PARTIAL FRACTIONS: AN INTEGRATIONIST PERSPECTIVE

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 8.5.

What students should already know: The integrals for $1/x$, $1/(x^2+1)$, and $f'(x)/f(x)$, and the key trigonometric substitution of arctan.

What students should definitely get: The statement about the existence of a partial fraction decomposition. How such a decomposition can be obtained, and how each of the pieces can be integrated.

Executive summary

Words ...

(1) For most practical purposes, we can study *monic* polynomials instead of arbitrary polynomials. A monic polynomial is a polynomial whose leading coefficient is 1. The reason we can restrict attention to monic polynomials is that any nonzero polynomial can be expressed as a nonzero constant times a monic polynomial.
(2) A nonconstant monic quadratic polynomial is irreducible (i.e., cannot be expressed as a product of polynomials of smaller degree) if and only if it has negative discriminant.
(3) Every nonconstant monic polynomial with real coefficients is a product of monic linear polynomials and irreducible monic quadratics, and this factorization is unique. Thus, all irreducible monic polynomials are either linear or quadratics with negative discriminant.
(4) The partial fractions approach breaks up any rational function as the sum of a polynomial and rational functions of the form R/Q^k where Q is a monic irreducible factor of the original denominator and R is a polynomial of degree strictly less than the degree of Q.
(5) Each of these partial fraction pieces is easy to integrate. The case where Q is linear, it is of the form $x - \alpha$, and the numerator is a constant, so this is a straightforward power integration. In the case where Q is quadratic, we break R as the sum of a constant and the derivative of Q. The constant part is handled by a trigonometric substitution, and the derivative of Q part is handled by the u-substitution $u = Q$.
(6) The partial fractions approach shows that every rational function can be integrated, and we obtain an antiderivative that involves ln (evaluated at some linear function of x), arctan (again, evaluated at some linear function of x), and other rational functions.
(7) Using the partial fractions approach and the equivalence of repeated integrability with the integrability of x times a function, we can show that any rational function can be *repeatedly* integrated, with the final answer in terms of arctan, ln, and rational functions.

Actions ... Please go through the notes on partial fractions as well as the discussion of these in the book. We here list only some salient points:

(1) Before beginning, make the denominator monic, and use the Euclidean algorithm to reduce to a problem where the degree of the numerator is less than the degree of the denominator.
(2) The general approach is to first factorize the denominator and then break it up into partial fractions with unknown numerators. The coefficients of the numerator need to be determined. One way of doing this is to take a common denominator, multiply out, compare coefficients, and solve the resultant system of linear equations.
(3) Instead of equating coefficients, we can also use a strategy of plugging in values. We plug in values so that a large number of the expressions that we are evaluating become zero.
(4) In particular, if we want to write:

$$\frac{r(x)}{(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)} = \frac{c_1}{x-\alpha_1} + \dots + \frac{c_n}{x-\alpha_n}$$

where all the α_is are distinct and the degree of r is less than n, then we get:

$$c_i = \frac{r(\alpha_i)}{\text{product of } \alpha_i - \alpha_j, \text{ all } j \neq i}$$

We can use this to very rapidly write any fraction with denominator a product of distinct linear factors in terms of partial fractions, and then integrate it.

(5) To handle:

$$\frac{r(x)\,dx}{(q(x))^k}$$

where q is an irreducible quadratic, we do repeated division, taking quotients and remainders, and obtain the result in terms of partial fractions.

(6) A thorough understanding of the partial fractions approach should allow you to predict, simply by looking at a rational function, whether the antiderivative expression for it will be (i) a rational function, (ii) something involving rational functions and arctan, (iii) something involving rational functions and ln, or (iv) something involving rational functions, arctan, and ln. For some practice of these, refer to the integration quiz.

1. The goal: integrate any rational function

Today's goal is to obtain a general strategy that will allow us to integrate any rational function. If we are able to accomplish this, then by some remarks made earlier, it should also be possible to *repeatedly* integrate any rational function.

What we are doing is based on many deep results in algebra about polynomials, most of which we cannot explore in full detail because you don't have the requisite background.

1.1. **A theorem about factorization of polynomials.**

For convenience, we will deal here only with *monic* polynomials. A monic polynomial is a polynomial whose leading coefficient is 1, i.e., the coefficient of the highest degree term is 1. Every nonzero polynomial can be expressed as a constant multiple of a monic polynomial. Since constants can be pulled out of integration and differentiation problems, do not affect the nature of the zeros, and have a clear and easy-to-work-out effect on the graph and behavior of a function, we do not really lose out on much by restricting our analysis to monic polynomials.

To further simplify matters, we will deal with polynomials of degree 1 or more, i.e., with nonconstant polynomials. Thus, our focus is on *nonconstant monic polynomials*.

We are now in a position to state the grand theorem valid *over the real numbers*:

> Every nonconstant monic polynomial p can be expressed as a product of nonconstant monic polynomials $p_1, p_2, \dots, p_n$, where each p_i is either linear (i.e., degree 1) or quadratic with negative discriminant. In other words, every nonconstant monic polynomial can be expressed as a product of nonconstant monic irreducible polynomials of degree 1 or 2. Moreover, the decomposition is unique up to ordering of the p_is. The p_is need not be distinct.

This result essentially states that there is a uniqueness of factorization for polynomials and all the irreducible polynomials are of degree either 1 or 2.

1.2. **The two kinds of irreducible pieces.**

It is worth examining a little more closely the nonconstant monic polynomials that are irreducible, i.e., cannot be factorized further. From the above theorem, the only such types are the linear polynomials (polynomials of the form $x - a$) and the quadratic polynomials with negative discriminant (polynomials of the form $x^2 + Bx + C$ where $B^2 - 4C < 0$).

By the square completion technique, we can write:

$$x^2 + Bx + C = (x + (B/2))^2 - D/4$$

where $D = B^2 - 4C$ is the discriminant.

Since D is negative, $-D/4$ is positive, and is the square of something. Setting $\beta = -B/2$ and $\gamma = \sqrt{-D/4}$, we obtain:

$$x^2 + Bx + C = (x - \beta)^2 + \gamma^2$$

where $\gamma > 0$.

1.3. **Coming home.** We next look at the question: how do we integrate:

$$\int \frac{dx}{q(x)}$$

where q is an *irreducible* nonconstant monic polynomial? We already have the two cases above, so we consider both of them:

(1) If $q(x) = x - a$, i.e., q is linear, then the integral is $\ln|x - a| + C$. It's a nice coincidence that we have already tackled this integral.
(2) If q is quadratic with negative discriminant, we first write $q(x) = (x - \beta)^2 + \gamma^2$, with $\gamma > 0$. Then we use the integration formula to obtain that the integral is $(1/\gamma)\arctan((x - \beta)/\gamma) + C$. It's again a nice coincidence that we were talking about these integrals recently.

Now, what about integrating something like:

$$\int \frac{a(x)\,dx}{q(x)}$$

where q is an irreducible nonconstant monic polynomial, and a is any polynomial? We first use polynomial long division (yes!) to write:

$$a(x) = b(x)q(x) + r(x)$$

where b is the quotient and r is the remainder, with either $r = 0$ or the degree of r less than the degree of q (note that r need not be monic). We thus obtain:

$$\frac{a(x)}{q(x)} = b(x) + \frac{r(x)}{q(x)}$$

To integrate the left side, we can integrate the right side. Integrating the polynomial b is straightforward, so what we are left with is an integration of the form:

$$\int \frac{r(x)\,dx}{q(x)}$$

where the degree of r is strictly smaller than the degree of q. We make three cases:

(1) q has degree one: In this case, r is a constant, and can be pulled out of the integration, and we are reduced to integrating $dx/q(x)$, which we discussed above.
(2) q has degree two but r is still a constant: In this case, we pull r out and are again reduced to integrating $dx/(q(x))$, which we discussed above.
(3) q has degree two and r has degree one: This needs more thought, and we will now turn our attention to it.

1.4. **Integating a linear over an irreducible quadratic.** Consider the integration problem, with $\gamma > 0$:

$$\int \frac{(Ax + B)\,dx}{(x - \beta)^2 + \gamma^2}$$

We use the following two known integrals:

$$\int \frac{1\,dx}{(x - \beta)^2 + \gamma^2} = \frac{1}{\gamma}\arctan\left(\frac{x - \beta}{\gamma}\right)$$

and:

$$\int \frac{2(x-\beta)\,dx}{(x-\beta)^2+\gamma^2} = \ln((x-\beta)^2+\gamma^2)$$

The latter integral is based on the f'/f formulation. Note that we do not need an absolute value sign on the right because the expression in parentheses is always positive.

We now try to write the integrand that we need to integrate in terms of the two integrands that we know how to integrate. In other words, we try to write:

$$Ax + B = c_1(1) + c_2(2(x-\beta))$$

where c_1 and c_2 are constants. We then try to solve for c_1 and c_2, and then split the integrand as a linear combination of the known integrands, that we then integrate.

For instance:

$$\int \frac{x\,dx}{x^2+x+1}$$

Note that the denominator has negative discriminant, and we can use square completion to write it as $(x+(1/2))^2 + (\sqrt{3}/2)^2$. We obtain:

$$\int \frac{x\,dx}{(x+(1/2))^2+(\sqrt{3}/2)^2}$$

The derivative of the denominator is $2x+1$. We now want to write:

$$x = c_1(1) + c_2(2x+1)$$

Expanding and comparing coefficients, we obtain that $2c_2 = 1$ and $c_1 + c_2 = 0$, which yields that $c_1 = -1/2$, $c_2 = 1/2$. We thus get:

$$\frac{-1}{2}\int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2} + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$$

This simplifies to:

$$\frac{-1}{\sqrt{3}}\arctan\left(\frac{x+(1/2)}{\sqrt{3}/2}\right) + \frac{1}{2}\ln(x^2+x+1)$$

With some practice, you should be able to see the splitting by inspection, and hence will not need to set up a system of simultaneous linear equations to determine the constants c_1 and c_2.

1.5. **Partial fractions.** Suppose we want to integrate a rational function of the form:

$$\frac{a(x)}{q(x)}$$

where q is nonconstant and monic. Note that if q is constant, this is just polynomial integration. If q is not monic, we can pull out a constant to make it monic. Thus, we can, *without loss of generality*, restrict to the case where q is a nonconstant monic polynomial.

We can first use long division to reduce this problem to integrating:

$$\frac{r(x)}{q(x)}$$

where the degree of r is strictly less than the degree of q. The claim is that the above can be written as a sum of expressions of the form:

$$\frac{R(x)}{Q(x)^k}$$

where Q is nonconstant monic irreducible (hence, degree either 1 or 2), the degree of R is less than the degree of Q, k is a positive integer, and Q^k divides q. We then devise strategies to:

(1) find a way of taking a rational function and writing it as a sum of terms of the above form. This process is called *partial fraction decomposition*.
(2) integrate each of the terms we obtain as a result of the decomposition.

The unappealing idea behind (1) is setting up and solving a monstrous system of linear equations (although there are shortcuts in some cases). The unappealing idea behind (2) is to use trigonometric substitutions in the rare cases where we cannot directly obtain a result from one of the canned formulas.

1.6. **Denominator a product of two distinct linear terms.** Consider:

$$\int \frac{dx}{x^2 - 5x + 6}$$

The denominator can be factorized as $(x-2)(x-3)$, and we obtain:

$$\int \frac{dx}{(x-2)(x-3)}$$

An astute observation at this stage yields that:

$$\frac{1}{(x-2)(x-3)} = \frac{(x-2)-(x-3)}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$$

Thus, the original integral expression becomes:

$$\int \frac{dx}{x-3} - \int \frac{dx}{x-2}$$

which simplifies to:

$$\ln|x-3| - \ln|x-2| + C$$

which can also be written as:

$$\ln\left|\frac{x-3}{x-2}\right| + C$$

In this case, the partial fraction decomposition that helped us was:

$$\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$$

In general, if we are given:

$$\frac{Ax+B}{(x-\alpha_1)(x-\alpha_2)}$$

We want to write it as:

$$\frac{Ax+B}{(x-\alpha_1)(x-\alpha_2)} = \frac{c_1}{x-\alpha_1} + \frac{c_2}{x-\alpha_2}$$

We now want to determine c_1 and c_2 in terms of A and B. We do so by cross-multiplying and comparing coefficients. Specifically, we get:

$$Ax + B = c_1(x-\alpha_2) + c_2(x-\alpha_1)$$

This yields $A = c_1 + c_2$ and $B = -(c_1\alpha_2 + c_2\alpha_1)$. We then proceed to solve for c_1 and c_2 in terms of A and B.

In our case, $Ax + B$ was 1, so $A = 0$ and $B = 1$, while $\alpha_1 = 2$, $\alpha_2 = 3$. Solving this system yielded $c_1 = -1$, $c_2 = 1$, which is what we used.

Aside: Speed to boast about. If you want to get really quick at integrating such expressions, you should solve these above systems in general (i.e., find expressions for c_1 and c_2 in terms of A, B, α_1, and α_2, and solve through to the end in the *general case*), and then memorize the final expressions you get. You can then jump straight from the statement of the question to the final answer. However, it is recommended that before mastering such shortcuts, you should solve out a few problems in full detail.

Of particular interest is the case where the numerator is 1. In this case, the formula is easy to master and fairly intuitive.

1.7. **Denominator a product of more than two distinct linear terms.** In this subsection, we explore the linear equations approach. In the next subsection, we consider a slight variant of the approach that works well for products of distinct linear terms, and is considerably faster in many such cases.

Suppose we want to integrate:

$$\int \frac{dx}{x^3 - 2x}$$

We first note that the denominator is expressible as a product of three linear terms, so we rewrite as:

$$\int \frac{dx}{x(x - \sqrt{2})(x + \sqrt{2})}$$

Our goal is to now find constants c_1, c_2, and c_3 such that:

$$\frac{1}{x^3 - 2x} = \frac{c_1}{x} + \frac{c_2}{x - \sqrt{2}} + \frac{c_3}{x + \sqrt{2}}$$

One way to do this is to take a common denominator on the right side and simplify it. We get:

$$\frac{1}{x^3 - 2x} = \frac{c_1(x^2 - 2) + c_2 x(x + \sqrt{2}) + c_3 x(x - \sqrt{2})}{x^3 - 2x}$$

We now equate the numerators to get:

$$1 = c_1(x^2 - 2) + c_2 x(x + \sqrt{2}) + c_3 x(x - \sqrt{2})$$

Since these are equal as polynomials in x, we equate them coefficient-wise, obtaining a system of simultaneous linear equations in the variables c_1, c_2, and c_3, that we then dutifully solve. Once we determine all the values, we write the answer, which is:

$$c_1 \ln|x| + c_2 \ln|x - \sqrt{2}| + c_3 \ln|x + \sqrt{2}| + C$$

Let's do the actual finding of c_1, c_2, and c_3. We get the following system of simultaneous linear equations:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ \sqrt{2}c_2 - \sqrt{2}c_3 &= 0 \\ -2c_1 &= 1 \end{aligned}$$

Solving, we obtain that $c_1 = -1/2$, $c_2 = 1/4$, and $c_3 = 1/4$. The answer is thus:

$$\frac{-1}{2} \ln|x| + \frac{1}{4} \ln|x - \sqrt{2}| + \frac{1}{4} \ln|x + \sqrt{2}| + C$$

We can combine the second and third terms to get:

$$\frac{-1}{2} \ln|x| + \frac{1}{4} \ln|x^2 - 2| + C$$

We see how this can be generalized. If the denominator is a product of n distinct linear terms, say $(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$, we try to write the function as:

$$\frac{c_1}{x - \alpha_1} + \frac{c_2}{x - \alpha_2} + \dots + \frac{c_n}{x - \alpha_n}$$

We then take a common denominator, simplify the numerator, and equate it as a polynomial to the numerator of the original integrand. We equate coeffiicent-wise as polynomials in x, creating a system of simultaneous linear equations with variables c_i, that we then solve. The final answer is:

$$c_1 \ln|x - \alpha_1| + c_2 \ln|x - \alpha_2| + \cdots + c_n \ln|x - \alpha_n|$$

1.8. **Another way of finding the coefficients for partial fractions.** Let us reconsider the problem of splitting into partial fractions:

$$\int \frac{r(x)\,dx}{(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)}$$

where all that α_is are distinct.

We want to write this as:

$$\frac{r(x)}{(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)} = \frac{c_1}{x - \alpha_1} + \frac{c_2}{x - \alpha_2} + \cdots + \frac{c_n}{x - \alpha_n}$$

To do this, we take a common denominator and equate the numerators, getting:

$$r(x) = c_1(x - \alpha_2) \ldots (x - \alpha_n) + c_2(x - \alpha_1)(x - \alpha_3) \ldots (x - \alpha_n) + \cdots + c_n(x - \alpha_1) \ldots (x - \alpha_{n-1})$$

Earlier, we had proceeded from this point onward by expanding out, comparing coefficients of each power of x, and equating these coefficients. This gave us a system of linear equations in the variables $c_1, c_2, \ldots, c_n$. We then proceeded to solve this system to find the c_is.

For $n \leq 2$, this procedure is feasible and quick. However, for $n \geq 3$, the procedure becomes messy because we first need to do a lot of tedious term multiplications to find coefficients, and then we need to solve a tedious system of linear equations.

The approach we have used above is based on the key idea that *if two polynomials are equal, they are equal coefficient-wise.*

The approach we will now use is based on the key idea that *if two polynomials are equal, their values at every number are equal.*

Consider:

$$r(x) = c_1(x - \alpha_2) \ldots (x - \alpha_n) + c_2(x - \alpha_1)(x - \alpha_3) \ldots (x - \alpha_n) + \cdots + c_n(x - \alpha_1) \ldots (x - \alpha_{n-1})$$

This equality is an *identity* in x, in the sense that it is true for all x. In particular, it is true if we set $x = \alpha_1$. Doing this, we note that the only term on the right side that survives is the first product, and we get:

$$r(\alpha_1) = c_1(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n)$$

We thus get:

$$c_1 = \frac{r(\alpha_1)}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n)}$$

We can obtain analogous expressions for the other c_is.

With specific values, we can compute this easily. This method is preferable (from a speed perspective) when $n \geq 3$. Please see numerical examples in the book.

The idea is also important from a conceptual perspective. The key idea is that when a given equation is an *identity* in one variable (i.e., it is true for all possible values that that one variable can take) we can plug in specific values of that variable to get specific equations in the other variables.

1.9. **Denominator a higher power of a linear expression.** Consider the integration:

$$\int \frac{x^2 + 5x + 6}{(x-1)^3}\, dx$$

Here, we can do a u-substitution with $u = x - 1$, so $x = u + 1$, and the expression now becomes a sum of powers of u. Let's do this. Note that $dx = du$, so we get:

$$\int \frac{(u+1)^2 + 5(u+1) + 6}{u^3}\, du$$

This simplifies to:

$$\int \frac{u^2 + 7u + 12}{u^3}\, du$$

Dividing each term by u^3 and splitting up, we get:

$$\int \frac{du}{u} + 7\int \frac{du}{u^2} + 12\int \frac{du}{u^3}$$

which becomes:

$$\ln|u| - \frac{7}{u} - \frac{6}{u^2}$$

Substituting back $u = x - 1$, we obtain:

$$\ln|x-1| - \frac{7}{x-1} - \frac{6}{(x-1)^2} + C$$

The above approach is equivalent to the following superficially different approach, where we try to rewrite the original function as:

$$\frac{\text{first constant}}{x-1} + \frac{\text{second constant}}{(x-1)^2} + \frac{\text{third constant}}{(x-1)^3}$$

where each part is easy to integrate using the integration of power functions.

1.10. **Denominator a product of linear and quadratic irreducibles.** Integration is a thankless task. Accomplishments in integration invite more integration problems rather than applause.

So far, we have tackled situations where the denominator is a single irreducible, where it is a product of distinct linear terms, and where it is a power of a linear term. We now deal with the case where the denominator is a product involving distinct linear and quadratic irreducibles. This is *almost* the end. The final straw will be the case where the denominator is a product of possibly repeated linear and quadratic terms.

We consider the integral:

$$\int \frac{x^3 + x + 1}{x^3 + x}\, dx$$

First, note that here, the degree of the numerator is *not* strictly less than the degree of the denominator, which means that the first step is long division. We do this, and obtain:

$$\int \left[1 + \frac{1}{x^3 + x}\right] dx$$

The 1 integrates to x, so we need to integrate:

$$\int \frac{dx}{x^3 + x}$$

The denominator can be factorized as $x(x^2 + 1)$. It cannot be factorized further because $x^2 + 1$ is an irreducible quadratic.

We try now to write it as:

$$\frac{1}{x^3+x} = \frac{Ax+B}{x^2+1} + \frac{c}{x}$$

Again, we cross multiply and simplify the right side and equate numerators, getting:

$$1 = x(Ax+B) + c(x^2+1)$$

Equating coefficients, we obtain:

$$\begin{aligned} A + c &= 0 \\ B &= 0 \\ c &= 1 \end{aligned}$$

We thus get $A = -1$, $B = 0$, and $c = 1$. Plugging in, we obtain:

$$\frac{1}{x^3+x} = \frac{-x}{x^2+1} + \frac{1}{x}$$

We now try integrating both pieces. Note that we can do each piece separately, because it is of the form r/q where q is irreducible and r has smaller degree. In our case, the answer is:

$$\frac{-1}{2}\ln(x^2+1) + \ln|x|$$

Finally, we should remember that the original problem was different, and we need to add a x that comes from integrating the 1. So the answer to the original problem is:

$$x - \frac{1}{2}\ln(x^2+1) + \ln|x| + C$$

1.11. **Situation where the denominator is a power of an irreducible quadratic.** Suppose we want to carry out the integration:

$$\int \frac{Ax^3+Bx^2+Cx+D}{[(x-\beta)^2+\gamma^2]^2}\,dx$$

The first step is to break up the integrand as:

$$\frac{\text{constant or linear}}{(x-\beta)^2+\gamma^2} + \frac{\text{constant or linear}}{[(x-\beta)^2+\gamma^2]^2}$$ Note: Correction made

This can be done by long division. For instance, consider the problem:

$$\int \frac{x^3+2x^2+3x+4}{(x^2+1)^2}\,dx$$

We use polynomial division to divide the numerator by x^2+1. We obtain that:

$$x^3+2x^2+3x+4 = (x^2+1)(x+2) + (2x+2)$$

Dividing both sides by $(x^2+1)^2$, we obtain:

$$\frac{x^3+2x^2+3x+4}{(x^2+1)^2} = \frac{x+2}{x^2+1} + \frac{2x+2}{(x^2+1)^2}$$

Now, for each piece, we use the integration strategy we learned earlier: write the numerator as a linear combination of 1 and $2(x-\beta)$. In this case, for instance, $2(x-\beta) = 2x$, so the first fraction becomes:

$$\frac{1}{2}\frac{2x}{x^2+1} + 2\cdot\frac{1}{x^2+1}$$

The second integrand becomes:

$$1 \cdot \frac{2x}{(x^2+1)^2} + 2 \cdot \frac{1}{(x^2+1)^2}$$

We have thus broken up the original integrand as a combination of four terms, each of which has numerator either equal to 1 or equal to the derivative of the quadratic in the denominator. For the cases where the numerator is 1, we use the trigonometric substitution idea. For the cases where the numerator is the derivative of the quadratic in the denominator, we use a u-substitution where u is the expression in the denominator. In this case, our answer is:

$$\frac{1}{2}\ln(x^2+1) + 2\arctan(x) - \frac{1}{x^2+1} + 2\int \frac{dx}{(x^2+1)^2}$$

The fourth integral requires the trigonometric substitution $\theta = \arctan x$, which reduces to integrating $\cos^2\theta$, which we in turn simplify and substitute back in terms of x.

1.12. **The overall strategy.** Here is a summary of the overall strategy (as before, we assume that the denominator is nonconstant and monic):

(1) First, factorize the denominator.
(2) Collect all repeated factors together.
(3) If the numerator and denominator have common factors, cancel.
(4) If the degree of the numerator is greater than that of the denominator, use polynomial division to reduce to a problem where the degree of the numerator is less than that of the denominator.
(5) Write the new rational function r/q as a sum of rational functions of the form R/Q^k where Q is one of the irreducible factors, k is at most equal to the multiplicity of Q in q, and the degree of R is less than the degree of Q. This may be accomplished by polynomial division along with some setting up of simultaneous linear equations.
(6) For those cases where Q is linear, integration is straightforward and gives an answer that is a rational function if $k > 1$ and a logarithmic function if $k = 1$.
(7) For those cases where Q is quadratic, write R as a linear combination of 1 and the derivative of Q. The 1 part integrates using an arc tangent substitution, while the derivative of Q part integrates by setting $u = Q$. For the derivative of Q part, the answer is a rational function if $k > 1$ and $\ln(Q)$ if $k = 1$. Note that we do not need the absolute value here because the quadratic is always positive.

2. Integrating between limits

2.1. **Interval red alert.** One important thing to remember about rational functions is that they need not be defined everywhere. Specifically, a rational function *blows up* at the zeros of the denominator, i.e., the points where the denominator becomes zero. Thus, it is meaningless to integrate a rational function over an interval if the interval contains any of the zeros of the denominator.

2.2. **Sporadic manna from heaven: symmetry.** When integrating rational functions, it is useful to be on the lookout for symmetries (even/odd/mirror symmetry/half-turn symmetry) that may result in a particular integral being zero without our having to compute it as an indefinite integral. Here are some quick reminders:

(1) A polynomial is an even function if and only if it is an *even polynomial*: all its terms with nonzero coefficients have even degree.
(2) A polynomial is an odd function if and only if it is an *odd polynomial*: all its terms with nonzero coefficients have odd degree. Thus, x^3 is an odd polynomial, and so is $x^3 - 46x$, but $x^3 + 1$ is not odd.
(3) The quotient of two even polynomials is even, and the quotient of two odd polynomials is even. The quotient of an even polynomial by an odd polynomial, or the quotient of an odd polynomial by an even polynomial, is odd.

In particular, suppose we are trying to compute:

$$\int_{-1}^{1} \frac{x^5 - x}{(x^2+2)^3}\,dx$$

the answer is 0, because the function is odd and the interval of integration is symmetric about 0.

This method may sometimes allow us to simplify one or more of the many partial fractions that we obtain using the partial fraction decomposition, allowing us to concentrate on the others.

2.3. **Computation as the last resort: boring but dependable.** When shortcuts fail, we simply compute the indefinite integral and evaluate it between limits. At the end of this, we get an answer that possibly involves lns and arctans. Often, these cannot be further simplified for the particular limits of integration. Sometimes, they can, but the answers are still messy. In any case, we *have* reduced an integration problem to a problem that involves looking up natural logarithm and arc tangent tables. This is no small feat.

For instance, consider the problem:

$$\int_1^3 \frac{dx}{x^2 + x + 1}$$

An antiderivative that we obtain using our method is:

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{x + (1/2)}{\sqrt{3}/2}\right)$$

Evaluating between limits, we obtain:

$$\frac{2}{\sqrt{3}}\left[\arctan(7/\sqrt{3}) - \arctan(\sqrt{3})\right]$$

The second arctan is precisely $\pi/3$, but the first one does not correspond to any angle that we already know. Nonetheless, we *do* know that it is between $\pi/3$ and $\pi/2$. The difference is thus between 0 and $\pi/6$, and the final answer is thus between 0 and $\pi/3\sqrt{3}$, which is less than 0.62. If we had a better understanding of arctan (something we will achieve by the end of this course), we could probably estimate the integral even better.

3. Summaries and miscellanea

3.1. **Transcendental mess and repeated integration.** The term *transcendental* is used for functions that don't arise from algebraic functions through the usual processes of combination, composition, and takin inverses. It includes the exponential and logarithmic functions, as well as the trigonometric and inverse trigonometric functions.

As we have seen, integrating a rational function can give rise to a transcendental function. However, only a very limited collection of transcendental functions arise this way. In particular, only a small number of ways of combining ln and arctan can arise in this way.

Further, recall that to *repeatedly integrate* a rational function f, we use the fact that $f(x)$, $xf(x)$, $x^2 f(x)$ and so on are all rational functions and hence can all be integrated. All of these rational functions have the same denominator (possibly a smaller one due to cancellations).

Here are some observations:

- If the denominator can be completely factorized into *distinct* monic linear factors, then the antiderivative we get is a linear combination of lns (with constant coefficients) at these linear factors (absolute values), plus a polynomial. There is no appearance of arctan.

  Repeated integration of such a rational function yields a combination of lns with polynomial coefficients, plus a polynomial.
- If the denominator can be completely factorized into linear factors, possibly with repetition, then the antiderivative is a linear combination of lns, reciprocals of powers of these linear factors, plus a polynomial.

  Repeated integration of such a rational function yields a combination of lns with polynomial coefficients, plus a polynomial.

In particular, arctan starts appearing only once we introduce irreducible *quadratic* factors. Some of you may be wondering why arctan should appear at all, and why everything could not be done with ln. In fact, *if we were allowed to use the complex numbers instead of the real numbers*, then we could break everything down into linear factors, and everything could be effectively accomplished using ln. However, the complex numbers pose their own problems. Back in the real world, we need both ln and arctan, which is a kind of imaginary-twisted version of ln. This relationship is based on the relationship between circular trigonometric functions and hyperbolic trigonometric functions alluded to in the past.

3.2. **A summary of techniques for various degrees of numerators and denominators.** Here is a summary for most basic cases – we assume that the denominator is monic:

(1) The degree of the numerator is greater than the degree of the denominator: Use polynomial long division to convert to a problem where the degree of the numerator is less than the degree of the denominator.

(2) The denominator is linear, and the numerator is constant: In this case, we use the formula:

$$\int \frac{c}{x-\alpha}\,dx = c\ln|x-\alpha| + C$$

(3) The denominator is quadratic *irreducible* and the numerator is constant or linear. We break the numerator into two pieces, one of which integrates to a logarithmic expression and the other one integrates to an arc tangent expression:

$$\int \frac{Ax+B}{(x-\beta)^2+\gamma^2}\,dx = \frac{A}{2}\ln\left[(x-\beta)^2+\gamma^2\right] + \frac{B+A\beta}{\gamma}\arctan\left(\frac{x-\beta}{\gamma}\right) + C$$

(4) The denominator is quadratic with two distinct factors α and β:

$$\int \frac{Ax+B}{(x-\alpha)(x-\beta)}\,dx = \frac{A\alpha+B}{\alpha-\beta}\ln|x-\alpha| + \frac{A\beta+B}{\beta-\alpha}\ln|x-\beta|$$

(5) The denominator is a quadratic with a repeated linear factor: Here, we simply put u as that linear factor and do a u-substitution. We can also work out an explicit formula:

$$\int \frac{Ax+B}{(x-\alpha)^2}\,dx = A\ln|x-\alpha| - \frac{A\alpha+B}{x-\alpha} + C$$

(6) The denominator is cubic with three distinct linear factors α_1, α_2, α_3: We follow the partial fractions method and get a final answer as $c_1\ln|x-\alpha_1| + c_2\ln|x-\alpha_2| + c_3\ln|x-\alpha_3|$. As discussed in an earlier section, if we numerator is the polynomial $r(x)$, then $c_1 = r(\alpha_1)/(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)$, $c_2 = r(\alpha_2)/(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)$, and $c_3 = r(\alpha_3)/(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)$.

(7) The denominator is cubic and splits as a product of a linear factor and an irreducible quadratic: We use partial fractions to try to write the fraction as:

$$\frac{\text{constant}}{\text{the linear factor}} + \frac{\text{constant or linear}}{\text{the irreducible quadratic factor}}$$

We know how to integrate each part. (One could write out a general formula along the lines above, but it is probably unilluminating).

(8) The denominator is cubic and splits as a product of a linear factor repeated twice ($(x-\alpha)^2$) and an isolated linear factor $x-\beta$: In this case, we use partial fractions to write the fraction as:

$$\frac{\text{constant or linear}}{(x-\alpha)^2} + \frac{\text{constant}}{x-\beta}$$

We can then solve each part separately. Alternatively, we may choose to *directly* write the original as:

$$\frac{\text{constant}}{(x-\alpha)^2} + \frac{\text{constant}}{x-\alpha} + \frac{\text{constant}}{x-\beta}$$

(9) The denominator is the cube of $(x-\alpha)$: Then, set $u = x-\alpha$, substitute, and solve.

(10) The denominator is of degree 4 and is a product of two distinct irreducible quadratics: Obtain a partial fraction decomposition as:

$$\frac{\text{constant or linear}}{\text{First irreducible quadratic}} + \frac{\text{constant or linear}}{\text{Second irreducible quadratic}}$$

We know how to tackle each part.

(11) The denominator is of degree 4 and is the square of an irreducible quadratic: Obtain a partial fraction decomposition, using polynomial long division, as:

$$\frac{\text{constant or linear}}{\text{irreducible quadratic}} + \frac{\text{constant or linear}}{\text{square of irreducible quadratic}}$$

We know how to integrate the first fraction. For the second fraction, we break up the numerator as a linear combination of the derivative of the denominator and 1. The derivative part is solved by taking u equal to the denominator. The 1 part is handled by a trigonometric substitution.

(12) The denominator is of degree 4 and is the product of an irreducible quadratic and two distinct irreducible linear factors: Use partial fractions to get:

$$\frac{\text{constant or linear}}{\text{irreducible quadratic}} + \frac{\text{constant}}{\text{first linear factor}} + \frac{\text{constant}}{\text{second linear factor}}$$

Try figuring out the remaining cases for a degree 4 polynomial: (i) product of an irreducible quadratic and the square of a linear factor, (ii) product of four distinct linear factors , (iii) product of two linear factors, each squared, (iv) product of one linear factor cubed and another linear factor (v) product of a squared linear factor and two other linear factors (vi) a single linear factor to the fourth power.

If you're ambitious, you might want to work out all the cases for polynomials of degree five.