REVIEW SHEET FOR MIDTERM 1: BASIC

MATH 153, SECTION 55 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet, the advanced review sheet, AND the integration worksheet to the review session.

This is the **basic** part of the review sheet and you are expected to go through it by yourself. There is a separate **advanced** part of the review sheet that contains error-spotting exercises that we'll be doing in the review session.

The document is arranged as follows. The initial sections/subsections correspond to topics. Each subsection has two sets of points, "Words" which includes basic theory and definitions, and "Actions" which provides information on strategies for specific problem types. In some cases, there are additional points. The lists of points are largely the same as the executive summaries at the beginning of the lecture notes. Sometimes, I missed out some points in the original executive summaries and have added them, clearly indicating that the point has been added.

There may be parts of the lectures that are not mentioned in this review sheet. Sometimes, these are the parts of lectures that were optional and were included for "fun" purposes, such as an explanation of present value, the separation of secular trends from seasonal trends, etc. The review sheet is an attempt to strike a balance between brevity and relevance on the one hand and comprehensiveness on the other. If you're keen on more, go back to the lecture notes. If you're keen on less, be selective in what you read.

To maximize efficiency in the review session, here is what I suggest. Go through all the lists of points. For each point, make sure you understand it by jotting down a relevant example or illustration or providing a brief justification. If you have difficulty, go back to the lecture notes and read them in detail. You might also want to look at more worked examples in the book, and check out homework and quiz problems.

We will not go over this review sheet in the review session unless you have specific questions. Instead, we will concentrate on the advanced part and also do a lot of integration practice.

1. Exponential growth and decay

1.1. **Basics of exponential growth and decay.** Words ...

(1) A function f is said to have exponential growth if $f'(t) = kf(t)$ for all t. Such a function must be of the form $f(t) := Ce^{kt}$. Here, $C = f(0)$, and can be thought of as the initial value. k is a parameter controlling the *rate of growth*. When $k > 0$, we have growth, and when $k < 0$, we have *decay*. When $k = 0$, there's no growth or decay.
(2) For a function with exponential growth and growth rate k, the time taken for the function value to multiply by a number $q > 0$ depends only on q and k. Specifically, the time interval is $(\ln q)/k$. In particular, for growth, the doubling time is $(\ln 2)/k$. Note that if $\ln q$ and k have opposite signs, the time taken is negative – which means that we need to go *back* in time to multiply by a factor of q.
(3) If a function takes a time interval t_{d1} to multiply by a factor of q_1 and a time interval t_{d2} to multiply by a factor of q_2, we have the relation: $\ln(q_1)/t_{d1} = \ln(q_2)/t_{d2}$. Thus, given three of these quantities, we can calculate the fourth.
(4) Exponential functions grow faster than all positive power functions (and hence all polynomial functions) while logarithmic functions grow slower than all positive power functions.
(5) Exponential functions *decay* slower than linear functions.
(6) *Added*: The graph of the logarithm of a function Ce^{kt} is $kt + \ln C$. In particular, it is a straight line with slope k and intercept $\ln C$. In particular, the fact that describing an exponential growth function requires two parameters or two observation points is equivalent to the fact that you need two pieces of information to describe a straight line, or that given any two points, there is a unique straight line joining them.

Actions ...

(1) Suppose we know that f is a function of the form $f(t) := Ce^{kt}$, but we do not know the values of the constants C and k. One way of determining these values is to determine the value of f and f' at some point t_0. We can then get k as $f'(t_0)/f(t_0)$ and then solve to get $C = f(t_0)/e^{kt_0}$. This type of specification is termed an *initial value specification* and a problem with such a specification is termed an *initial value problem*.

(2) Another way we can determine C and k is if we are given the value of f at two points t_1 and t_2. In this case, we solve to obtain that $k = \frac{1}{t_2 - t_1} \ln[f(t_2)/f(t_1)]$, and we can plug back to get $C = f(t_1)/e^{kt_1}$. Note that this is a reformulation of the formula that the time taken to multiply by a factor of q is $(\ln(q))/k$. This kind of specification is somewhat related to the notion of *boundary value specification*.

(3) But in many real-world situations, we do not need to actually determine the constants C and k. Rather, we use the fact that $\ln(q_1)/t_{d1} = \ln(q_2)/t_{d2}$ to compare the rates of growth in two intervals.

(4) Even nicer, in many cases, we do not need to know the actual values of $\ln(q_1)$ and $\ln(q_2)$, because the only thing that matters is their *quotient* $\ln(q_2)/\ln(q_1)$, which can also be viewed as the relative logarithm $\log_{q_1}(q_2)$. Thus, for instance, if q_2 is a rational power of q_1, we know the quotient precisely even though we may not know $\ln(q_1)$ and $\ln(q_2)$. For instance, $\ln(8)/\ln(4) = 3/2$.

(5) *Added*: To check whether a function has exponential growth, we can plot the graph of its logarithm, and check whether that is a straight line. Plotting just two points gives us no evidence either way, though it is sufficient for finding the values of k and C *assuming* that growth is exponential. Plotting three or more points and finding them to be collinear provides affirmative evidence.

1.2. **Compound interest.**

(1) Compound interest: This is written as $A(t) = A_0 e^{rt}$ where r is the *continuously compounded interest rate*, A_0 is the initial principal or initial amount, and $A(t)$ is the amount at time t. The corresponding differential equation is $A'(t) = rA(t)$. Continuously compounded interest differs from simple interest, where $A(t) = A_0(1+rt)$ and from discretely compounded interest, where the interest earned is added to the principal at periodic intervals.

(2) The time taken for the amount to double under continuously compounded interest with rate r is $(\ln 2)/r$, which is approximately $0.7/r$. When r is expressed as a percentage, we need to divide 70 by that percentage to get the doubling time. (Times here are typically measured in years). This is called the *rule of 70*. *Note*: The rule of 70 also applies to discretely compounded interest rates when r is very small, but that is a topic for next quarter.

1.3. **Radioactive decay.**

(1) A radioactive material undergoes *decay*, i.e., its quantity goes *down* with time. The constant k in the expression Ce^{kt} is thus a negative number.

(2) The fraction that remains is 1 minus the fraction that decays. Thus, if 1/3 of the material decays, then the relevant q to plug into formulas is $q = 2/3$, *not* 1/3.

(3) The rate of decay of radioactive materials is typically measured by their half-life, which is the time taken for half the material to decay and half to remain. (1/2 is the only number that is equal to 1 minus itself). We have the formula $k = (-\ln 2)/\text{half-life}$.

2. Inverse trigonometric functions

2.1. **Main points.** Words ...

(1) The functions sin, cos, tan, and their reciprocals are all periodic functions. While tan and cot have a period of π, the other four have a period of 2π each. While sin and cos are continuous and defined for all real numbers, the other four functions have points of discontinuity where they approach infinities of different signs from both sides. Also, tan and cot are one-to-one on a single period domain, while the other functions are usually two-to-one.

(2) To construct inverses to these functions, we take intervals small enough such that the function is one-to-one restricted to that interval, but the range of the function restricted to that interval is the

whole range. We also try to make our choice in such a manner that the *other function in the square sum/difference relationship* is nonnegative on the domain.

(3) The choices are: $[-\pi/2, \pi/2]$ for sin (note that cos is nonnegative on this), $[0, \pi]$ for cos (note that sin is nonnegative on this), $(-\pi/2, \pi/2)$ for tan (note that sec is nonnegative on this), and $(0, \pi)$ for cot (note that csc is nonnegative on this).

(4) We define the corresponding inverse trigonometric functions arcsin, arccos, arctan and arccot. The domains for both arcsin and arccos equal $[-1, 1]$ while the domains for both arctan and arccot equal all of $\mathbb{R}$. The range of arcsin is $[-\pi/2, \pi/2]$ and the range of arccos is $[0, \pi]$. The range of arctan is $(-\pi/2, \pi/2)$ and the range of arccot is $(0, \pi)$.

(5) arcsin is an increasing function with vertical tangents at the endpoints, and arccos is a decreasing function with vertical tangents at the endpoints. For all x, we have $\arcsin x + \arccos x = \pi/2$. arctan is an increasing function with horizontal asymptotes valued at $-\pi/2$ and $\pi/2$ and arccot is a decreasing function with horizontal asymptotes valued at π and 0.

(6) We define the arc secant function and the arc cosecant function as $\operatorname{arcsec}(x) = \arccos(1/x)$ and $\operatorname{arccsc}(x) = \arcsin(1/x)$. These are defined for all x outside $(-1, 1)$.

(7) Using the formula for differentiating the inverse function, we obtain that $\arcsin'(x) = 1/\sqrt{1 - x^2}$ and $\arccos'(x) = -1/\sqrt{1 - x^2}$. Thus, $\int dx/\sqrt{1 - x^2} = \arcsin x + C$. Also, we obtain $\int dx/\sqrt{a^2 - x^2} = \arcsin(x/a) + C$.

(8) Similarly, we obtain that $\arctan'(x) = 1/(1 + x^2)$ and $\operatorname{arcsec}'(x) = 1/(|x|\sqrt{x^2 - 1})$. Note that in the case of arc secant, we need the absolute value to account for the fact that tangent is not nonnegative on the range of arc secant.

Actions ...

(1) $\sin(\arcsin x) = x$ if x lies in the domain of the arcsin function. Note that otherwise $\sin(\arcsin x)$ does not make sense. Similar observations hold for the other trigonometric functions.

(2) *Solving equations*: The solutions to $\sin x = \alpha$, where $\alpha \in [-1, 1]$ come in two families: $\{2n\pi + \arcsin \alpha : n \in \mathbb{Z}\}$ and $\{2n\pi + (\pi - \arcsin \alpha) : n \in \mathbb{Z}\}$. Similarly, the solutions to $\cos x = \alpha$ where $\alpha \in [-1, 1]$ come in two families: $\{2n\pi + \arccos \alpha : n \in \mathbb{Z}\}$ and $\{2n\pi - \arccos \alpha : n \in \mathbb{Z}\}$. In the special case where $\alpha = 1$ (respectively, $\alpha = -1$), the two solution families for sin (respectively, cos) collapse into one solution family.

(3) $\arcsin(\sin x)$ need not be equal to x – they are equal if and only if x is in the range of arcsin. *Added*: The function $\arcsin \circ \sin$ is a piecewise linear function with a sawtooth shape and a period of 2π. See the lecture notes for a more detailed description.

(4) We often want to compute things like $\cos(2 \arctan x)$. To tackle these situations, we set $\theta = \arctan x$, so we obtain $\tan \theta = x$. The problem now reduces to determining $\cos(2\theta)$ in terms of $\tan \theta$, which is some elementary trigonometry. (It's elementary only if you know at least some of the double-angle formulas).

(5) The integration formulas for $1/\sqrt{1 - x^2}$ etc. give rise to many slightly more general integration formulas. The list is given below. Don't just memorize it, make sure you internalize it.

Note: In class, we discussed a heuristic based on homogeneous degree. That heuristic was not mentioned in the lecture notes, and in fact it is covered in the lecture notes for integrating radicals, which we plan to cover on January 21 and which is *not* in the midterm syllabus. If you already understand the homogeneous degree stuff, that's great, but you don't need to spend time thinking about it for now.

2.2. **The formulas for indefinite integration.** We have the following formulas for indefinite integration:

$$\begin{aligned} \int \frac{dx}{\sqrt{1 - x^2}} &= \arcsin(x) + C \\ \int \frac{dx}{1 + x^2} &= \arctan(x) + C \\ \int \frac{dx}{|x|\sqrt{x^2 - 1}} &= \operatorname{arcsec}(x) + C \end{aligned}$$

We now consider slight variants of these. In all the formulas below, $a > 0$ is a constant. We can obtain all these formulas from the previous ones via the u-substitution $u = x/a$:

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \arcsin\left(\frac{x}{a}\right) + C \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \\
\int \frac{dx}{|x|\sqrt{x^2 - a^2}} &= \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C
\end{aligned}$$

We consider yet more variants of these. Here, $a > 0$ and b is arbitrary, but both are constants:

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - (x-b)^2}} &= \arcsin\left(\frac{x-b}{a}\right) + C \\
\int \frac{dx}{a^2 + (x-b)^2} &= \frac{1}{a}\arctan\left(\frac{x-b}{a}\right) + C \\
\int \frac{dx}{|x-b|\sqrt{(x-b)^2 - a^2}} &= \frac{1}{a}\operatorname{arcsec}\left(\frac{x-b}{a}\right) + C
\end{aligned}$$

Next, we see how these can be combined with other ideas. As before, $a > 0$ is a constant:

$$\begin{aligned}
\int \frac{f'(x)\,dx}{\sqrt{a^2 - (f(x))^2}} &= \arcsin\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f'(x)\,dx}{(f(x))^2 + a^2} &= \frac{1}{a}\arctan\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f'(x)\,dx}{|f(x)|\sqrt{(f(x))^2 - a^2}} &= \frac{1}{a}\operatorname{arcsec}\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f(\arctan(x/a))}{a^2 + x^2}\,dx &= \frac{1}{a}\int f(u)\,du \text{ where } u = \arctan(x/a) \\
\int \frac{f(\arcsin(x/a))}{\sqrt{a^2 - x^2}}\,dx &= \int f(u)\,du \text{ where } u = \arcsin(x/a)
\end{aligned}$$

3. Hyperbolic functions

(1) We define *hyperbolic cosine* $\cosh x := (e^x + e^{-x})/2$ and *hyperbolic sine* $\sinh x := (e^x - e^{-x})/2$. cosh is the *even part* of the exponentiation function (and in particular, is an even function) while sinh is the *odd part* of the exponentiation function (and in particular, is an odd function).
(2) cosh and sinh are derivatives of each other, and hence also antiderivatives of each other.
(3) cosh is even and positive, decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, concave up throughout, goes to ∞ as $x \to \pm\infty$, and its local and absolute minimum value of 1 are attained at 0.
(4) sinh is odd, increasing on all of $\mathbb{R}$, negative and concave down on $(-\infty, 0)$, and positive and concave up on $(0, \infty)$. It passes through $(0, 0)$ where it has its unique point of inflection. Note that at $(0, 0)$, the derivative takes its minimum value, which is 1. In this important respect, the graph does *not* look like x^3, where we have a horizontal tangent at $x = 0$.
(5) $\cosh^2 x - \sinh^2 x = 1$. A lot of the identities involving hyperbolic sine and hyperbolic cosine look very similar to the corresponding identities involving the trigonometric (circular) sine and cosine. In fact, we can move back and forth between the circular and the hyperbolic using the following rule: change the sign in front of any term that involves a product of two sine terms. This rule is termed *Osborne's rule*.

4. Integration by parts

Words ...

(1) Integration by parts is a technique that uses the product rule to integrate a product of two terms. If F and g are the two functions, and G is an antiderivative of g, we obtain:

$$\int F(x)g(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx$$

This basically follows from the product rule, which states that:

$$\frac{d}{dx}[F(x)G(x)] = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + F'(x)G(x)$$

In particular, we *integrate* one function and *differentiate* the other.

(2) Applying integration by parts twice stupidly tells us nothing. In particular, if we choose to re-integrate the piece that we just obtained from differentiation, we get nowhere.

(3) The definite integral version of this is:

$$\int_a^b F(x)g(x)\,dx = [F(x)G(x)]_a^b - \int_a^b F'(x)G(x)\,dx$$

In particular, note that the part outside the integral sign is simply evaluated between limits.

(4) We can use integration by parts to show that integrating a function f twice is equivalent to integrating f and the function $xf(x)$. More generally, integration a function f k times is equivalent to integrating $f(x)$, $xf(x)$, $x^2f(x)$, and so on up till $x^{k-1}f(x)$. *Added*: Please beware that being able to integrate $xf(x)$ alone is not equivalent to being able to integrate f twice. Rather, being able to integrate both $f(x)$ and $xf(x)$ is equivalent to being able to integrate f twice.

(5) To integrate $e^x g(x)$, we can use integration by parts, typically taking e^x as the second part. We could also do this integral by finding a function f such that $f + f' = g$, and then writing the answer as $e^x f(x) + C$. The latter approach is feasible and sometimes quicker in case g is a polynomial function.

(6) *Added*: Integrating the inverse function: Integration by parts can be used to integrate the inverse of a function that we know how to integrate. Specifically, the formula is:

$$\int f^{-1}(x)\,dx = xf^{-1}x - \int x(f^{-1})'(x)\,dx$$

The latter integral can be rewritten as $\int f(u)\,du$ after the substitution $u = f^{-1}(x)$. In particular, this means that knowing how to integrate f allows us to integrate f^{-1}.

Actions ...

(1) Integration by parts is *not* the first or best technique to consider upon seeing a product. The first thing to attempt is the u-substitution/chain rule. In cases where such a thing fails, we move to integration by parts.

(2) For products of trigonometric functions, it is usually more fruitful to apply the trigonometric identities, such as $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, than to use integration by parts.

(3) To apply integration by parts, we need to express the function as a product. The *part to integrate* should always be chosen as something that we *know* how to integrate. Beyond this, we should try to make sure that: (i) the part to differentiate gets *simpler in some sense* after differentiating, and (ii) the part to integrate does not get too much more complicated upon integration.

(4) Beware of the circular trap when doing integration by parts. In particular, when using integration by parts twice, you should always make sure that the part to integrate is *not* chosen as the thing you just got by differentiating.

(5) For polynomial times trigonometric or exponential, always take the polynomial as the first part (the part to differentiate). The trigonometric or exponential thing is the thing to integrate. After enough steps, the polynomial is reduced to a constant, and the trigonometric part (hopefully) does not become any more complex.

(6) The ILATE/LIATE rule is a reasonable precedence rule for doing integration by parts.

(7) In some cases, we may use integration by parts once or twice and then relate the integral we get at the end to the original integral in some other way (for instance, using a trigonometric identity) to solve the problem. Examples include $e^x \cos x$ and $\sec^3 x$.
(8) For functions such as $\ln(x)$, we typically take 1 as the part to integrate and the given function as the part to differentiate.
(9) In general, for functions of the form $f(\ln x)$ or $f(x^{1/n})$, we can first do a u-substitution (setting $u = \ln x$ or $u = x^{1/n}$ respectively). This converts it to a product, on which we can apply integration by parts. *We can also apply integration by parts directly, but this tends to get messy.*

5. Induction

Words...

(1) Induction is a powerful tool that allows us to prove a statement for all positive integers (sometimes, for all positive integers $\geq$ some given positive integer) by proving it in just two special cases. These are the *base case* (proving it for the smallest positive integer in the set, usually 1) and the *induction step*. The induction step is a *conditional implication* that shows that if the statement is true for the positive integer k, then it is true for $k + 1$.
(2) *Statement* here could be some equality or inequality depending on the positive integer. Usually, it is something like a sum of n terms or a product of n terms being equal to some nice polynomial or rational function in n. Sometimes, we have an inequality instead. There are other forms of statement too, such as divisibility statements, but we aren't dealing with them as of now.

Actions (try to recall problems on induction)...

(1) Proving the base case is straightforward, as long as you remember to do it.
(2) To prove the induction step, write what it means for the statement to be true for k, and write what it means for the statement to be true for $k + 1$. Try to figure out a way to prove the *conditional implication*: assuming true for k, prove true for $k + 1$.
(3) With summations, we usually start with the expression for k and add the $(k + 1)^{th}$ term to both sides. Then, we do some algebraic manipulation and we're done. With products, we multiply instead of add.
(4) When dealing with inequality instead of equality, it is usually required to prove an *auxiliary inequality*. Basically, the right side that you get from the k assumption needs to be shown to be related to the right side you need to get for the $k + 1$ conclusion.
(5) Sometimes, you may want to make an educated guess about what you should prove before proving it by induction. We saw some examples involving $(1 - 1/n)$ and $(1 - 1/n^2)$. These are all nice tricks, and this kind of cancellation of successive terms is called *telescoping*. But you will not be expected to guess what to prove – you'll be told. Proving it by induction is largely procedural

Caution ...

(1) Always clearly indicate that statements that you want to show and have not yet established are statements that you want to show.
(2) Please make sure that you show the base case correctly.

Frills ...

(1) Induction is a bit like differentiation/integration. Specifically, the inductive step is an analogue of the derivative, and the base case is an analogue of a specific value of the $+C$ that we see in indefinite integration.
(2) To prove the inductive step in an induction problem, we could try using induction again. This is analogous to differentiating/integrating twice.
(3) There is a concept of induction for sufficiently large integers, where we try to establish a statement only for natural numbers $n \geq n_0$. Both the base case and inductive step need to be suitably modified (the base case is n_0 and the inductive step can assume $k \geq n_0$).
(4) In some variants on induction, we show that $P(k)$ and $P(k - 1)$ implies $P(k + 1)$. If using such a variant, we need to make the base case correspondingly thicker, i.e., we need to show $P(1)$ and $P(2)$. In yet another variant of induction, we assume the truth of P for *all* smaller natural numbers.

(5) It is possible to induct on several parameters, either simultaneously or sequentially. This is a bit like differentiating a function of multiple variables in terms of each of the variables one by one.
(6) Induction can also be used to prove statements that are qualitatively different for different congruence classes modulo d. For such statements, we can either do the usual induction $k \rightsquigarrow k+1$, making cases based on congruence class, or a jump induction $k \rightsquigarrow k+d$, again making cases based on congruence class. In the latter case, we need to establish the first d natural numbers as base cases.