectangle, we can choose either order of integration, as long as the integration is feasible. For other types of regions, we need to first determine whether the region is Type I or Type II, and break it up into pieces of those types.
(2) For a multiplicatively separable function over a rectangular region (or for a sum of such multiplicatively separable functions), things are particularly easy.
(3) Sometimes, an integral cannot be computed using a particular order of integration – we might get stuck on the inner or the outer stage. However, it may be computable using the other order of integration.
(4) We can often use symmetry-based techniques to argue that certain parts of the integrand integrate to zero.
(5) Even in cases where the integral cannot be computed, we can bound it between limits using maximum or minimum values of function and/or using bigger or smaller regions on which the integral can be computed.

1. Double integral and iterated integral

1.1. **What's a double integral?** We will study the theory of double integrals (Section 16.1 of the book) a little later in the course. For now, we provide an intuitive idea of a double integral. Suppose f is a function of two variables (x,y). The *double integral* of f over a subset D of $\mathbb{R}^2$ on which f is defined is the *total contribution* of the f-values at all points in the domain. One way of thinking of it is as follows: we divide D up into a lot of small regions, we pick a point in each region, multiply the f-value at that point with the area of the region and add up. This total gives the integral of f over the region D.

The notation for the double integral of a function f over a region D is:

$$\int\int_D f(x,y)\,dA$$

The dA here represents an area element or area differential, and is the two-dimensional analogue of dx in one dimension. A detailed exploration of the meaning is possible, but beyond our current scope.

The double integral does integration over a *region* in the same way that the ordinary (single) integral does integration over an *interval*. The region over which integration is being done is termed the *region of integration* or *domain of integration* and the function being integrated is termed the *integrand*.

For a function with nonnegative values, the double integral over a region can also be interpreted as a volume. We will see this interpretation a little later here.

1.2. **Linearity.** The double integral of a sum of two functions is the sum of their double integrals:

$$\int\int_D [f(x,y) + g(x,y)]\,dA = \int\int_D f(x,y)\,dA + \int\int_D g(x,y)\,dA$$

Also, scalars can be pulled out of double integrals:

$$\int\int_D cf(x,y)\,dA = c\int\int_D f(x,y)\,dA$$

1.3. **What's an iterated integral?** An iterated integral is an expression that involves an integral inside another integral (and possibly even more integrals. For instance:

$$\int_a^b \left(\int_{p(x)}^{q(x)} f(x,y)\,dy\right)\,dx$$

What this means is:

- We first compute the inner integral by integrating with respect to y, treating x as a constant. If $F(x, y)$ is an antiderivative, then the definite integral is $F(x, q(x)) - F(x, p(x))$.
- The final answer computed above now depends only on x, the variable y has been integrated over and thus discarded. We now integrate this function of x between the limits a and b.

A special case of this kind of iterated integral is one where the limits for the inner function do not depend on the outer variable, i.e., an integration of the form:

$$\int_a^b \left(\int_p^q F(x, y) \, dy \right) \, dx$$

Note that we could also consider an iterated integral where the inner variable of integration is x and the outer variable of integration is y.

When things are reasonably clear, we can drop the parenthesization for iterated integrals, so the above can be written as:

$$\int_a^b \int_p^q F(x, y) \, dy \, dx$$

1.4. **Fubini's theorem relating double and iterated integrals on rectangles.** Consider the filled rectangle $R = [a, b] \times [p, q]$ in the xy-plane. This is a rectangle with vertices (a, p), (b, p), (a, q), and (b, q). The region can be described as $\{(x, y) : x \in [a, b], y \in [p, q]\}$. Fubini's theorem for rectangles says that if F is a continuous function of two variables defined on this filled rectangle, then:

$$\int \int_R F(x, y) dA = \int_a^b \int_p^q F(x, y) \, dy \, dx = \int_p^q \int_a^b F(x, y) \, dx \, dy$$

In other words, the double integral equals the iterated integral computed in either order.

The assumption of continuity can be weakened somewhat: we only need to assume that f is bounded on R, and the set of points where it is discontinuous is contained in a union of a finite number of smooth curves. This generalization will help us deduce an important corollary for functions whose domains are not rectangular.

1.5. **Intuitive explanation of Fubini's theorem.** Recall that the double integral of a function $F(x, y)$ can be thought of as follows: $F(x, y)$ denotes the value at point (x, y), and the double integral is the total contribution of all points. For instance, $F(x, y)$ could denote the pressure at the point (x, y), and the double integral over the rectangle/region is the total force exerted on the region.

Iterated integration serves to break this integration up by *slicing* horizontally or vertically. Let's be more specific:

- The iterated integral $\int_a^b \left(\int_p^q F(x, y) \, dy \right) \, dx$ can be interpreted as follows: the inner integral $\int_p^q F(x, y) \, dy$ is integrating *along a vertical slice* for a fixed value of x (i.e., along a line parallel to the y-axis). The outer integral is then adding up the contributions of all the vertical slices.
- The iterated integral $\int_p^q \left(\int_a^b F(x, y) \, dx \right) \, dy$ can be interpreted as follows: the inner integral $\int_a^b F(x, y) \, dx$ is integrating *along a horizontal slice* for a fixed value of y (i.e., along a line parallel to the x-axis). The outer integral is then adding up the contributions of all the horizontal slices.

That all these values are the same is some infinite version of the idea that addition is commutative and associative, i.e., we can regroup summations by collecting all things with one common coordinate and then adding up over that coordinate.

1.6. **The special case of multiplicatively separable functions.** A case worth noting is where $F(x, y)$ is of the form $F(x, y) = f(x)g(y)$, i.e., we can separate it as the product of a function purely of x and a function purely of y.

Using the notation established above, if f is continuous on $[a, b]$ and g is continuous on $[p, q]$, then F is continuous on $R = [a, b] \times [p, q]$ and:

$$\int\int_R F(x,y)\,dA = \left(\int_a^b f(x)\,dx\right)\left(\int_p^q g(y)\,dy\right)$$

This is a corollary of Fubini's theorem, and can be deduced by using either of the iterated integrals.

In particular, this means that if F can be written as a *sum* of multiplicatively separable functions, then its integral is a sum of the products of integrals of these functions. In fancy notation, if $f = \langle f_1, f_2, \ldots, f_n \rangle$ and $g = \langle g_1, g_2, \ldots, g_n \rangle$, with all the f_is continuous on $[a,b]$ and all the g_is continuous on $[p,q]$, and if $F(x,y) = \sum_{i=1}^n f_i(x)g_i(y)$, then:

$$\int\int_R F(x,y)\,dA = \sum_{i=1}^n \left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_p^q g_i(y)\,dy\right)\right]$$

Note also that when calculating the integral of a multiplicatively separable function, if either of the integrals of the pieces is zero, the other one does not need to be computed and the product is zero. We will see related ideas a little later when we cover symmetry.

1.7. **A concept of antiderivative.** Suppose G is a function with the property that $G_{xy} = F$, i.e., F is the mixed second-order partial derivative of G. Then, the integral of F over a rectangle $[a,b] \times [p,q]$ is:

$$G(b,q) - G(a,q) - G(b,p) + G(a,p)$$

Basically, the top right and bottom left values get added and the bottom right and top left values get subtracted.

This is sort of like an antiderivative. But the approach is rarely used for explicit computations and we usually try to find definite integrals.

2. Double integrals over regions other than rectangles

2.1. **Defining such a double integral using a rectangle.** Suppose D is a closed bounded region in the plane. In particular, this means that D can be enclosed inside a rectangular region. Suppose R is such a rectangular region. Then, we define the double integral $\int\int_D f(x,y)\,dA$ as $\int\int_R F(x,y)\,dA$ where $F(x,y)$ is defined as:

$$F(x,y) := \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

In other words, we integrate the function that's f on D and 0 outside. Note that F need not be continuous, even if f is. So, we might be skeptical of applying results such as Fubini's theorem to F. If, however, the boundary of D is a piecewise smooth curve, then by the slightly more general formulation of Fubini's theorem, it turns out that the continuity of f within D allows us to apply Fubini's theorem to F. This is great news because it means that we can compute double integrals as iterated integrals.

2.2. **Type I and Type II regions.** For simplicity, we assume that the regions we are dealing with are all connected, closed, bounded regions and their boundary curves are piecewise smooth.

We call a region D in the xy-plane a *Type I region* if its intersection with every line parallel to the y-axis is either empty, or a point, or a line segment, i.e., the intersection is always connected. Such a region can be described as the region enclosed by the graphs of two continuous functions $y = g_1(x)$ and $y = g_2(x)$, with $g_1(x) \le g_2(x)$, for x in an interval $[a,b]$. The function g_2 is simply the y-coordinate value of the upper endpoint of the line segment and the function g_1 is the y-coordinate value of the lower endpoint of the line segnment. In other words:

$$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

We can compute double integrals over Type I regions using iterated integration. To integrate $f(x,y)$ over the Type I region of the kind given above:

$$\int\int_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

A *Type II region* is a region whose intersection with every horizontal line is either empty or a point or a line segment. Such a region can be described as the region enclosed by the the graphs of continuous functions with x expressed in terms of y, i.e., something of the form:

$$D = \{(x, y) : p \le y \le q, g_1(y) \le x \le g_2(y)\}$$

To integrate over the Type II region of the kind given above, we can do the integration:

$$\int\int_D f(x, y)\, dA = \int_p^q \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy$$

Note that both these results follow from the general version of Fubini's theorem for rectangles, using the trick of transitioning to $F(x, y)$ from $f(x, y)$.

2.3. **Convex regions.** A *convex region* is a region with the property that for any two points in the region, the line segment joining those two points lies completely inside the region. Convex regions are both Type I and Type II. In particular, this means we can use either of the integration methods to compute integrals over convex regions.

Circular disks, triangular regions, and rectangular regions are all examples of convex regions. A heart-shaped region is *not* a convex region.

2.4. **Breaking up a region into Type I and Type II regions.** If a region D is closed, connected, and bounded with a smooth bounding curve, and f is a continuous function of D, it may still happen that D is neither Type I nor Type II. There are still some ways out. The first is to partition D into finitely many pieces (chambers) such that:

- Each piece is Type I or Type II
- The intersection of any two of the pieces is one-dimensional and hence the restriction of the double integral over that intersection is zero.
- The double integral on D is now the sum of the values of double integrals on each piece, and each of the individual double integrals can be computed as an iterated integral by Fubini's theorem.

This is a two-dimensional analogue of chopping up an interval into sub-intervals using a partition. Here, instead of sub-intervals, we use subregions.

In the one-dimensional case, the slight overlap (isolated points) between the partitioned pieces does not result in any double-counting, i.e., the integral on the whole interval is the sum of the integrals on the parts. In the two-dimensional cases, the slight overlap at boundary curves (which are one-dimensional) does not result in any double-counting, because the boundary curves are infinitesimal/negligible.

3. In practice: computing iterated and double integrals

3.1. **Theory versus practice: the one-variable nightmare.** Let's recall the situation in one variable first and then we'll discuss how the situation changes with more variables. We know that any continuous function in one variable is integrable. This knowledge does not always translate to actually being able to find expressions for the integrals. There are three levels of difficulty:

- First, there are many functions expressible in terms of elementary functions but which do not have antiderivatives expressible in terms of elementary functions. To give names to the antiderivatives, we need to invent new branches of mathematics. For instance, logarithms were invented to integrate $1/x$, and trigonometry was invented to integrate $1/(x^2 + 1)$. But there's a lot more work to do – some functions slip through the cracks and integrating them requires us to invent more branches of mathematics.

  Examples of elementarily expressible functions that do not have elementarily expressible antiderivatives are e^{-x^2}, $\sin(x^2)$, $(\sin x)/x$, $(e^x - 1)/x$, $1/\sqrt{x^4 + 1}$, and many others.
- Second, the procedure for integrating a function does not break down into a bunch of deterministic rules. This is in sharp contrast with differentiation, where if we know how to differentiate a bunch of functions, we know how to differentiate all functions generated from them using the processes of pointwise combination, composition, inverses, and piecewise combination. For integration, all we have are heuristics. Thus, even if a neat antiderivative does exist, it can be hard to find.

- Third, even if we are able to find antiderivatives, computing their values between limits can be difficult. Even to integrate a rational function, we need ln and arctan and computing the values of these is hard.

Each of these challenges continues to operate in many variables. With multiple variables, there is some further bad news and some mitigating good news. We turn to these.

3.2. **The further bad news.** The inner-most step of an iterated integral is something like:

$$\int_a^b f(x, y)\,dx$$

Here, we are treating y as a constant temporarily while doing this integration. However, we cannot put an actual value on y – it's an *unknown known* for now, and in fact, when we have completed this integration and are willing to move on outward, it will become a variable again. Thus, this integration really is not integrating a plain vanilla function but rather trying to do a large number of integrations – one for each fixed value of y – *simultaneously* by getting a generic expression.

Now, it may turn out that there is no uniform general expression for y. Consider the example:

$$\int_2^3 \frac{dx}{x^2 + \sin y}$$

When $\sin y > 0$, the integral becomes:

$$\frac{1}{\sqrt{\sin y}} \left[\arctan(x/\sqrt{\sin y})\right]_2^3$$

When $\sin y = 0$, the integral becomes:

$$[-1/x]_2^3 = (1/2) - (1/3) = 1/6$$

When $\sin y < 0$, the integral becomes:

$$\frac{1}{2\sqrt{-\sin y}} \left[\ln((x - \sqrt{-\sin y})/(x + \sqrt{-\sin y}))\right]_2^3$$

So, even though the original function had a single piece description, the new function we get after integrating has a *piecewise description*.

This will occur only rarely, and not in the routine examples that we will see. Also, although it complicates matters, it does not make the task any more impossible. To perform the outer integration for the resultant piecewise function, we simply break the domain (for the outer integration) into the various pieces and perform the integration separately in each piece.

3.3. **More bad news for non-rectangular regions.** Another piece of bad news, that applies particularly to non-rectangular regions, is that complications could arise not only from the nature of the integrand, but also from the shape of the region. For Type I or Type II regions, the nature of the bounding functions that determine how the inner variable varies in terms of the outer variable determine the expression to be integrated on the outside. Thus, even for very easy functions $f(x, y)$, the actual integration procedure may become difficult because of the complexity arising from the shape of the region.

3.4. **The good news: use Fubini's to change order of integration.** The good news is that sometimes, an integral is impossible to do when written as an iterated integral with a particular ordering of x and y, but can be done if the ordering of x and y were reversed. Luckily, by Fubini's theorem, the answers have the same value.

Let's consider a couple of examples.

Our first example is the function x^y on the interval $[0, 1] \times [0, 1]$. The domain is a square region with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Note that the function is undefined at the bottom left vertex $(0, 0)$. It takes the value 1 on the lower edge, 0 on the left edge, 1 on the right edge, and is equal to the function x on the top edge. Note that everywhere in the square where it is defined, the function takes a value in $[0, 1]$. We want to integrate it over the square.

We could set up the integral as an iterated integral in either of these two ways:

$$\int_0^1 \int_0^1 x^y \, dy \, dx, \qquad \int_0^1 \int_0^1 x^y \, dx \, dy$$

Let's consider the first formulation of the integral:

$$\int_0^1 \int_0^1 x^y \, dy \, dx$$

The inner integral is:

$$\int_0^1 x^y \, dy$$

This simplifies to:

$$\left[\frac{x^y}{\ln x}\right]_0^1 = \frac{x-1}{\ln x}$$

The new integral that we need to compute is thus:

$$\int_0^1 \frac{x-1}{\ln x} \, dx$$

The *indefinite integral* of the integrand is not possible to compute. So we're basically stuck.

On the other hand, if we use the other formulation (*FIXED ERROR BELOW!*:

$$\int_0^1 \int_0^1 x^y \, dx \, dy$$

The inner integral is:

$$\int_0^1 x^y \, dx$$

This simplifies to:

$$\left[\frac{x^{y+1}}{y+1}\right]_0^1 = \frac{1}{y+1}$$

We can now integrate this:

$$\int_0^1 \frac{1}{y+1} \, dy = [\ln(y+1)]_0^1 = \ln 2$$

Note that in this example, integrating in the wrong order got us into problems at the *outer stage*, not at the inner stage. In some cases, integrating in the wrong order can prevent us from getting started. Here is an example:

$$\int_0^1 \int_x^1 \exp(-y^2) \, dy \, dx$$

This is the integral of the function $\exp(-y^2)$ over the triangular region for the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$, i.e., the upper left half triangle in the unit square $[0,1] \times [0,1]$. Unfortunately, as written here, the inner integral cannot be computed in elementary terms.

Note that the region here is both a Type I and a Type II region. This means that it can be sliced either vertically or horizontally. If we slice horizontally instead, then for any fixed y, the constraint on x is $0 \le x \le y$, and we get:

$$\int_0^1 \int_0^y \exp(-y^2) \, dx \, dy$$

The inner integral is now:

$$\int_0^y \exp(-y^2)\,dx = y\exp(-y^2)$$

The outer integral now becomes:

$$\int_0^1 y\exp(-y^2)\,dy = \left[\frac{-1}{2}\exp(-y^2)\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

This is similar to Example 3 in the book.

3.5. **Integrating polynomials.** Polynomials are very easy to integrate *over rectangular regions* because every polynomial is a sum of monomials, every monomial is multiplicatively separable as a product of power functions, and each power function can be integrated.

For instance, to integrate the polynomial $xy + 2x^5y^3$ over the interval $[1,3] \times [4,6]$, we do:

$$\int_1^3 x\,dx \int_4^6 y\,dy + 2\int_1^3 x^5\,dx \int_4^6 y^3\,dy$$

In other words, it is a sum of products of integrals of power functions of one variable. The rest is just straightforward arithmetic.

To integrate a polynomial over a non-rectangular region is a little trickier, and may not be feasible for all regions. First, note that we can still additively separate the polynomial as a sum of monomials, so it suffices to integrate each monomial, i.e., each expression of the form x^ay^b. However, because the region is no longer rectangular, we cannot use multiplicative separability.

Here's an example. Consider integrating x^2y^2 on the circular disk $x^2 + y^2 \le 1$. This is both Type I and Type II. If we go for horizontal slicing, then for $x \in [-1,1]$, we have $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. The integral thus becomes:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$$

The inner integral becomes $x^2y^3/3$ which between limits is $2x^2(1-x^2)^{3/2}/3$. This needs to be integrated on $[-1,1]$. Note how, even though we started only with polynomials, the integrand for the outer integration involves fractional powers. The fractional powers are arising from the shape of the domain of integration.

It is possible to complete the question in the case of circular disks, but this is best done using double integrals in polar coordinates, covered in Section 16.4 of the book. We will, however, not cover this topic formally as part of the syllabus, although I will explain it in class and give a few examples.

3.6. **Integrating rational functions.** We first consider integrating rational functions over rectangular regions. If the denominator of the rational function is of the form cx^ay^b (i.e., it is a monomial) then the rational function is a sum of multiplicatively separable functions and can be integrated using the same idea discussed above for polynomials.

More generally, if the denominator of the rational function can be factorized as the product of a polynomial in x and a polynomial in y, we can use the multiplicatively separable approach.

For instance, for the rational function:

$$\frac{x^2 + y^2 - 2xy + 3}{x^2y^2 + 2x^2 + y^2 + 2}$$

The denominator can be factored as $(x^2+1)(y^2+2)$ and hence the rational function can be written as:

$$\frac{x^2}{x^2+1}\frac{1}{y^2+1} + \frac{1}{x^2+1}\frac{y^2}{y^2+2} - 2\frac{x}{x^2+1}\frac{y}{y^2+2} + 3\frac{1}{x^2+1}\frac{1}{y^2+2}$$

This is a multiplicatively separable form, and can be integrated over a rectangular region. Note: We are assuming knowledge of how to integrate rational functions of one variable, something you saw in one variable calculus.

In other cases, it is not completely obvious how to do the integration, so we just try iterated integration and see how it works out. For instance, consider:

$$\int_1^2 \int_1^2 \frac{1}{x+y}\, dy\, dx$$

The inner integral is $\ln(x+2) - \ln(x+1)$. The outer integral thus becomes:

$$\int_1^2 \ln(x+2) - \ln(x+1)\, dx$$

After some integration by parts (we skip steps) this becomes:

$$[(x+2)\ln(x+2) - (x+1)\ln(x+1)]_1^2 = 4\ln 4 - 3\ln 3 - 3\ln 3 + 2\ln 2 = 10\ln 2 - 6\ln 3$$

In fact, it is possible to give a sketch for why this kind of integration procedure will work for a wide variety of (all?) rational functions.

3.7. **Exponential and trigonometric functions.** Again, for these, one thing to look for is multiplicative separability, or expressibility as a sum of multiplicatively separable functions, and hope that each of the constituent functions of one variable can be integrated.

Consider $f(x,y) = \sin(x+y)$. We want to calculate:

$$\int_a^b \int_p^q \sin(x+y)\, dy\, dx$$

We could do this directly, integrating first with respect to y, to get $-\cos(x+q) + \cos(x+p)$ and then integrating with respect to x to get $-\sin(b+q) + \sin(a+q) + \sin(b+p) - \sin(a+p)$.

Alternatively, we could rewrite $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and integrate by additively and then multiplicatively splitting, to get:

$$(\cos a - \cos b)(\sin q - \sin p) + (\sin b - \sin a)(\cos p - \cos q)$$

It's possible to work out that both of these are the same. (Note: This is easier to see if we use the "antiderivative" concept mentioned earlier: the antiderivative by iterated integration is $-\sin(x+y)$ and the antiderivative by multiplicative separation is $-\cos x \sin y - \sin x \cos y$ which becomes the same thing.

For exponential functions, note that $\exp(f(x) + g(y)) = \exp(f(x))\exp(g(y))$ and is hence multiplicatively separable.

4. Area and volume interpretations

To make this course simpler, we will refrain from complicated volume computations for the surfaces that are graphs of functions, but we will go over the theoretical facts just in case you need them for the future.

4.1. **Double integral equals volume.** Consider a function $z = f(x,y)$ on a closed connected bounded domain D such that $z \geq 0$ for all $(x,y) \in D$. Then, the integral $\int\int_D f(x,y)\, dA$ equals the volume of the region between the surface $z = f(x,y)$, the xy-plane. On the sides, this region is bounded by line segments joining points in the boundary of D and the corresponding points on the graph of the surface above them.

The three-dimensional region can also be described as follows: it is the union of all the line segments obtained by joining each point $(x,y,0)$ with the point $(x,y,f(x,y))$ where $(x,y) \in D$.

The fact that the double integral value equals the volume is the three-dimensional analogue of the fact that the single integral value equals the area under the graph of the function.

4.2. **Interpretation of slicing and iterated integration.** We can now interpret the horizontal and vertical slicing.

Computing the integral along a horizontal slice, i.e., a line parallel to the xy-plane, correspondings to computing the area of the intersection of the region with a plane parallel to the xz-plane through that line. Specifically, computing the integral:

$$\int_a^b f(x, y_0)\, dx$$

means computing the area of the intersection of the region with the plane $y = y_0$, or equivalently, computing the area under the graph of the function $x \mapsto f(x, y_0)$ between $x = a$ and $x = b$.

The outer part of the integration then integrates this area function along the other axis, to give the total volume.

If we perform the integration in the other order, we are computing the areas of intersection with planes parallel to the yz-plane, and then integrating this area function along the x-axis.

Note that all this fits in with the cross sectional method of determining volume as the integral of the areas of the cross sections along planes as we move along an axis perpendicular to these planes.

5. Properties of double integrals

5.1. **Inequalities that can be used for estimation.** These inequalities are a lot like those of single integrals:

- If $f(x, y) \geq 0$ on a domain D, then $\int\int_D f(x, y)\,dA \geq 0$.
- If $f(x, y) \geq g(x, y)$ on a domain D, then $\int\int_D f(x, y)\,dA \geq \int\int_D g(x, y)\,dA$.
- If $D_1 \subseteq D_2$, and f is a *nonnegative function* defined on D_2, then $\int\int_{D_1} f(x, y)\,dA \leq \int\int_{D_2} f(x, y)\,dA$: This last one is important because it means that to calculate integrals over an irregularly shaped region, we can bound from above and below by calculating integrals over a region contained inside it and over a region containing it.
- If $D = D_1 \cup D_2$ and $D_1 \cap D_2$ is one-dimensional, then $\int\int_D f(x, y)dA = \int\int_{D_1} f(x, y)\,dA + \int\int_{D_2} f(x, y)\,dA$ (this was already discussed earlier).
- The integral of the function 1 over a domain D is the area of D.
- If $f(x, y)$ on a domain D is bounded from above and below by M and m respectively, and D has area A, then the integral of $f(x, y)$ over D is between mA and MA.

5.2. **Symmetry-based ideas.** These all build on the corresponding symmetry-based ideas for functions of one variable:

- If f is odd in the variable x, and the domain of integration is symmetric about the y-axis, then the integral is zero: If we do integration using horizontal slices, we see that each horizontal slice integrates to zero, so the overall integral is zero.

  Note that for a multiplicatively separable function, what matters is that the part depending on x be odd, and the part depending on y does not matter.
- If f is odd in the variable y, and the domain of integration is symmetric about the x-axis, then the integral is zero: If we do integration using vertical slices, we see that each vertical slice integrates to zero, so the overall integral is zero.

  Note that for a multiplicatively separable function, what matters is that the part depending on y be odd, and the part depending on x does not matter.

Often, a given function can be expressed as a sum of functions some of which are odd in x or in y, and hence, using symmetry of domain, can be declared to be zero. Others may need to be computed.

Consider, for instance, the case $f(x, y) = x^3y^2 + \ln(x^2 + x + 1)\sin(y^3)$, being integrated over the circular disk $x^2 + y^2 \leq 1$. Note that f as given is not odd in either variable. However, it is the sum of the functions x^3y^2 (which is odd in x) and $\ln(x^2 + x + 1)\sin(y^3)$ (which is odd in y). Moreover, the domain is symmetric about both axes. Thus, the integral for both these functions is zero, hence the overall integral for f is zero.