TAKE-HOME CLASS QUIZ: DUE WEDNESDAY FEBRUARY 20: INTEGRATION TECHNIQUES (ONE VARIABLE)

MATH 195, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________________

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE ONLY ENTER FINAL ANSWER OPTIONS THAT YOU PERSONALLY ENDORSE. DO NOT ENGAGE IN GROUPTHINK.

(1) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? Please see Option (E) before answering.
   (A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
   (B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
   (C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
   (D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
   (E) None of the above, i.e., they are all necessarily true.

   Your answer: ____________________________

(2) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
   (A) $k - 2$
   (B) $k - 1$
   (C) k
   (D) $k + 1$
   (E) There is no bound in terms of k.

   Your answer: ____________________________

(3) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).
   (A) $F(1) = F(0)$.

(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Your answer: ____________________________

(4) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see options (D) and (E) before answering.
(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Your answer: ____________________________

(5) Suppose F, G are continuously differentiable functions defined on all of $\mathbb{R}$. Suppose a, b are real numbers with $a < b$. Suppose, further, that $G(x)$ is identically zero everywhere except on the open interval (a, b). Then, what can we say about the relationship between the numbers $P = \int_a^b F(x)G'(x)\,dx$ and $Q = \int_a^b F'(x)G(x)\,dx$?
(A) $P = Q$
(B) $P = -Q$
(C) $PQ = 0$
(D) $P = 1 - Q$
(E) $PQ = 1$

Your answer: ____________________________

(6) Consider the integration $\int p(x)q''(x)\,dx$. Apply integration by parts twice, first taking p as the part to differentiate, and q as the part to integrate, and then again apply integration by parts to avoid a circular trap. What can we conclude?
(A) $\int p(x)q''(x)\,dx = \int p''(x)q(x)\,dx$
(B) $\int p(x)q''(x)\,dx = \int p'(x)q'(x)\,dx - \int p''(x)q(x)\,dx$
(C) $\int p(x)q''(x)\,dx = p'(x)q'(x) - \int p''(x)q(x)\,dx$
(D) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) + \int p''(x)q(x)\,dx$
(E) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) - \int p''(x)q(x)\,dx$

Your answer: ____________________________

(7) Suppose p is a polynomial function. In order to find the indefinite integral for a function of the form $x \mapsto p(x)\exp(x)$, the general strategy, which always works, is to take $p(x)$ as the part to differentiate and $\exp(x)$ as the part to integrate, and keep repeating the process. Which of the following is the best explanation for why this strategy works?
(A) exp can be repeatedly differentiated (staying exp) and polynomials can be repeatedly integrated (giving polynomials all the way).
(B) exp can be repeatedly integrated (staying exp) and polynomials can be repeatedly differentiated, eventually becoming zero.
(C) exp and polynomials can both be repeatedly differentiated.
(D) exp and polynomials can both be repeatedly integrated.
(E) We need to use the recursive version of integration by parts whereby the original integrand reappears after a certain number of applications of integration by parts (i.e., the polynomial equals one of its higher derivatives, up to sign and scaling).

Your answer: ____________________________

(8) Consider the function $x \mapsto \exp(x)\sin x$. This function can be integrated using integration by parts. What can we say about how integration by parts works?

(A) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once to get the answer directly.
(B) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(C) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice to get the answer directly.
(D) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(E) We choose exp as the part to integrate and sin as the part to differentiate, and we apply integration by parts four times to get the answer directly.

Your answer: ____________________

(9) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________

(10) Suppose f is a continuous function on $(0, \infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0, \infty)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________

(11) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.
(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Your answer: ____________________

(12) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin \sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0, 1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**?
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.

(D) f_2 and f_3.
(E) f_2 and f_4.

Your answer: ______________________________

(13) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?
(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2 e^{-x^2}$
(E) $x \mapsto x^4 e^{-x^2}$

Your answer: ______________________________

(14) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?
(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, ... If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Your answer: ______________________________

(15) Which of these functions of x is *not* elementarily integrable?
(A) $x\sqrt{1+x^2}$
(B) $x^2\sqrt{1+x^2}$
(C) $x(1+x^2)^{1/3}$
(D) $x\sqrt{1+x^3}$
(E) $x^2\sqrt{1+x^3}$

Your answer: ______________________________

(16) Consider the function $f(k) := \int_1^2 \frac{dx}{\sqrt{x^2+k}}$. f is defined for $k \in (-1, \infty)$. What can we say about the nature of f within this interval?
(A) f is increasing on the interval $(-1, \infty)$.
(B) f is decreasing on the interval $(-1, \infty)$.
(C) f is increasing on $(-1, 0)$ and decreasing on $(0, \infty)$.
(D) f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.
(E) f is increasing on $(-1, 0)$, decreasing on $(0, 2)$, and increasing again on $(2, \infty)$.

Your answer: ______________________________

(17) For which of these functions of x does the antiderivative necessarily involve *both* arctan *and* ln?
(A) $1/(x+1)$
(B) $1/(x^2+1)$

(C) $x/(x^2+1)$
(D) $x/(x^3+1)$
(E) $x^2/(x^3+1)$

Your answer: ____________________________

(18) Suppose F is a (not known) function defined on $\mathbb{R} \setminus \{-1, 0, 1\}$, differentiable everywhere on its domain, such that $F'(x) = 1/(x^3 - x)$ everywhere on $\mathbb{R} \setminus \{-1, 0, 1\}$. For which of the following sets of points is it true that knowing the value of F at these points **uniquely** determines F?
(A) $\{-\pi, -e, 1/e, 1/\pi\}$
(B) $\{-\pi/2, -\sqrt{3}/2, 11/17, \pi^2/6\}$
(C) $\{-\pi^3/7, -\pi^2/6, \sqrt{13}, 11/2\}$
(D) Knowing F at any of the above determines the value of F uniquely.
(E) None of the above works to uniquely determine the value of F.

Your answer: ____________________________

(19) Suppose F is a continuously differentiable function whose domain contains (a, ∞) for some $a \in \mathbb{R}$, and $F'(x)$ is a rational function $p(x)/q(x)$ on the domain of F. Further, suppose that p and q are nonzero polynomials. Denote by d_p the degree of p and by d_q the degree of q. Which of the following is a **necessary and sufficient condition** to ensure that $\lim_{x\to\infty} F(x)$ is finite?
(A) $d_p - d_q \geq 2$
(B) $d_p - d_q \geq 1$
(C) $d_p = d_q$
(D) $d_q - d_p \geq 1$
(E) $d_q - d_p \geq 2$

Your answer: ____________________________

For the next two questions, build on the observation: For any nonconstant monic polynomial $q(x)$, there exists a finite collection of transcendental functions $f_1, f_2, \ldots, f_r$ such that the antiderivative of any rational function $p(x)/q(x)$, on an open interval where it is defined and continuous, can be expressed as $g_0 + f_1 g_1 + f_2 g_2 + \cdots + f_r g_r$ where $g_0, g_1, \ldots, g_r$ are rational functions.

(20) For the polynomial $q(x) = 1 + x^2$, what collection of f_is works (all are written as functions of x)?
(A) $\arctan x$ and $\ln|x|$
(B) $\arctan x$ and $\arctan(1+x^2)$
(C) $\ln|x|$ and $\ln(1+x^2)$
(D) $\arctan x$ and $\ln(1+x^2)$
(E) $\ln|x|$ and $\arctan(1+x^2)$

Your answer: ____________________________

(21) For the polynomial $q(x) := 1 + x^2 + x^4$, what is the size of the smallest collection of f_is that works?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________________