SEQUENCES OF REAL NUMBERS

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 11.2.

What students should definitely get: The two definitions of sequence, the three ways of describing sequences, switching back and forth between different ways, the key types/properties of sequences.

What students should hopefully get: Discrete calculus and its relationship with continuous calculus.

Executive summary

0.1. The basics.
Words ...

(1) A sequence in a set is a function from $\mathbb{N}$ to that set.
(2) A sequence of reals can be described in three ways: as an ordered list of real numbers (where we write only the first few members due to space and time considerations), as a closed form expression for the general term of the sequence (i.e., thinking of it as a function), and in terms of a recurrence relation.
(3) The *range* of a sequence is the set of values that it takes. The range of a sequence differs from the sequence in the following two senses: (i) it ignores repetition (ii) it ignores the ordering or sequencing of the elements.
(4) There are many properties that we can talk of in the context of sequences: increasing, decreasing, non-increasing, non-decreasing, monotonic, constant, periodic, bounded from below, bounded from above, and bounded. Note that the boundedness-related properties depend *only* on the range of the sequence, whereas the other properties depend on the sequence as a whole.
(5) For any property p that can be evaluated on each sequence, we can talk of the property *eventually* p, which means that some left shift of the sequence has the property p. For instance, we can talk of eventually increasing, eventually decreasing, eventually constant, eventually monotonic, and eventually periodic.
(6) Eventually bounded is the same as bounded.
(7) There are various operations we can do on sequences similar to the corresponding operations on functions: add, subtract, multiply, divide, multiply by a scalar, and compose with a function defined on the range of the sequence.
(8) We can do left shifts, right shifts (though this requires us to throw in more new terms), splicing, and other fancy operations.
(9) If a sequence is defined recursively, i.e., using a recurrence relation, then we need to separately specify initial values. The number of initial values that we need to specify depends on how far back the recurrence relation reaches. This is related both to the principle of mathematical induction and the idea of free parameters and initial value specifications for differential equations.
(10) We can define a discrete derivative, called the *forward difference operator*, defined as follows: for a sequence $f : \mathbb{N} \to \mathbb{R}$, the forward difference operator is $(\Delta f)(n) = f(n+1) - f(n)$. This is analogous to the derivative of a continuous function.
(11) The *integration equivalent* for the forward difference operator is the summation operator. (Try to find the formula for this in the notes/class discussion).
(12) The forward difference operator behaves analogously to differentiation (though the formulas differ) for constants, polynomial sequences, and periodic sequences. (Please see the notes for a list of points).
(13) Periodic sequences can be defined using a case-wise definition based on the remainder modulo the period. They can alternatively be defined as combinations of trigonometric functions. In the special case of a period 2, we can also express in terms of $(-1)^n$.

(14) A sequence with periodic derivative can be expressed as the sum (pointwise) of a linear sequence and a periodic sequence.

1. Sequences of real numbers

1.1. **Definition of a sequence.** Suppose S is a set. A *sequence* in S is a function $f : \mathbb{N} \to S$, i.e., a function from the set of natural numbers to the set S. Alternatively, we can think of a sequence as an infinite list $a_1, a_2, \dots, a_n, \dots$ indexed by $n \in \mathbb{N}$. Here's how we go back and forth between the two notions:

- *From function to list*: Given a function $f : \mathbb{N} \to S$, the corresponding list is $f(1), f(2), \dots$.
- *From list to function*: Given a list $a_1, a_2, \dots, a_n, \dots$, the corresponding function is the function $n \mapsto a_n$.

The value $f(n)$ or a_n is termed the n^{th} *term* of the sequence. We often describe the sequence with the shorthand $(a_n)_{n \in \mathbb{N}}$, or just (a_n). The position in the sequence, which is given by the *subscript*, is sometimes termed the *index*.

For the purpose of this course, we deal with sequences in $\mathbb{R}$, i.e., *sequences of real numbers*.

Examples are:

(1) The sequence $1, 2, 3, \dots$ corresponds to the function $f(n) = n$.
(2) The sequence $1, 4, 9, 16, 25, 36, 49, \dots$ corresponds to the function $f(n) = n^2$.
(3) The sequence $1, -1, 1, -1, 1, -1, \dots$ corresponds to the function $f(n) = (-1)^{n+1}$.

We will talk later about rules for pattern recognition in sequences.

1.2. **The range of a sequence.** The range of a sequence is defined as the set of values that it takes. Equivalently, it is the range of the sequence viewed as a function. Thus, the range of the sequence (a_n) is the set:

$$\{a_n : n \in \mathbb{N}\}$$

The range of a sequence is *not* the same thing as the sequence, because the range is *just* a set, and does not capture the order in which the elements appear. Moreover, the sequence itself can have repetitions whereas the underlying set does not have repetitions.

1.3. **Types of sequences.**

(1) A sequence is said to be *free of repetition* if the corresponding function is one-to-one.
(2) A sequence (a_n) is termed *constant* if it is constant as a function, so the list just lists the same element again and again.
(3) A sequence is termed *eventually constant* if there exists a natural number n_0 such that $a_n = C$ for some fixed number C for all $n \geq n_0$.
(4) A sequence (a_n) is termed *periodic* if there exists a natural number k such that $a_{n+k} = a_n$ for all natural numbers n.
(5) A sequence (a_n) is termed *eventually periodic* if there exist natural numbers n_0, k, such that $a_{n+k} = a_n$ for all $n \geq n_0$.
(6) A sequence (a_n) is termed *increasing* or *strictly increasing* if $a_{n+1} > a_n$ for all n. Note that unlike the previous notions, increasing makes sense only for sequences in a set which has an ordering. Since we're dealing with sequences of real numbers, this is not an issue. An increasing sequence is free of repetition.
(7) A sequence (a_n) is termed *non-decreasing* or *weakly increasing* if $a_{n+1} \geq a_n$ for all n. Note that any increasing sequence is non-decreasing. A constant sequence is also non-decreasing.
(8) A sequence (a_n) is termed *decreasing* or *strictly decreasing* if $a_{n+1} < a_n$ for all n. A decreasing sequence is free of repetition.
(9) A sequence (a_n) is termed *non-increasing* or *weakly decreasing* if $a_{n+1} \leq a_n$ for all n. Note that any decreasing sequence is non-increasing. A constant sequence is also non-increasing.
(10) A sequence (a_n) is termed *monotonic* if it is either non-increasing or non-decreasing.
(11) There are "eventually" versions of all these notions: increasing, decreasing, non-increasing, and non-decreasing.

(12) A sequence (a_n) is termed *bounded* if its range is bounded. Note that a monotonic sequence is automatically bounded from one side, and we therefore only need to check whether it is bounded from the other side.

1.4. **Operations on sequences.** Recall that there are plenty of operations that we can perform on functions. These give rise to operations on sequences, because sequences are functions on the natural numbers. Specifically, the *pointwise combinations* all give rise to analogous operations on sequences.

(1) For a sequence (a_n) and a real number λ, we can define the sequence $(\lambda(a_n))$, whose n^{th} term is λ times the n^{th} term of the original sequence.
(2) For two sequences (a_n) and (b_n), we can define sequences $(a_n + b_n)$, $(a_n - b_n)$, $(a_n b_n)$, and (a_n/b_n) (this last one makes sense only if $b_n \neq 0$ for all n). If the sequences are interpreted as functions, these correspond to the pointwise addition, subtraction, multiplication, and division respectively.
(3) For a function $g : \mathbb{R} \to \mathbb{R}$ and a sequence (a_n), we can define a sequence $(g(a_n))$, whose n^{th} term is the image under g of the original n^{th} term of the sequence. In fact, g itself need not be defined for all real numbers – it only needs to be defined on the range of the sequence.

Here are some other operations:

(1) Left and right shifts: Given a sequence (a_n) and a natural number k, we can define a sequence (b_n) by $b_n = a_{n+k}$. What we have done is basically moved all terms of the sequence k units to the left. This is similar to what we've noticed for functions: the graph of $f(x + h)$ is obtained from the graph of f by shifting to the left by h. We can also do a *right shift*, where we define $c_n = a_{n-k}$, but we have the problem that we need to provide a new definition for the first k terms of the new sequence.
(2) Splicing two sequences: Suppose a_n and b_n are two sequences. We can define a new sequence by splicing the terms together. The new sequence, as a list, goes like $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$. We will come back to this later.

2. Describing and identifying sequences

2.1. **Three descriptions.**

(1) The *general term description* or *closed form description* is an explicit description of the function underlying a sequence. It is a rule that directly allows us to, given n, compute the n^{th} term of the sequence. Closed form expressions are useful because they *directly* allow us to compute the n^{th} term without going through the rigmarole of computing previous terms.
(2) *Description by example*, which is what people use before they learn algebra, just lists the first few terms of the sequence and lets people guess the pattern. The advantage of this is that for sequences that are very easy to describe, description by example is compact and also can be understood by people with no formal exposure to algebra. The drawback is that as soon as the sequence becomes more complicated, description by example fails. Also, the first few terms in a sequence may fit many different patterns so there is always some ambiguity.
(3) *Description by a recurrence relation*: Here, the n^{th} term of the sequence is described in terms of the $(n-1)^{th}$ term and perhaps the $(n-2)^{th}$ term and some earlier terms. In other words, each term is defined in temrs of its predecessors. This is also termed a *recursive definition* or an *inductive definition*. Recursive definitions are useful in many cases because closed form expressions are hard to construct. They also allow us to quickly compute the *next term* if we have already computed a given term.

2.2. **Pattern recognition: from example to concrete description.** When a sequence is described by listing the first few terms, we need to infer a general pattern that allows us to compute future terms. In some cases, we may infer the general pattern in terms of a closed form expression (general term description) while in other cases, we might infer the general pattern in terms of a recursive definition. Either way, we are then able to predict successive terms.

Consider, for instance, the sequence:

$$1, 2, 3, 4, 5, \ldots$$

We easily predict that the general term is $a_n = n$. For a kid who is just learning to count and add, it might also be more helpful to describe the sequence recursively, by:

$$a_n = a_{n-1} + 1$$

Consider now the sequence:

$$1, 3, 5, 7, 9, 11, \ldots$$

If you give this to a smart and arithmetically enriched but algebraically deprived child, the child will notice that each term is obtained by adding 2 to the preceding term. On the basis of this, the child can predict that the next term is 13, the term after that is 15, and so on. In other words, what the student notes is that:

$$a_n = a_{n-1} + 2$$

This inductive definition allows the student to predict the n^{th} term given the $(n-1)^{th}$ term. But to compute, say, the 1000^{th} term requires computing the first 999 terms. A closed form expression would be much nicer.

Using the profound fact that multiplication is repeated addition, we can notice that, in fact:

$$a_n = 2n - 1$$

we can *prove* using induction that the inductive definition is equivalent to this closed form expression.

Now, we can compute a_{1000} without computing the first 999 terms. Multiplication rocks! And so does the idea of a closed form expression.

Here is another example:

$$2, 5, 10, 17, 26, \ldots$$

There are two different ways of determining the underlying rule behind the problem. One of these requires a thorough memory of *multiplication tables*, while the other can be gleaned merely by a knowledge of *addition and subtraction*. Seeing why these two rules are equivalent, on the other hand, requires the use of *algebraic manipulation*, and the rigor is provided by the *principle of mathematical induction*.

Let's first consider the multiplication table approach. We note that $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, and so on. We thus note that the n^{th} term os given by:

$$a_n = n^2 + 1$$

This means that the 15^{th} term is $15^2 + 1 = 226$, while the 29^{th} term is $29^2 + 1 = 842$. And so on.

What if you don't know your multiplication tables? Then, you look for increments. You note that $a_2 - a_1 = 3$, $a_3 - a_2 = 5$, $a_4 - a_3 = 7$, and $a_5 - a_4 = 9$. Thus, it seems that the increments themselves are getting incremented by 2. We thus get:

$$a_n - a_{n-1} = a_{n-1} - a_{n-2} + 2$$

or:

$$a_n = 2a_{n-1} - a_{n-2} + 2$$

We can go somewhere midway, and notice that:

$$a_n = a_{n-1} + (2n - 1)$$

2.3. **A recursive definition requires a specification of initial values.** There's a little point that needs to be clarified. This goes back to how we formulate the principle of mathematical induction. It is that if a sequence is defined recursively, it is essential to specify initial values. More specifically:

(1) If a sequence is defined by a recurrence relation that expresses the n^{th} term in terms of the $(n-1)^{th}$ term, then we need to specify the first term. Otherwise, we don't know how to start.
(2) If a sequence is defined by a recurrence relation that expresses the n^{th} term in terms of the $(n-1)^{th}$ term and $(n-2)^{th}$ term, then we need to specify the first two terms. Otherwise, we again don't know how to start.
(3) If a sequence is defined by a recurrence relation that expresses the n^{th} term in terms of the previous m terms, then we need to specify the first m terms, because the recurrence relation can be successfully applied only from the $(m+1)^{th}$ term onward.

2.4. **Applying induction.** To prove that a given recursive definition of a sequence is equivalent to a given closed form definition, we often use the *principle of mathematical induction*. Essentially:

(1) Case where the first term is given and the n^{th} term is defined in terms of the $(n-1)^{th}$ term: In this case, we check the base case of $n = 1$ satisfies the closed form expression and then use the recurrence relation to show that if the $(n-1)^{th}$ term satisfies the closed form expression, so does the n^{th} term.
(2) Case where the first m terms are given and the n^{th} term is defined in terms of the m preceding terms: In this case, our base cases are the first m terms and the recurrence relation allows us to do the induction step on the n^{th} term.

We have done this in the past. For instance, we proved by the principle of mathematical induction that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

What we are essentially doing is showing that the sequence (a_n) defined by:

$$a_n = a_{n-1} + 1, \qquad a_1 = 1$$

is the same as the sequence:

$$a_n = \frac{n(n+1)}{2}$$

All this reminds us of ... differential equations.

3. Discrete calculus

We now tie together a lot of ideas from the continuous and discrete realms. While the ideas are being developed here *only at a conceptual level*, and you are not expected to master any of the details, *these are very very important ideas*. Understanding the analogy as well as the concrete give-and-take between continuous and discrete calculus is very important for a careful quantitative analysis in any discipline.

3.1. **The derivative.** In old-fashioned calculus, we define the derivative of f at x as:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

There are two one-sided notions of derivative. The *right hand derivative* is the right hand limit:

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

On the other hand (literally), the left hand derivative is the left hand limit:

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

The quotient whose limit we are trying to compute is a *difference quotient*, and it is the slope of the secant line joining two points on the graph of f (namely, $(x, f(x))$ and $(x+h, f(x+h))$. The limiting quantities are the one-sided derivatives, and they are the slopes of one-sided tangent lines to the graph at $(x, f(x))$. If these two one-sided derivatives are equal, then the function has a tangent line at the point.

The crucial property of the reals that we are using in defining the derivative is that we can keep taking real numbers closer and closer to a given number on either side. Thus, the derivative really *is* a limit and cannot be computed simply as a difference quotient.

Contrast this with the natural numbers or integers. The natural numbers (as also the integers) are discrete. For any integer, there is a unique *next* integer and a unique *preceding* integer. This suggests the following definitions for left hand derivative and right hand derivative: the right hand derivative of a function $f : \mathbb{N} \to \mathbb{R}$ at $n \in \mathbb{N}$ is $f(n+1) - f(n)$, while the left hand derivative is $f(n) - f(n-1)$. If we graph f and then connect successive points by straight lines, the slopes of these straight lines give the values of the derivatives.

Note that the left hand derivative at n is not defined for $n = 1$, and equals the right hand derivative at $n - 1$ for $n > 1$. Thus, we can, for most practical purposes, consider the right hand derivative only. We denote this by Δf, so:

$$(\Delta f)(n) := f(n+1) - f(n)$$

The operator Δ is often called the *forward difference operator*. In sequence terms:

$$(\Delta a)_n = a_{n+1} - a_n$$

This *discrete derivative* behaves in a manner remarkably similar to differentiation of functions on the reals. Note, however, that unlike the real numbers, *every* function is differentiable.

3.2. **Repeated derivatives, polynomials, and "integration".** We can apply the forward difference operator multiple times to a function, with the k^{th} application denoted Δ^k. What happens under such application? Some obvious things:

(1) Applying the forward difference operator to a constant sequence gives the zero sequence.
(2) Applying the forward difference operator to a polynomial function (i.e., a polynomial sequence) of degree $d > 0$ gives a polynomial of degree $d - 1$.
(3) In particular, repeated application of the forward difference operator to any polynomial sequence gives the zero sequence.
(4) Applying the forward difference operator to a periodic sequence gives a periodic sequence.

Can we define a notion of integration? In other words, given Δf, can we recover f? Yes, up to a constant:

(1) Knowing Δf allows us to construct a recursive definition of f, given by $f(n) = f(n-1)+(\Delta f)(n-1)$. Thus, the only free parameter we have to choose is $f(1)$. This is analogous to the $+C$ in indefinite integration.
(2) More explicitly, if $\Delta f = g$, then:

$$f(n) = f(1) + g(1) + g(2) + \cdots + g(n-1) = f(1) + \sum_{i=1}^{n-1} g(i)$$

where the $f(1)$ is the freely varying parameter, that we could christen C in analogy with indefinite integration.
(3) If Δf is a polynomial sequence of degree d, f is a polynomial sequence of degree $d + 1$.
(4) More generally, if we know $\Delta^m(f)$, we know f modulo the values of f at the first m natural numbers. This is because knowing $\Delta^m(f)$ allows us to write down an explicit recurrence relation for $f(n)$ in terms of the values of f on the preceding m values. This is analogous to the fact that knowing the m^{th} derivative of a function determines the function uniquely up to additive polynomials of degree less than m (a total of m free parameters or m degrees of freedom).

3.3. **Discrete differential equations.** Here is the analogy:

(1) A recursive definition of a sequence (i.e., of a function on $\mathbb{N}$) is analogous to a *differential equation*. The number of previous terms that are needed to define the n^{th} term is the *order* of the differential equation.

(2) A closed form definition of a sequence is thus analogous to a solution of the differential equation.
(3) The *general solution* to a recursive definition that harks back m terms allows us to pick in a largely unconstrained manner the first m terms. This is analogous to the fact that a differential equation of order m has a general solution with m degrees of freedom.

For the simplest kind of discrete differential equations, there is a complete solution strategy, that we will not go into here, but that you can pick up if the need arises. We turn instead to a somewhat trickier story: periodic sequences and exponential sequences.

Aside: the analogue to autonomous differential equation. Recall that an autonomous differential equation is one where the independent variable does not explicitly appear. In our discrete analogue, the independent variable (the time-line variable) is n, i.e., the index of location in the sequence. The corresponding notion to autonomous differential equation is that of a recurrence relation that does not explicitly reference n except as the subscript/index. Any left shift of a solution to such a recurrence relation is also a solution to the recurrence relation. What this also means is that each term can be predicted by looking at the preceding terms without caring about how far out in the sequence we are.

3.4. **Periodic sequences.** What goes around comes around, and periodic sequences are an example. Periodic sequences are often easy to spot, but how to express them in closed form is not completely obvious. First, an example:

$$1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$$

The pattern is clear. But how do we describe it mathematically? Here are two attempts:

(1) *An attempt at a closed form expression*: We note that a_n is equal to 1 if $n-1$ is a multiple of 3, 2 if $n-2$ is a multiple of 3, and 3 is n is a multiple of 3. we can also put it this way: a_n is the remainder on dividing n by 3, except when the remainder is 0, in which case it is 3.
(2) *An attempt at a recursive definition*: $a_1 = 1$, and $a_n = a_{n-1} + 1$ if a_{n-1} is not 3, and $a_n = 1$ and $a_{n-1} = 3$.

Let us try to understand the closed form expression more closely. What we are doing is noting that the sequence has period 3. If a sequence (a_n) has period p, then the value a_n depends *only* on the value of the remainder on dividing n by p. There are only finitely many possible remainders: 0, 1, and so on up to $p-1$. By specifying what happens for each remainder, we have completely specified the function.

In the *special case* that the period is 2, there is another way of creating a closed form expression. In other words, if our sequence looks like:

$$\alpha, \beta, \alpha, \beta, \alpha, \beta, \ldots$$

Then the n^{th} term is given by:

$$\frac{\alpha + \beta}{2} + \frac{(-1)^n(\beta - \alpha)}{2}$$

We are using the special fact that the sequence $(-1)^n$ itself has period 2. This trick does not work for larger n.[1]

3.5. **Sequences with periodic derivative.** Consider this:

$$10, 9, 8, 9, 8, 7, 8, 7, 6, 7, 6, 5, \ldots$$

We notice here that the derivative sequence is given by:

$$-1, -1, 1, -1, -1, 1, -1, -1, 1, \ldots$$

Thus, even though the original sequence is not periodic, the derivative sequence is periodic. This is the discrete analogue of functions such as $x - \sin x$, where the function is not periodic but the derivative is. Many of the comments made in the context of the $x - \sin x$ apply in this context as well. In particular, any such sequence can be expressed as a sum of a linear sequence and a periodic sequence. The linear sequence

[1] Actually, it does, but we need to use complex numbers or trigonometry to make it work.

describes the secular trend, and the periodic sequence describes the “seasonal” fluctation. We will not pursue this further right now as it would take us too far afield.