REVIEW SHEET FOR MIDTERM 1: ADVANCED

MATH 153, SECTION 55 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet, the basic review sheet, AND the integration worksheet to the review session.

1. Exponential growth and decay

Combined error-spotting exercises...

(1) The growth of a population is exponential. In one year, the population increases by 300%. The population increase in two years should thus be the square of 300%, which is 900%.
(2) With exponential growth, the time taken for a quantity to increase by 10% is five years. Thus, the time taken for the quantity to increase by 30% must be fifteen years.
(3) With exponential decay, the time taken for a quantity to decrease by 10% is five years. Thus, the time taken for the quantity to decrease by 30% must be fifteen years.
(4) For a function undergoing exponential growth, the ratio of its tripling time to its doubling time is 3/2.
(5) To determine whether a quantity has exponential growth with respect to time and to find the rate of exponential growth, it suffices to observe the quantity at two points in time.

2. Inverse trigonometric functions

2.1. Main points.
Error-spotting exercises ...

(1) The function $f(x) := \arcsin(\sin x)$ coincides with the function $x \mapsto x$ everywhere.
(2) The function $f(x) := \arccos(\sin x)$ coincides with the function $x \mapsto \sqrt{1 - x^2}$ everywhere.
(3) The function $f(x) := \cos(\arcsin x)$ coincides with the function $x \mapsto (\pi/2) - x$ everywhere.

2.2. The formulas for indefinite integration.
Error-spotting exercises ...

(1) We have:

$$\int_1^{\sqrt{3}} \frac{dx}{1 + x^2} = [\arctan x]_1^{\sqrt{3}} = \arctan \sqrt{3} - 1 = \frac{\pi}{6} - 1$$

(2) We have:

$$\int_0^1 \frac{dx}{\sqrt{2 - x^2}} = (1/2)[\arcsin(x/2)]_0^1 = (1/2)(\pi/6) = \pi/12$$

(3) Consider:

$$\int \frac{\cos x \, dx}{3 - 2\cos^2 x} = \int \frac{\cos x \, dx}{3 - 2(1 - \sin^2 x)} = \int \frac{\cos x \, dx}{1 + \sin^2 x}$$

Now, with the u-substitution $u = \sin x$, we get:

$$\int \frac{du}{1 + u^2}$$

This becomes $\arctan u = \arctan(\sin x)$.

3. Hyperbolic functions

Error-spotting exercises ...

(1) We have $\cosh(\ln x) - \sinh(\ln x) = \exp(-\ln x) = -\exp(\ln x) = -x$.
(2) We have $\cosh(2x) = 2\cosh^2 x - 1 = 1 - 2\sinh^2 x = \cosh^2 x - \sinh^2 x$

4. Integration by parts

Error-spotting exercises ...

(1) Using integration by parts, knowledge of how to integrate both f and g is sufficient to know how to integrate the product functin fg.
(2) The u-substitution method for integration is the correct strategy for integrating the composite of two functions. Integration by parts is the correct strategy for integrating the composite of two functions.
(3) We can use integration by parts to show that integrating a function f twice is equivalent to integrating the function $xf(x)$.
(4) We can use integration by parts to show that integrating a function f on the interval $[a, b]$ is equivalent to integrating f^{-1} on the same interval $[a, b]$.
(5) The function $x \mapsto \ln(\sin x)$ can be integrated using the u-substitution $u = \sin x$ and then performing integration by parts (recursive version).

5. Induction

No error-spotting exercises.

6. Quickly

This section lists things you should be able to do quickly.

6.1. Our common values.

Preferably remember these (or be capable of computing quickly) to at least one digit. Generally, you will *not* be asked to do any numerical computations using these. In practice, the main way this is useful is to figure out whether something is positive or negative. For instance, is $3\sqrt{2} - 4$ positive? What about $e^2 - 8$? Often, there are other ways of answering such questions, but remembering the numerical values is a quick and dirty approach.

(1) Square roots of 2, 3, 5, 6, 7, 10.
(2) Natural logarithms of 2, 3, 5, 7, and 10.
(3) Value of π, $1/\pi$, $\sqrt{\pi}$, and π^2.
(4) Value of e, $1/e$.
(5) Some relative logarithms, such as $\log_2 3$ or $\log_2(10)$. Although you don't need these values to a significant degree of precision, it is useful to have some idea of their magnitude.

6.2. Adding things up: arithmetic.

You should be able to:

(1) Do quick arithmetic involving fractions.
(2) Sense when an expression will simplify to 0.
(3) Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
(4) Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

6.3. Computational algebra.

You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3. *This could be an area for potential improvement for many people.*

(4) Factorize an even polynomial of degree four. *This could be an area for potential improvement for many people.*
(5) Do polynomial long division.
(6) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

6.4. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi \pm x)$ and $\cos(\pi \pm x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles or other larger angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is $1/2$.
(6) Formulas for double angles, half angles: $\sin(2x)$, $\cos(2x)$ in terms of sin and cos; also the reverse: $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$.
(7) Remember the formulas for $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, and $\cos(A-B)$.
(8) Convert between products of sin and cos functions and their sums: for instance, the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$. You don't have to remember these identities separately since they follow from the identities covered in the previous point, but you should be comfortable going back and forth.

6.5. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

6.6. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) once or twice on sight, without rough work.
(2) Differentiate sums of powers of x on sight (without rough work).
(3) Differentiate rational functions with a little thought.
(4) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a\sin x + b\cos x$ or $a\exp(-x)$.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic up to constant factors, e.g. $a\exp(mx+b)$ or $a\sin(mx+\varphi)$.
(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).
(7) Differentiate ln, exp, and expressions of the form f^g and $\log_f(g)$.

6.7. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a sum of power functions quickly.
(4) Know that the integral of sine or cosine on any quadrant is ± 1.
(5) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a+bx)^m$.
(6) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$, $\sin^3$, $\cos^3$, $\tan^3$, $\sec^3$, $\cot^3$, $\csc^3$, and other higher powers of the basic trigonometric functions.
(7) Integrate on sight things such as $x\sin(x^2)$, getting the constants right without much effort.
(8) *By parts*: Integrate $\int xf(x)\,dx$ on sight if it is easy to integrate f twice, without having to think much (the answer is $x\int f - \int\int f$). Similarly, do $\int x^2 f(x)\,dx$ if you know how to integrate f twice.
(9) Using the previous point, integrate $\int f(\sqrt{x})\,dx$ with minimal stress and effort.
(10) *By parts*: Integrate $\int f(x)\,dx$ on sight if $xf'(x)$ is easy to integrate, e.g., $\int \ln x\,dx$.
(11) Remember the integrals for formats such as $e^x\cos x$ and $e^x\sin x$.
(12) If there is an easy choice of f such that $f + f' = g$, integrate $\int e^x g(x)\,dx = e^x f(x)$ on sight and similarly integrate $\int g(\ln x)\,dx = xf(\ln x)$ on siht.

6.8. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

6.9. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: $1/3$, $2/3$, $4/3$, $5/3$, $1/2$, 1, 2, 3, -1, $-1/3$ $-2/3$.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.
(8) Graph the inverse trigonometric functions arctan, arcsin, and arccos.

6.10. **Graphing: transformations.** Given the graph of f, you should be able to quickly graph the following:

(1) $f(mx)$, $f(mx+b)$: pre-composition with a linear function; how does $m < 0$ differ from $m > 0$?
(2) $Af(x) + C$: post-composition with a linear function, how does $A > 0$ differ from $A < 0$?
(3) $f(|x|)$, $|f(x)|$, $f(x^+)$, and $(f(x))^+$: pre- and post-composition with absolute value function and positive part fnuction.
(4) More slowly: $f(1/x)$, $1/f(x)$, $\ln(|f(x)|)$, $f(\ln|x|)$, $\exp(f(x))$, and other popular composites.