REVIEW SHEET FOR FINAL: ADVANCED

MATH 153, SECTION 55 (VIPUL NAIK)

1. Series and convergence

Prior to trying the exercises, please review the corresponding sections on "Words", "Actions", and "Cautionary Notes" in the basic review sheet.

Error-spotting exercises...

(1) The sum $\sum_{k=0}^{\infty} 1/k^2$ converges. One way of seeing this is that when $k = \infty$, $1/k^2 = 1/\infty^2 = 0$. Hence, we know that the terms approach 0. We know that for a series to converge, the terms must go to zero. Hence, the terms go to zero. Hence, the series converges.

(2) The sum $\sum_{k=1}^{\infty} 1/x^2$ converges. One way of seeing this is to use the integral test. We know that $1/x^2$ is a nonnegative continuous decreasing function of x. So we can apply the integral test to it, and we get:

$$\sum_{k=1}^{\infty} 1/x^2 = \int_1^{\infty} dx/x^2 = -1/\infty - (-1/1) = 1$$

So the sum is 1, which is a finite number.

(3) The sum $\sum_{k=0}^{\infty} 1/(k^3 + k^2)$ converges, because as we all know:

$$\sum_{k=0}^{\infty} \frac{1}{k^3 + k^2} = \sum_{k=0}^{\infty} \frac{1}{k^3} + \sum_{k=0}^{\infty} \frac{1}{k^2}$$

The sum on the right side converges.

(4) The sum $\sum_{k=1}^{\infty} \frac{k^{5/2}}{k^4 - 4k - 8}$ diverges, because the degree difference $4 - 5/2 = 3/2$ is less than 2.

2. Root and ratio tests

Error-spotting exercises ...

(1) We can use the root test to show that $\sum_{k=1}^{\infty} 1/k^2$ is convergent as follows. The k^{th} root of the k^{th} term is $(1/k)^{2/k}$. As $k \to \infty$, $1/k \to 0^+$, so $(1/k)^{2/k} \to 0$. Hence, the root test applies and the series converges.

(2) By the ratio test, we can show that $\sum_{k=1}^{\infty} 1/k^2$ converges as follows. The ratio of the $(k+1)^{th}$ term to the k^{th} term is $k^2/(k+1)^2$. This is clearly less than 1. By the ratio test, since the ratio is less than 1, the series converges.

(3) Consider the series $\sum_{k=0}^{\infty} \frac{(-3)^k}{k^2+1}$. Applying the ratio test, we get that the limit of ratio of successive terms is -3. This limit is less than 1, so the series converges.

3. Absolute and conditional convergence

Error-spotting exercises ...

(1) Consider the series $\sum (-1)^k k^3/(k^2 + 1)$. We know that the terms are alternating in sign. Hence, by the alternating series theorem, the summation converges.

(2) Consider the series $1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots$. We know that the series converges by the alternating series theorem, and by Abel's theorem, it converges to $\ln 2$. Suppose $A = 1 + 1/2 + 1/3 + 1/4 + \dots$. Then $A/2 = 1/2 + 1/4 + 1/6 + \dots$. Thus $A - A/2 = 1 + 1/3 + 1/5 + 1/7 + \dots$. So we get:

$$A/2 = 1 + 1/3 + 1/5 + 1/7 + \dots$$

We also had:

$$A/2 = 1/2 + 1/4 + 1/6 + 1/8 + \dots$$

Subtracting, we get:

$$0 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots$$

But we already noted that the sum is $\ln 2$. Thus, $0 = \ln 2$.

4. Taylor series

4.1. **Taylor series at 0.** Error-spotting exercises:

(1) The n^{th} Taylor polynomial for a function that is n times differentiable at 0 is a polynomial of degree n.
(2) The Taylor series for $\sin x$ is just x: The first derivative of sin is cos, the second derivative is $-\sin$. We see that the second derivative of sin is 0 at 0. Differentiating 0 further gives 0, so all higher derivatives are also zero. So, the Taylor series is just P_1, which is just the polynomial x.
(3) The Taylor series for $x^{34/3}$ is just the zero polynomial: We know that all higher derivatives of $x^{34/3}$ are powers of x, but since $34/3$ is not an integer, we never get to x^0, and hence all the powers evaluated at 0 give the value 0. So the Taylor series is just 0.
(4) The Taylor polynomial P_2 for $e^x \sin x$ is the product of the Taylor polynomials P_2 for e^x and for $\sin x$.

4.2. **Taylor series in $x - a$.** No error-spotting exercises.

5. Power series

Error-spotting exercises ...

(1) Suppose the radius of convergence of a power series is a positive real number c. Then, the interval of convergence is the *closed* interval $[-c, c]$ if and only if the power series is *absolutely* convergent at c. If the power series is conditionally convergent at c, then the interval of convergence is $(-c, c]$. Similarly, if the power series is conditionally convergent at $-c$, then the interval of convergence is $[-c, c)$. Finally, if the power series is not convergent at either endpoint, the interval of convergence is $(-c, c)$.
(2) Consider the function:

$$f(x) := \frac{1}{(2-x)(3-x)}$$

Then f has a power series about 0 with radius of convergence 1, because it is a rational function. We all know that rational functions have radius of convergence 1.
(3) Consider the function:

$$f(x) := \frac{1}{(x^2+4)(x-3)(x+5)}$$

The radius of convergence of f is 3 (because that's the smallest number at which the function blows up. Further, the interval of convergence is the open interval $(-3, 3)$.
(4) Consider the power series $\sum_{k=0}^{\infty} x^k/2^{k^2}$. The radius of convergence for this is $\lim_{k\to\infty} 1/(1/(2^{k^2}))^{1/k}$, which is 2^k. So, the power series has radius of convergence 2^k.
(5) Consider the power series $\sum_{k=0}^{\infty} 2^{2^k} x^k$. The radius of convergence is $\lim_{k\to\infty} 1/(2^{2^k})^{1/k} = 1/2^2 = 1/4$.
(6) arctan is a function defined and infinitely differentiable on all of $\mathbb{R}$. So, the Taylor series of arctan must have radius of convergence equal to ∞.
(7) Consider the function $f(x) := \sum_{k=0}^{\infty} 2^{k^2} x^k$. The Taylor series for f converges to f on all of $\mathbb{R}$.

(8) Consider the power series $\sum_{k=0}^{\infty} \frac{k^{11/4} 3^k x^k}{p(k)}$ where p is a polynomial with positive leading coefficient that is not zero at any integer. The radius of convergence of this power series is 3. Moreover, the endpoints $+3$ and -3 are included if and only if the degree of p is at least two more than $11/4$. Since p is a polynomial and has integer degree, this is equivalent to saying that the degree of p is at least 5.

6. Summation techniques

Error-spotting exercises...

(1) Consider the summation:

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$$

This picks out only the even degree terms in the power series of the exponential function, hence it return the *even part* of the exponential function. In other words, the summation gives the hyperbolic cosine function cosh.

(2) Consider the summation:

$$\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$$

We can rewrite the terms as:

$$x^k \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

The series then telescopes, and the infinite sum is just $x^1/1 = x$.

(3) The summation:

$$\sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$$

gives $\cosh \sqrt{x}$. To see this, substitute $u = \sqrt{x}$ into the summation, and we obtain:

$$\sum_{k=0}^{\infty} \frac{u^{2k}}{(2k)!}$$

This is $\cosh u = \cosh \sqrt{x}$.

7. Approximations all in one place

To make life easier for you, we list here the various approximation techniques used.

(1) *Approximating a sum by an integral*: This is done using a numerical version of the integral test. This allows us to approximate the values of the zeta function. Please see the notes for how this is done. Note that this gives both an upper bound and a lower bound.

(2) *Approximating a function by Taylor polynomials*: For a function that is globally analytic or analytic about a point, we can approximate its value by Taylor polynomials. The higher the degree we allow for the Taylor polynomial, the better the approximation in general. The magnitude of possible error is determined using the max-estimate version of the Lagrange formula. *It is important to note that the error estimate could vary quite a bit from function to function. Some power series converge more slowly than others.* A good rule of thumb is that the more quickly the terms go to zero, the fewer the number of terms we need to take to get a good approximation.

In some cases, the Taylor approximation applies nicely only in a small range. For instance, the Taylor series for sin converges to sin globally, but the convergence is quick only for small values of x. The Taylor series for arctan converges to it only on $[-1, 1]$. However, we can use various identities such as those relating $\sin x$ and $\cos(\pi/2 - x)$ and those relating $\arctan x$ and $\arctan(1/x)$ to reduce to the case of a rapidly converging Taylor series.

(3) *Approximating an integral computation using Taylor polynomials*: Even computing the value of plain vanilla functions like sin at specific points requires the use of Taylor series. Miraculously, we can do with Taylor series what we cannot always do with functions – integrate term wise. This allows us to calculate definite integrals of globally and locally analytic functions by first integrating the Taylor series term wise and then using a Taylor polynomial approximation. Examples of functions that cannot be integrated in the language of elementary functions, but whose integrals can be computed to a fair degree of accuracy using this approach, are $(\sin x)/x$, $(e^x - 1)/x$, e^{-x^2}, $\sin(x^2)$, and similar functions.

Note that, at least for cases where the power series is easy to write down, this approach is a lot less cumbersome than the approach of using upper and lower sums.

(4) *Trying to find where a function is zero*, particularly when there is no algebraic method to solve this: We use techniques like the intermediate value theorem and the mean value theorem to show the existence of zeros in certain intervals. We also use derivative behavior and other techniques to further narrow down the intervals under consideration.

8. Limit computations and order of zero

(1) Suppose f is a function that has a zero at a. The order r of the zero at a is the least upper bound of the set of values β such that $\lim_{x \to a} |f(x)|/|x - a|^\beta = 0$. For $\beta < r$, the limit is 0. For $\beta > r$, the limit is undefined (∞-types).

(2) To simplify notation, we concentrate on the case where $a = 0$ (the location of a does not matter, because we can always translate).

(3) If f is infinitely differentiable at 0 and takes the value 0 at 0, the order of f at 0 is the smallest n such that $f^{(n)}(0) \neq 0$. Note that it is possible, but rare, for a function to have a zero of order ∞ at 0 (an example is the e^{-1/x^2} function). We ignore such examples.

(4) In particular, for an infinitely differentiable function that is 0 at 0, the order of the zero (if finite) if always a positive integer. Moreover, if the order is r, them $\lim_{x \to 0} f(x)/x^r = f^{(r)}(0)/r!$, which is the corresponding Taylor coefficient.

(5) This is particularly intuitive for analytic functions, because if we replace a function by its Taylor series we readily see that the order of its zero is the lowest order with nonzero coefficient. We also see that the limit of the quotient by x^r is that coefficient.

(6) We can have a function with a zero at a point that has fractional order, but the function cannot be infinitely differentiable. For instance, consider $x^{p/q}$, where p and q are positive and q is odd, with q not dividing p. This is differentiable k times where k is the greatest integer less than p/q. On the other hand, the order of the zero is p/q, which is a fraction. What happens is that we cannot differentiate the $(k+1)^{th}$ time at 0, since that exceeds the order.

(7) We can also have a situation of a function f with zero of order r such that $\lim_{x \to \infty} f(x)/x^r$ is zero. Again, however, f cannot be infinitely differentiable. Examples include $x/(\ln x)$, which has a zero order order 1. Intuitively, the order of the zero here is 1^+. (Note that ln in the denominator has a positive effect near zero though it has a negative effect near ∞. Ponder why).

(8) *Important note on order of zero of log*: Note that $\ln x \to -\infty$ as $x \to 0^+$, so log has an *anti-zero* at zero of order infinitesimally less than 0, i.e., 0^-. Thus, for $r > 0$, $x^r \ln x$ has a zero at 0 of degree r^-.

Thus, the rule for logarithm near 0 *(where the zero has degree* 0^-*) is somewhat opposite to the rule for logarithm out to* ∞ *(where the growth is* 0^+*).*

(9) We can also have a situation of a function f with zero of order r such that $\lim_{x \to \infty} f(x)/x^r$ is undefined or infinite. Again, however, f cannot be infinitely differentiable. Examples include $x(\ln x)$, which has a zero order order 1 at 0. Intuitively, the order of the zero here is 1^-.