REVIEW SHEET FOR MIDTERM 2: BASIC

MATH 153, SECTION 55 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of the basic and advanced review sheet AND the previous review sheet to the review session.

This review sheet does *not* repeat review of material in the midterm 1 syllabus, although some of the "Quickly" stuff does not overlap. So, please go through the midterm 1 review sheet too. In the review session, we will concentrate on the midterm 2 review sheet (advanced version), but we will review some of the formulas from the basic version. However, we will not review every point and you should do that review on your own time.

This is the basic review sheet. The error-spotting exercises are in the advanced review sheet.

1. Left-overs from integration

1.1. Integrating radicals.

Words ...

(1) Expressions of the form $a^2 + x^2$ (with $a > 0$) in the denominator or under the radical sign suggest the substitution $\theta = \arctan(x/a)$. With this substitution, $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$, $a^2 + x^2 = a^2\sec^2\theta$, and $\sqrt{a^2 + x^2} = a\sec\theta$. In the end, when substituting back, we use $\theta = \arctan(x/a)$, $\tan\theta = x/a$, $\sec\theta = \sqrt{a^2 + x^2}/a$, $\cos\theta = a/\sqrt{a^2 + x^2}$, and $\sin\theta = x/\sqrt{a^2 + x^2}$. The first sentence of substitutions is useful when converting the given integrand into a trigonometric integrand. The second sentence is useful when converting the integrated answer back at the end. (This latter step is unnecessary when we are dealing with a definite integral and we transform limits simultaneously).

(2) For $a^2 - x^2$ under a square root, we have a similar substitution $\theta = \arcsin(x/a)$. For $x^2 - a^2$, we take $\theta = \arccos(a/x)$. It is useful to work out the forward and backward substitutions for these. (See the notes for the details of these substitutions). *It is strongly suggested that you internalize both the forward and the backward substitutions to the point where they become automatic. Memorization helps, but you should also be able to re-derive things on the spot as the need arises.*

(3) There is a little subtlety in these substitutions. When we take θ as arcsin, we know that $\cos\theta$ is nonnegative. Hence, when we simplify $\sqrt{a^2 - x^2}$, we get $\sqrt{a^2\cos^2\theta}$. Because by assumption a is positive, and because $\cos\theta$ is nonnegative, we can write the answer as $a\cos\theta$. In other words, we know how exactly we can lift off the square root. Something similar happens when we are dealing with the tangent and secant functions: secant is nonnegative on the range of arc tangent. Unfortunately, tangent is *not* nonnegative on the entire range of arc secant, so we need to actually look at the region where we are carrying out the integration. In case both the upper and lower bounds of integration are greater than a, we know that we will in fact get $\tan\theta$.

Note: Some of you may find it useful to draw right triangles, as suggested in the book, if reading trigonometric ratios off triangles is easier for you than algebraic manipulation of trigonometric expressions.

Actions ...

(1) Trigonometric substitutions allow us to integrate things like $x^m(a^2 + x^2)^{n/2}$. However, some special cases of these can be integrated without resort to trigonometric substitutions. For instance, when n is a nonnegative even integer, this is a sum of powers of x and can be integrated term wise. Also, if m is odd, we can do a u-substitution with $u = a^2 + x^2$.
(2) Similar remarks apply to expressions involving $\sqrt{a^2 - x^2}$ and $\sqrt{x^2 - a^2}$.
(3) To apply this or similar techniques to more general quadratics, we need to use a technique known as *completing the square*. Here, we rewrite:

$$Ax^2 + Bx + C = A(x + (B/2A))^2 + (C - B^2/4A)$$

The special case where $A = 1$ is given by:

$$x^2 + Bx + C = (x + (B/2))^2 + (C - (B/2)^2)$$

Note that the left-over constant term after completing the square is $-D/4A$ where D is the discriminant of the quadratic polynomial. In the case $A = 1$, when the polynomial has positive discriminant, this left-over term is negative, whereas when the polynomial has negative discriminant, this left-over term is positive. In the latter case, we can write it as the square of something. We would thus have written our original polynomial as $(x - \beta)^2 + \gamma^2$, whereupon we can make the substitution $\theta = \arctan((x - \beta)/\gamma)$ (or directly apply the integration formula).

1.2. **Partial fractions.** Words ...

(1) For most practical purposes, we can study *monic* polynomials instead of arbitrary polynomials. A monic polynomial is a polynomial whose leading coefficient is 1. The reason we can restrict attention to monic polynomials is that any nonzero polynomial can be expressed as a nonzero constant times a monic polynomial.
(2) A nonconstant monic quadratic polynomial is irreducible (i.e., cannot be expressed as a product of polynomials of smaller degree) if and only if it has negative discriminant.
(3) Every nonconstant monic polynomial with real coefficients is a product of monic linear polynomials and irreducible monic quadratics, and this factorization is unique. Thus, all irreducible monic polynomials are either linear or quadratics with negative discriminant.
(4) The partial fractions approach breaks up any rational function as the sum of a polynomial and rational functions of the form R/Q^k where Q is a monic irreducible factor of the original denominator and R is a polynomial of degree strictly less than the degree of Q.
(5) Each of these partial fraction pieces is easy to integrate. The case where Q is linear, it is of the form $x - \alpha$, and the numerator is a constant, so this is a straightforward power integration. In the case where Q is quadratic, we break R as the sum of a constant and the derivative of Q. The constant part is handled by a trigonometric substitution, and the derivative of Q part is handled by the u-substitution $u = Q$.
(6) The partial fractions approach shows that every rational function can be integrated, and we obtain an antiderivative that involves ln (evaluated at some linear function of x), arctan (again, evaluated at some linear function of x), and other rational functions.
(7) Using the partial fractions approach and the equivalence of repeated integrability with the integrability of x times a function, we can show that any rational function can be *repeatedly* integrated, with the final answer in terms of arctan, ln, and rational functions.

Actions ... Please go through the notes on partial fractions as well as the discussion of these in the book. We here list only some salient points:

(1) Before beginning, make the denominator monic, and use the Euclidean algorithm to reduce to a problem where the degree of the numerator is less than the degree of the denominator.
(2) The general approach is to first factorize the denominator and then break it up into partial fractions with unknown numerators. The coefficients of the numerator need to be determined. One way of doing this is to take a common denominator, multiply out, compare coefficients, and solve the resultant system of linear equations.
(3) Instead of equating coefficients, we can also use a strategy of plugging in values. We plug in values so that a large number of the expressions that we are evaluating become zero.
(4) In particular, if we want to write:

$$\frac{r(x)}{(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)} = \frac{c_1}{x - \alpha_1} + \dots + \frac{c_n}{x - \alpha_n}$$

where all the α_is are distinct and the degree of r is less than n, then we get:

$$c_i = \frac{r(\alpha_i)}{\text{product of } \alpha_i - \alpha_j, \text{ all } j \neq i}$$

We can use this to very rapidly write any fraction with denominator a product of distinct linear factors in terms of partial fractions, and then integrate it.

(5) To handle:

$$\frac{r(x)\,dx}{(q(x))^k}$$

where q is an irreducible quadratic, we do repeated division, taking quotients and remainders, and obtain the result in terms of partial fractions.

(6) A thorough understanding of the partial fractions approach should allow you to predict, simply by looking at a rational function, whether the antiderivative expression for it will be (i) a rational function, (ii) something involving rational functions and arctan, (iii) something involving rational functions and ln, or (iv) something involving rational functions, arctan, and ln. For some practice of these, refer to the integration quiz.

1.3. **Improper integrals.** Words ...

(1) The integral $\int_a^\infty f(x)\,dx$ is defined as the limit $\lim_{L\to\infty}\int_a^L f(x)\,dx$. If F is an antiderivative of f, this equals $\lim_{L\to\infty} F(L) - F(a)$.
(2) The integral $\int_{-\infty}^a f(x)\,dx$ is defined as the limit $\lim_{L\to-\infty}\int_L^a f(x)\,dx$. If F is an antiderivative of f, this equals $F(a) - \lim_{L\to-\infty} F(L)$.
(3) The integral $\int_{-\infty}^\infty f(x)\,dx$ is defined as a double limit, where the upper limit of integration is limited to infinity, while the lower limit of integration is limited to negative infinity. If F is an antiderivative of f, this equals $\lim_{L\to\infty} F(L) - \lim_{M\to-\infty} F(M)$.
(4) Another kind of improper integral occurs where the function is integrated over an interval and is not defined at an endpoint of the interval. Here, we take the limit over intervals of integration where the interval gradually tends towards the trouble points. For instance, if a function f is to be integrated over $[a, b]$ but b is a trouble point, we take $\lim_{c\to b}\int_a^c f(x)\,dx$. If F is an antiderivative of f, then if F extends continuously to b, this is just equal to $F(b) - F(a)$.
(5) In general, if there are multiple trouble points, we first partition the interval of integration so that all the trouble points are at the partition boundaries. We then use the limiting procedure on each piece and add up across the pieces.

Actions ...

(1) The most straightforward way of computing an indefinite integral is to compute the corresponding antiderivative and take the difference between the upper and lower limits.
(2) In some cases, this is either infeasible or terribly messy. In these cases, we may use the various other methods for computing definite integrals that bypass computing the antiderivative. These include the use of symmetry and a combination of u-substitution plus noticing that after the substitution, the upper and lower limits of integration become the same.
(3) In yet other cases, taking the limit of the antiderivative may be hard, and we may need to use all the techniques discussed in preceding subsections for computing this antiderivative.

2. Differential equations

2.1. **Solving differential equations at large.** Words ...

(1) A differential equation with dependent variable y and independent variable x is something of the form $F(x, y, y', y'', \dots) = 0$.
(2) The *order* of a differential equation is the *largest* k for which the k^{th} derivative appears in the differential equation. In particular, a *first-order differential equation* only involves x, y, and y', and does not involve y'' or higher derivatives. A *second-order differential equation* only involves x, y, y', and y''.
(3) A *polynomial differential equation* is one where F looks like a polynomial in y and its derivatives. A *linear differential equation* is a differential equation of the form:

$$f_k(x)y^{(k)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$

We can clear out the coefficient of $y^{(k)}$ by dividing throughout by $f_k(x)$. The *homogeneous* case is where the right side is zero.

(4) A *particular solution* is a relation $R(x, y) = 0$ that, when plugged into the differential equation, satisfies it. (Here, higher derivatives are computed using *implicit differentiation*). A particular solution in *functional form* is one where we explicitly find a function f with $y = f(x)$ that satisfies the differential equation.
(5) A *solution family* is a family with one or more parameters such that for every permissible value of the parameter, we obtain a particular solution.
(6) The *general solution* is a solution family that contains all particular solutions.
(7) A general principle is that the number of freely varying parameters in the general solution, also described as the number of *degrees of freedom*, equals the order of the original differential equation. The reason for this is roughly that the number of integrations we do that introduce new degrees of freedom equals the order of the differential equation.
(8) An *autonomous* differential equation is a differential equation where the independent variable does not appear explicitly (except as the thing in terms of which differentiation is carried out). The independent variable can be thought of as *time*. Autonomous differential equations have the property that any time translate of a solution is also a solution. This property is found in most physical laws, and essentially states that the formulation of the physical law does not depend on when we started measuring time, i.e., there is no natural time origin.
(9) To solve a second-order differential equation we usually do a substitution to break it up into solving two first-order differential equations.
(10) Of first-order differential equations, there are two broad classes that we know how to solve: *separable* differential equations and *linear* differential equations. For the latter case, the solution method isolates y as a function of x. In the former case, we can get a mixed bag situation.

Actions ...

(1) The separable case is where we have $y' = f(x)g(y)$. In this case, we rearrange to obtain $\int dy/g(y) = \int f(x)\,dx$, and integrate both sides. We need to put the $+C$ on only one side, because additive constants emanating from both integrals can be combined into one additive constant.
(2) In the autonomous separable case, we have $dy/dt = g(y)$, and we integrate to obtain $\int dy/g(y) = \int dt$. This is the case that arises when we look at the logistic equation and its many variants.
(3) In the linear case, we have $y' + p(x)y = q(x)$ (after dividing out by any coefficient of y'). Let $H(x) = \int p(x)\,dx$. The integrating factor that we choose is $e^{H(x)}$. When we multiply by this integrating factor the left side becomes the derivative of $ye^{H(x)}$. Thus, we obtain:

$$y = e^{-H(x)} \int q(x)e^{H(x)}\,dx$$

Note that the $+C$ arises in the *inner* integral, so the general solution is a particular solution plus $Ce^{-H(x)}$.
(4) When solving differential equations (particularly the separable case) we often get a solution involving logarithms. In some cases, it may be useful to *exponentiate* both sides. When we do so, the original additive constant C arising from indefinite integration becomes a multiplicative constant e^C. We can also absorb *sign uncertainty* into it and define a new constant $k = e^C \operatorname{sgn}(y)$ to get the answer in terms of a sign expression.
(5) In a similar vein to the above, if our answer involves an inverse trigonometric function, we can apply the trigonometric function to both sides. In this case, the additive constant *sticks inside*. For instance, if we get $\arctan y = x + C$, then applying tan to both sides yields $y = \tan(x + C)$. To simplify this further (if we so desire), we need to use the angle sum formula. The other major caveat that we need to bear in mind is that there is a *loss of information* when we apply the trigonometric substituion to both sides, because an inverse trigonometric function value is constrained to a particular range. This constraint needs to be kept track of separately.
(6) In some cases, before exponentiating or applying the inverse trigonometric function, it might help to use the initial value condition to pin down the freely varying parameters (see the next subsection).

2.2. **Graphical interpretation and initial value problems.** Words ...

(1) Any *particular solution* (whether expressed with y as an explicit function of x or in terms of a relation between x and y) can be plotted as a curve in the xy-plane. When it is an explicit function, this is the *graph of a function* – otherwise, it's just the set of points satisfying the relation. This picture in the plane is called an *integral curve* or a *solution curve* for the differential equation.
(2) The *general solution* is thus a picture which has all the particular solutions marked.
(3) Since solving a k^{th} order differential equation introduces k degrees of freedom, we expect that to pin down a unique solution, we need k pieces of information. In particular, to choose one particular solution for a first-order differential equation, we need (by and large) one piece of information. In an *initial value problem*, this is provided by specifying an initial value, which is one point (x_0, y_0) on the particular solution curve.
(4) Geometrically, we expect that the solution family to a first-order differential equation has one real parameter and that, except in some degenerate cases, knowing one point on the curve determines the curve. In other words we expect that by and large, the solution curves do not intersect.
(5) For higher-order differential equations, on the other hand, we expect that even after knowing one point on the curve, we have pinned down only one of many degrees of freedom, and we still have solution families to deal with rather than isolated solutions. More information, such as information about higher derivatives, or information about the curve passing through other points, is desirable.

Actions ... Nothing really, except that we plug in the initial value condition to pin down the constants.

3. The least upper bound axiom

Words ...

(1) The real numbers satisfy the *least upper bound property*: any nonempty subset of the set of real numbers that is bounded from above has a least upper bound *that is also a real number*. This property does *not* hold if we replace the real numbers by the rational numbers (i.e., the least upper bound of a set of rationals bounded from above exists as a real number, but need not be a rational number).
(2) The real numbers satisfy the *greatest lower bound property*: any nonempty subset of the set of real numbers that is bounded from below has a greatest lower bound *that is also a real number*. This property again does *not* hold if we replace the real numbers by the rational numbers.
(3) We can prove the greatest lower bound property using the least upper bound property. There are two proofs of this. One of these proofs involve *reflection*: replacing a set by its set of negatives. The other proof, which is there in the book, is also worth going through. Please go through it. I'll go through it in review session. You will not be asked the proof in the test, but it may be helpful for multiple choice questions and other conceptually based problems.
(4) The natural numbers satisfy a property that is somewhat similar to the greatest lower bound property for the reals, but stronger: any nonempty subset of the set of natural numbers has a least element. This is equivalent to the *principle of mathematical induction*.
(5) If a nonempty subset of the real numbers has a *maximum* element, then that element is also the least upper bound of the set. Conversely, if the least upper bound of a set is in the set, then that is also the maximum element of the set.
(6) If a nonempty subset of the real numbers has a *minimum* element, then that element is also the greatest lower bound of the set. Conversely, if the greatest lower bound of a set is in the set, then that is also the minimum element of the set.
(7) A nonempty finite subset always has a maximum and a minimum element. Thus, its greatest lower bound and least upper bound are both in the set.
(8) For an interval with lower endpoint a and upper endpoint b, the least upper bound is b and the greatest lower bound is a. Note that this holds for all the four possibilities for the interval: $[a, b]$, (a, b), $[a, b)$, and $(a, b]$.
(9) If T is a nonempty subset of a nonempty bounded subset S of $\mathbb{R}$, any lower bound for S remains a lower bound for T and any upper bound for S remains an upper bound for T. However, we may have an upper bound for T that is *not* an upper bound for S. Similarly, we may have a lower bound

for T that is *not* a lower bound for S. Thus, the least upper bound for T is $\leq$ the least upper bound for S, and the greatest lower bound for T is $\geq$ the greatest lower bound for S.

(10) A set does *not* have an upper bound if and only if it has arbitrarily large elements. Similarly, a set does *not* have a lower bound if and only if it has arbitrarily small elements (i.e., negative elements of arbitrarily large magnitude).

(11) If M is the least upper bound of a nonempty subset S of $\mathbb{R}$, then, for every $\epsilon > 0$, S has a nonempty intersection with the interval $(M-\epsilon, M]$. In particular, if $M \notin S$, then S has a nonempty intersection with the interval $(M - \epsilon, M)$. (See also the analogous theorem for greatest lower bounds, which is Theorem 11.1.4 in the book).

Actions ...

(1) To compute the greatest lower bound and least upper bound of a set, we first need to compute the set. Finding the set as a union of intervals is often useful.

(2) Given a set S, we can construct corresponding sets such as $S + \lambda$ (translation), $-S$ (reflection about 0), $f(S)$ (image of S under a function f), and $\text{abs}(S)$ (the set of absolute values of elements of S, i.e., folding about 0). Please review the results that relate bounds for S with bounds on these corresponding sets.

4. Sequences of reals

4.1. **Sequencs: basics.** Words ...

(1) A sequence in a set is a function from $\mathbb{N}$ to that set.

(2) A sequence of reals can be described in three ways: as an ordered list of real numbers (where we write only the first few members due to space and time considerations), as a closed form expression for the general term of the sequence (i.e., thinking of it as a function), and in terms of a recurrence relation.

(3) The *range* of a sequence is the set of values that it takes. The range of a sequence differs from the sequence in the following two senses: (i) it ignores repetition (ii) it ignores the ordering or sequencing of the elements.

(4) There are many properties that we can talk of in the context of sequences: increasing, decreasing, non-increasing, non-decreasing, monotonic, constant, periodic, bounded from below, bounded from above, and bounded. Note that the boundedness-related properties depend *only* on the range of the sequence, whereas the other properties depend on the sequence as a whole.

(5) For any property p that can be evaluated on each sequence, we can talk of the property *eventually* p, which means that some left shift of the sequence has the property p. For instance, we can talk of eventually increasing, eventually decreasing, eventually constant, eventually monotonic, and eventually periodic.

(6) Eventually bounded is the same as bounded.

(7) There are various operations we can do on sequences similar to the corresponding operations on functions: add, subtract, multiply, divide, multiply by a scalar, and compose with a function defined on the range of the sequence.

(8) We can do left shifts, right shifts (though this requires us to throw in more new terms), splicing, and other fancy operations.

(9) If a sequence is defined recursively, i.e., using a recurrence relation, then we need to separately specify initial values. The number of initial values that we need to specify depends on how far back the recurrence relation reaches. This is related both to the principle of mathematical induction and the idea of free parameters and initial value specifications for differential equations.

(10) We can define a discrete derivative, called the *forward difference operator*, defined as follows: for a sequence $f : \mathbb{N} \to \mathbb{R}$, the forward difference operator is $(\Delta f)(n) = f(n+1) - f(n)$. This is analogous to the derivative of a continuous function.

(11) The *integration equivalent* for the forward difference operator is the summation operator. (Try to find the formula for this in the notes/class discussion).

(12) The forward difference operator behaves analogously to differentiation (though the formulas differ) for constants, polynomial sequences, and periodic sequences. (Please see the notes for a list of points).
(13) Periodic sequences can be defined using a case-wise definition based on the remainder modulo the period. They can alternatively be defined as combinations of trigonometric functions. In the special case of a period 2, we can also express in terms of $(-1)^n$.
(14) A sequence with periodic derivative can be expressed as the sum (pointwise) of a linear sequence and a periodic sequence.

4.2. **Continuous-discrete interplay.** Words ...

(1) Given a function on $\mathbb{R}$, we can restrict the function to $\mathbb{N}$ and obtain a sequence. This restriction is unique.
(2) Conversely, given a sequence, i.e., a function on $\mathbb{N}$, we can extend it to a continuous function on $\mathbb{R}$. However, the extension is not unique, and there are a lot of different ways of extending. If the sequence is described by means of a nice closed form functional expression, we may be able to extend it by considering that functional expression for all real numbers.
(3) Usually, information about the function on the reals gives us corresponding information about the corresponding sequence, but we cannot get information in the reverse direction that easily. For instance, an increasing function gives an increasing sequence, but increasing sequences can arise from functions that are not increasing. A decreasing function gives a decreasing sequence, a monotonic function gives a monotonic sequence, and a bounded function gives a bounded sequence.
(4) A function with integer period gives a periodic sequence.
(5) The mean value theorem relates the derivative of a function to the discrete derivative (i.e., forward difference operator) of the corresponding sequence.
(6) We can define a notion of concave up and concave down for sequences based on the second discrete derivative. If a function is concave up, so is the corresponding sequence. If the function is concave down, so is the corresponding sequence.

5. Limit computation techniques

Words ...

(1) L'Hôpital's rule for $0/0$ form: Consider $\lim_{x \to c} f(x)/g(x)$ where $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$. If both f and g are differentiable around c, then this limit is equal to $\lim_{x \to c} f'(x)/g'(x)$.
(2) Analogous statements hold for one-sided limits and in the case $c = \pm\infty$.
(3) The $0/0$ form LH rule *cannot* be applied if the numerator does not limit to zero or if the denominator does not limit to zero.
(4) L'Hôpital's rule for ∞/∞ form: Consider $\lim_{x \to c} f(x)/g(x)$ where $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$. If both f and g are differentiable around c, then this limit is equal to $\lim_{x \to c} f'(x)/g'(x)$.
(5) Analogous statements hold for one-sided limits and for $c = \pm\infty$.
(6) For a function f having a zero at c, the *order* of the zero at c is the least upper bound of β such that $\lim_{x \to c} |f(x)|/|(x-c)|^\beta = 0$. At this least upper bound, the limit is usually finite and nonzero. For larger values of β, the limit is undefined.
(7) Given a quotient f/g for which we need to calculate the limit at c, the limit is zero if the order of the zero for f is greater than the order for g, and undefined if the order of the zero for f is less than the order for g. When the orders are the same, the limit could potentially be a finite nonzero number.

Actions ...

(1) For polynomial functions and other continuous functions, we can calculate the limit at a point by evaluating at the point. For rational functions, we can cancel common factors between the numerator and the denominator till one of them becomes nonzero at the point.
(2) There is a bunch of basic limits that translate to saying that for the following functions f: $f(0) = 0$ and $f'(0) = 1$, which is equivalent to saying that $\lim_{x \to 0} f(x)/x = 1$. These functions are $\sin x$, $x \mapsto \ln(1+x)$, $x \mapsto e^x - 1$, $x \mapsto \tan x$, $x \mapsto \arcsin x$, and $x \mapsto \arctan x$.

(3) For all the functions f of the above kind (that we call *strippable*), the following is true: in any multiplicative situation, if the input to the function goes to zero in the limit, the function can be stripped off.
(4) Two other basic limits are $\lim_{x \to 0}(1 - \cos x)/x^2 = 1/2$ and $\lim_{x \to 0}(x - \sin x)/x^3 = 1/6$. These can be obtained using the LH rule.
(5) Typically, to compute limits, we can combine the LH rule, stripping, and removing multiplicative components that we can calculate directly.
(6) Applying the LH rule 0/0 form pushes the orders of both the numerator and the denominator down by one.
(7) It is also useful to remember that logarithmic functions are dominated by polynomial functions, which in turn are dominated by exponential functions. These facts can be seen in various ways, including the LH rule.