CHAIN RULE AND SECOND DERIVATIVES

MATH 195, SECTION 59 (VIPUL NAIK)

The homework question is as follows: Suppose $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$.

(1) Show the formula for $\frac{\partial^2 z}{\partial t^2}$ as given in the book. It is a somewhat long formula.
(2) Find a similar formula for $\partial^2 z/(\partial s \partial t)$.

Last year, many people got it correct, but some people didn't clearly understand how to proceed. In anticipation of students facing similar difficulties this year, I include a detailed outline of the solution below. Even those students who are able to arrive at the correct answer by themselves may benefit from reading through this.

Similar one-variable question. Let's consider a simple situation, where we have $x = g(t)$ and $z = f(x)$, with no y and s appearing. We want a formula for d^2z/dt^2. In other words, we want to compute $(f \circ g)''$

You've all seen this, by the way, in a past quiz (January 9), where almost all of you got the question correct. The idea now is to carefully understand what we're doing so that it can readily be replicated in the multiple variable case.

We have, by the chain rule:

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = f'(x)g'(t) = f'(g(t))g'(t)$$

We now want to differentiate this with respect to t again:

$$\frac{d^2z}{dt^2} = \frac{d}{dt}[f'(g(t))g'(t)]$$

Note that the expression to be differentiated on the right side is a product, so we use the product rule, and obtain:

$$\frac{d^2z}{dt^2} = \frac{d}{dt}[f'(g(t))]g'(t) + f'(g(t))g''(t)$$

For the derivative of $f'(g(t))$ with respect to t, we *again* use the chain rule, and get $f''(g(t))g'(t)$, so overall:

$$\frac{d^2z}{dt^2} = f''(g(t))(g'(t))^2 + f'(g(t))g''(t)$$

Let's understand this step by step:

(1) We calculated the first derivative using the chain rule, and got a product of a derivative with respect to the intermediate value x (which became $f'(g(t))$) and a derivative with respect to the initial variable t (which became $g'(t)$).
(2) To differentiate this, we use the product rule.
(3) For one of the pieces in the product rule, we are differentiating $g'(t)$, which becomes $g''(t)$. This is the straightforward piece (second summand in our description above).
(4) For the other piece, we need to differentiate the composite $f'(g(t))$ with respect to t. For this, we *use the chain rule again.*

Second partial with a single variable. Return to the original question: Suppose $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$.

We have:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

With the subscript notation, this becomes:

$$\frac{\partial z}{\partial t} = f_x g_t + f_y h_t$$

So far, we are done with the equivalent of Step (1) for one variable. Note that instead of just a single produce, we have a sum of two products, indicating the two paths of dependence (via x and via y).

Now, we want to differentiate both sides with respect to t:

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(f_x g_t + f_y h_t\right)$$

We break additively and use the product rule on each piece (analogous to Step (2)), and get:

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial (f_x)}{\partial t} g_t + f_x \frac{\partial g_t}{\partial t} + \frac{\partial f_y}{\partial t} h_t + f_y \frac{\partial h_t}{\partial t}$$

Within each product rule, the second summand is easy to rewrite: $\partial g_t/\partial t$ becomes g_{tt} and $\partial h_t/\partial t = h_{tt}$. These simpler parts are analogous to Step (3) in the one-variable scenario.

The harder pieces are $\partial f_x/\partial t$ and $\partial f_y/\partial t$. These are analogous to Step (4). Let's look at these more carefully.

$f_x(x, y)$ is a function of the variables x and y, each of which in turn depends on s and t. Thus:

$$\frac{\partial f_x}{\partial t} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial t} = f_{xx} g_t + f_{xy} h_t$$

Similarly:

$$\frac{\partial f_y}{\partial t} = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial t} = f_{yx} g_t + f_{yy} h_t$$

Plugging these all back in the original expression, we get:

$$\frac{\partial^2 z}{\partial t^2} = (f_{xx} g_t + f_{xy} h_t) g_t + f_x g_{tt} + (f_{yx} g_t + f_{yy} h_t) h_t + f_y h_{tt}$$

There is a total of six terms, but using Clairaut's theorem, we see that the term $f_{xy} g_t h_t$ appears twice, so combining, we get a sum of five terms with a coefficient of 2 appearing on one of them.

The key difference relative to the situation with one variable: *dependence on two intermediate variables means that at Step (1), we have a sum of two products, and in Step (4), again each of the chain derivatives is a sum of two products. Hence we get a total of* 6 *terms.*

Mixed partial with both variables. Let's try to compute $\partial^2 z/(\partial s \partial t)$. We already have:

$$\frac{\partial z}{\partial t} = f_x g_t + f_y h_t$$

We differentiate both sides with respect to s, and get:

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s}[f_x g_t + f_y h_t]$$

Using additive splitting and the product rule (analogous to Step (2)), we get:

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial f_x}{\partial s} g_t + f_x \frac{\partial g_t}{\partial s} + \frac{\partial f_y}{\partial s} h_t + f_y \frac{\partial h_t}{\partial s}$$

The second and fourth summand are easy to simply: $\partial g_t/\partial s = g_{ts}$ and $\partial h_t/\partial s = h_{ts}$. The harder ones are $\partial f_x/\partial s$ and $\partial f_y/\partial s$. As before, we use the chain rule:

$$\frac{\partial f_x}{\partial s} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial s} = f_{xx}g_s + f_{xy}h_s$$

Similarly:

$$\frac{\partial f_y}{\partial s} = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial s} = f_{yx}g_s + f_{yy}h_s$$

Plugging these in, we get:

$$\frac{\partial^2 z}{\partial s \partial t} = (f_{xx}g_s + f_{xy}h_s)g_t + f_x g_{ts} + (f_{yx}g_s + f_{yy}h_s)h_t + f_y h_{ts}$$

As with the previous case, this is a sum of six terms.