ROOT AND RATIO TESTS

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 12.4.

What students should definitely get: The statements and typical applications of the root test and the ratio test.

What students should hopefully get: The notion of when a given test is indecisive, and the rationale behind these tests.

Throughout the material covered here, we deal with series of nonnegative terms.

Executive summary

Words ...

(1) *Not for discussion*: A geometric series with common ratio less than 1 is a discrete analogue of exponential decay. A geometric series with common ratio greater than 1 is a discrete analogue of exponential growth. The common ratio of the geometric series is a parameter controlling growth, just as the constant k controls growth in e^{kx}.
(2) If a series of nonnegative terms is eventually bounded from above by a geometric series with common ratio less than 1, then the series converges. This is the idea behind both the root test and the ratio test.
(3) The root test for a nonnegative series $\sum a_k$ looks at the limit $\lim_{k\to\infty} a_k^{1/k}$. If this limit is less than 1, the root test tells us that the series converges. If the limit is greater than 1, the series diverges. If the limit equals 1, the root test is indecisive (i.e., the series may converge or it may diverge).
(4) The ratio test for a nonnegative series $\sum a_k$ looks at the limit $\lim_{k\to\infty} a_{k+1}/a_k$. If this limit is less than 1, the series converges. If the limit is greater than 1, the series diverges. If the limit equals 1, the ratio test is indecisive (i.e., the series may converge or it may diverge).
(5) Here is a slight modification of the root test: if $a_k^{1/k}$ is greater than 1 for infinitely many k, the series diverges. This is for the simple reason that the terms cannot go to 0. On the other hand, if the sequence $a_k^{1/k}$ is *eventually bounded away from and below* 1 (i.e., bounded from above by a number strictly less than 1) then the series converges. The inconclusive case is thus where the sequence does eventually get below 1 but cannot be bounded away from 1 (i.e., it has terms arbitrarily close to 1).
(6) Here is a slight modification of the ratio test: if a_{k+1}/a_k approaches 1 from the right, or, more generally, if it is ≥ 1 for all sufficiently large k, the series diverges. This is because the terms do not go to 0. On the other hand, if a_{k+1}/a_k is bounded away from and bleow 1, the series converges. The inconclusive case is where the series comes really close to or overshoots 1 infinitely often.
(7) The root test is stronger than the ratio test. The reason is that the ratio test is highly sensitive to the precise orderings of the terms, while the root test can handle small permutations. [An example of this is in one of the advanced homework problems. Please look it up to refresh your memory.]

Actions ...

(1) The root test is more useful for power functions.
(2) The ratio test is more useful for factorials.
(3) For rational functions, both tests are indecisive, and we fall back on the rule covered earlier about the difference of degrees of numerator and denominator.
(4) In some cases, it is somewhat more convenient to massage the series a little before applying the root and ratio tests. As long as this massaging does not change the property of whether or not the series converges, that is perfectly fine.

1. Geometric series: some reflections

Our current goal is to devise more tests and techniques that can be used to quickly determine whether a given series converges. The idea is to compare the series with some known series behavior using such things as the *basic comparison theorem* (which is about the terms of one series being eventually bounded by the terms of the other series) and the *limit comparison theorem* (which is about the quotients of terms of two series having a finite nonzero limit).

The root test and the ratio test build on the ideas of basic comparison and limit comparison along with a heavy reliance on the properties of geometric series.

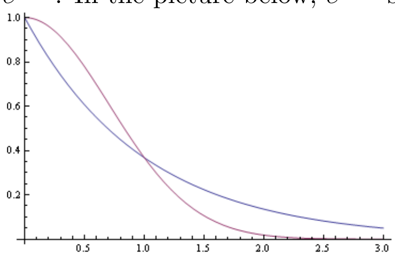
1.1. Geometric or exponential decay.

The geometric series with common ratio $r < 1$ is analogous to an exponential decay. The fact that such a geometric series converges is the discrete analogue of the fact that $\int_0^\infty e^{-kx}\,dx$ is finite.

How can we graphically characterize geometric or exponential decay? What is happening is that the graph of the *logarithm* function is linear with negative slope. Another way of thinking of it is that the *multiplicative* rate of decay is constant.

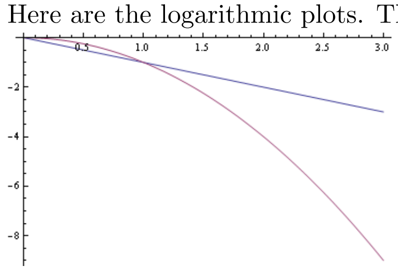
1.2. Decaying faster than exponential.

An example of a function that decays faster than exponential is e^{-x^2}. In the picture below, e^{-x^2} starts off decaying slower than e^{-x} but then overtakes it at $x = 1$:



We see that although e^{-x^2} starts off decaying more slowly than e^{-x}, it soon gets to decay a lot faster and reaches very close to zero rather quickly. A plot of the logarithm of this function reveals (the right half of) a downward facing parabola.

Here are the logarithmic plots. The straight line is $-x$, and the downward sloping parabola half is $-x^2$:



A discrete version of e^{-x^2} may be the function e^{-n^2}, for $n \in \mathbb{N}$. If e spooks you out, you can consider 2^{-n^2} – it has the same qualitative behavior.

It is worth thinking a bit about what it means qualitatively to decay faster than exponential.

Exponential decay occurs in situations where the current rate of decay is in constant proportion to the current quantity. Radioactivity and interest rates are examples. In the case of radioactivity, there is a fixed probability that a given atom undergoes radioactive decay in a given time frame, and so the fraction of the total number of atoms that undergo radioactive decay is a constant. This constant proportionality forces exponential decay.

The decay that we see in e^{-x^2} is a lot faster. Note that if $f(x) = e^{-x^2}$, then $f'(x) = -2xe^{-x^2}$. Thus, we get $f'(x)/f(x) = -2x$. The quotient $-2x$ is not a constant – it increases in magnitude as x gets larger. Thus, in proportional terms, e^{-x^2} decays faster than e^{-x}.

What this means, in practical terms, is that e^{-x^2} has much thinner tails than e^{-x}. For $f(x) = e^{-x^2}$, the value $f(5)/f(4)$ is much smaller than the value $f(4)/f(3)$.

Aside: Implications for understanding the normal distribution. The normal distribution is a mainstay of statistics and it arises a lot in the natural and the social sciences. The probability density function of the normal distribution is (up to scaling and translation) $e^{-x^2/2}$ for $x \in \mathbb{R}$, and the cumulative distribution function is $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}\,dt$.

The fast decay can be interpreted in the following way in statistics: the proportion of items that are two standard deviations above the median to those that are one standard deviation above the median is much bigger than the proportion at three standard deviations to two standard deviations. For height, for instance, if we assume a median of 5 feet 10 inches and a standard deviation of 2 inches, then the proportion of 6 feet 2 inches to 6 inches is much higher than the proportion of 6 feet 4 inches to 6 feet 2 inches.

1.3. **Bounded by a geometric implies summable (integrable).** The basic comparison theorem tells us that if the terms of one series (with nonnegative terms) are eventually bounded by a summable series, then the first series is also summable. The analogous statement for integration is also obvious.

In particular, if a series decays at a rate faster than geometric, it should be bounded by some geometric series, and hence should be summable. Analogously, if a nonnegative function decays at a rate faster than exponential, it is bounded from above by an exponentially decaying function and hence has a finite improper integral.

2. Root test and ratio test

2.1. **The ratio test.** Recall that for a geometric series of nonnegative terms, each term is given by a fixed multiple of the preceding term:

$$a_{k+1} = ra_k$$

and, if the series is convergent, then $0 < r < 1$.

This suggests that, for a series that is not a geometric series, we study the ratios of successive terms and see what these ratios look like. Specifically, for a series $\sum a_k$ with *nonnegative terms* we look at:

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

The ratio test says the following things:

(1) If the limit is less than 1, the series converges. The explanation is that we can find a geometric series with common ratio strictly less than 1 that bounds it eventually from above.
(2) If the limit is greater than 1, the series diverges. The explanation is that we can find a geometric series with common ratio strictly greater than 1 that bounds it eventually from below.
(3) If the limit is equal to 1, the test is inconclusive. In other words, it is possible that the series converges and it is possible that the series diverges. Note that *all* the p-series, some of which converge and some of which diverge, fall under this inconclusive case of the ratio test.

2.2. **The root test.** While the ratio test works by generalizing from the recursive description of geometric series, the root test works by generalizing from the closed form description. Recall that the closed form descroption of a geometric series is of the form $a_n = a_0 r^n$. Equivalently, $a_n^{1/n} = (a_0)^{1/n} r \to r$ as $n \to \infty$. We know that if $r < 1$, the series converges, and if $r > 1$, the series diverges.

The root test says that for a series $\sum a_n$ with nonnegative terms, we consider the limit:

$$\lim_{k \to \infty} (a_k)^{1/k}$$

(1) If the limit is less than 1, the series converges. This is because it can be bounded from above by a geometric series with common ratio less than 1.
(2) If the limit is greater than 1, the series diverges. This is because it can be bounded from below by a geometric series with common ratio greater than 1. An even more trivial reason is that the terms cannot go to 0, so the series cannot converge.
(3) If the limit equals 1, the test is inconclusive, i.e., the series may converge or diverge – we need to do something more to figure out exactly what's happening.

2.3. **Ratio test and root test.** The ratio test and root test have roughly the same power, and which test we choose to apply to a given situation depends only on which of the expressions $(a_k)^{1/k}$ and a_{k+1}/a_k is easier to compute and take the limit of.

Strictly speaking, the root test is more powerful than the ratio test. In other words, any series to which we can conclusively apply the ratio test is also a series to which we can conclusively apply the root test, and in fact, the limit of the sequence of ratios is the same as the limit of the sequence of roots. However, there do exist series where the root test works and the ratio test does not. This is essentially because the ratio test is very sensitive to the ordering of terms in the series while the root test depends only on the rough position. If we take a series that succumbs to both tests and shuffle the terms around slightly, the new series continues to satisfy the root test but no longer satisfies the ratio test. An example of such a series is in your advanced homework for this week.

2.4. **Oscillatory limits.** What if we try to apply the ratio test, and it turns out that the ratio isn't heading to any particular limit? We can still hope to use somewhat more general forms of the ratio test. Specifically, if the ratios a_{k+1}/a_k are eventually bounded below and away from 1 (i.e., there is some number $\alpha < 1$ and some k_0 such that $a_{k+1}/a_k < \alpha$ for all $k \geq k_0$) then the series converges. In other words, we aren't really interested in the ratios actually converging to a single number – we only care that they eventually settle down somewhere that is clearly below 1.

Similarly, of the ratios a_{k+1}/a_k are eventually bounded above and away from 1, the series diverges.

The same remarks apply to the root test.

2.5. **Approaching 1 from the left or the right.** For both the ratio and the root test, the limit equal to 1 is the inconclusive case. One subcase of this, however, is clear.

> If the sequence of ratios (respectively, roots) approaches 1 from the *right*, i.e., all terms of the sequence beyond a certain point are bigger than 1, then the series diverges. This is because if the ratios (respectively roots) are eventually bigger than 1, the series cannot go to 0.

Thus, the *genuinely* inconclusive case is the case where the sequence of ratios (or roots) approaches 1 either from the left side or in an oscillatory fashion. (For the root test, even infinitely many elements bigger than 1 are enough to declare divergence; for the ratio test, it is inconclusive).

In this case, the question is whether this sequence (of ratios/roots) approaches 1 slowly enough that the terms still go to 0 fast enough for the sums to converge. The slower the approach to 1, the faster the terms go to zero, and the better the conditions for the series sum to converge.

3. Modifying the series before applying root and ratio tests

We can apply any convergence-preserving modification before applying the root and ratio tests. Some of these modifications are discussed below:

- Left and right shifts: Applying these shifts does not affect the outcome of the root or ratio test, i.e., if we could use the root or ratio test after shifting, we could also use it, arriving at the same conclusion, prior to shifting.

  However, a left or right shift may make the root test easier to apply (in the sense of computational ease). For instance, for the series $2^{(n+1)^2}$, a shift might make it 2^{n^2} to which the root test is easier to apply.
- Basic comparison and limit comparison: If computing ratios or roots for the original series terms is hard, we can use basic comparison or limit comparison to shift to a new series for which the computation is easier.
- Permutations of terms: Applying permutations may make the root test easier to apply, but will not change the outcome. On the other hand, it may well change the outcome of the ratio test from inconclusive to conclusive. For example, consider a geometric series of positive terms with common ratio r between 0 and 1. Now, alter this series by flipping the $(2n-1)^{th}$ and $(2n)^{th}$ terms for every n. In the new series, the common ratios between $1/r$ and r^3, so the ratio test fails. But if we undid this flip operation, the common ratios would all become r, so the ratio test would be applicable. The root test, on the other hand, gives the same conclusion in both cases, but it is easier to apply for the actual geometric series rather than the messed-up one.

4. Summary of tests

We have seen the following rules/tests for nonnegative series:

(1) A *necessary but not sufficient condition* for a series of nonnegative terms to converge is that the terms approach zero. (In fact, this condition is also necesssary but not sufficient when the series has terms of mixed sign, something we shall talk about later).
(2) The rule for geometric series.
(3) The integral test, both in terms of a precise numerical bound and in its more general form of saying that the summation is finite if and only if the integral is finite.
(4) The basic comparison test, which says that if one series is eventually bounded by another, then there is a relationship (conclusive only in some directions) between the convergence/divergence of the series.

    We also developed some practical rules for the convergence of series constructed using rational functions and similar things. These practical rules can be proved rigorously using p-series and the basic comparison or limit comparison tests.
(5) The limit comparison test, which says that if, for two series, the quotients of correpsonding terms has a finite nonzero limit, then one series converges if and only if the other series does.
(6) The ratio test, which looks at the limit of the rations of successive terms of a series and uses that to predict convergence/divergence. The inconclusive case is where the ratio limits to 1.
(7) The root test, which looks at the limit of the n^{th} root of the n^{th} term, and uses that to predict convergence/divergence. The inconclusive case is where the root limits to 1.