

INVERSE TRIGONOMETRIC FUNCTIONS

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 7.7.

What students should already know: The definitions of the trigonometric functions and the key identities relating them.

What students should definitely get: The definitions of the inverse trigonometric functions for sine and cosine. The definitions of the other inverse trigonometric functions, and the differentiation rules for these functions. The application of these to some specific types of indefinite integration.

What students should hopefully get: The domain/range subtleties for inverse trigonometric functions. The one-sided nature of inverses, and how to compute $\arcsin \circ \sin$ and similar things in general. The “magic” of finding inverse trigonometric functions as antiderivatives of algebraic functions (rational functions and radical functions).

Note: If you would like a more detailed review of trigonometry concepts and ideas, please look at the notes titled *Trigonometry review* back from 152. These notes were not covered in class, but were included for reference.

EXECUTIVE SUMMARY

Words ...

- (1) The functions \sin , \cos , \tan , and their reciprocals are all periodic functions. While \tan and \cot have a period of π , the other four have a period of 2π each. While \sin and \cos are continuous and defined for all real numbers, the other four functions have points of discontinuity where they approach infinities of different signs from both sides. Also, \tan and \cot are one-to-one on a single period domain, while the other functions are usually two-to-one.
- (2) To construct inverses to these functions, we take intervals small enough such that the function is one-to-one restricted to that interval, but the range of the function restricted to that interval is the whole range. We also try to make our choice in such a manner that the *other function in the square sum/difference relationship* is nonnegative on the domain.
- (3) The choices are: $[-\pi/2, \pi/2]$ for \sin (note that \cos is nonnegative on this), $[0, \pi]$ for \cos (note that \sin is nonnegative on this), $(-\pi/2, \pi/2)$ for \tan (note that \sec is nonnegative on this), and $(0, \pi)$ for \cot (note that \csc is nonnegative on this).
- (4) We define the corresponding inverse trigonometric functions \arcsin , \arccos , \arctan and arccot . The domains for both \arcsin and \arccos equal $[-1, 1]$ while the domains for both \arctan and arccot equal all of \mathbb{R} . The range of \arcsin is $[-\pi/2, \pi/2]$ and the range of \arccos is $[0, \pi]$. The range of \arctan is $(-\pi/2, \pi/2)$ and the range of arccot is $(0, \pi)$.
- (5) \arcsin is an increasing function with vertical tangents at the endpoints, and \arccos is a decreasing function with vertical tangents at the endpoints. For all x , we have $\arcsin x + \arccos x = \pi/2$. \arctan is an increasing function with horizontal asymptotes valued at $-\pi/2$ and $\pi/2$ and arccot is a decreasing function with horizontal asymptotes valued at π and 0 .
- (6) We define the arc secant function and the arc cosecant function as $\text{arcsec}(x) = \arccos(1/x)$ and $\text{arccsc}(x) = \arcsin(1/x)$. These are defined for all x outside $(-1, 1)$.
- (7) Using the formula for differentiating the inverse function, we obtain that $\arcsin'(x) = 1/\sqrt{1-x^2}$ and $\arccos'(x) = -1/\sqrt{1-x^2}$. Thus, $\int dx/\sqrt{1-x^2} = \arcsin x + C$. Also, we obtain $\int dx/\sqrt{a^2-x^2} = \arcsin(x/a) + C$.
- (8) Similarly, we obtain that $\arctan'(x) = 1/(1+x^2)$ and $\text{arcsec}'(x) = 1/(|x|\sqrt{x^2-1})$. Note that in the case of arc secant, we need the absolute value to account for the fact that tangent is not nonnegative on the range of arc secant.

Actions ...

- (1) $\sin(\arcsin x) = x$ if x lies in the domain of the arcsin function. Note that otherwise $\sin(\arcsin x)$ does not make sense. Similar observations hold for the other trigonometric functions.
- (2) *Solving equations:* The solutions to $\sin x = \alpha$, where $\alpha \in [-1, 1]$ come in two families: $\{2n\pi + \arcsin \alpha : n \in \mathbb{Z}\}$ and $\{2n\pi + (\pi - \arcsin \alpha) : n \in \mathbb{Z}\}$. Similarly, the solutions to $\cos x = \alpha$ where $\alpha \in [-1, 1]$ come in two families: $\{2n\pi + \arccos \alpha : n \in \mathbb{Z}\}$ and $\{2n\pi - \arccos \alpha : n \in \mathbb{Z}\}$. In the special case where $\alpha = 1$ (respectively, $\alpha = -1$), the two solution families for sin (respectively, cos) collapse into one solution family.
- (3) $\arcsin(\sin x)$ need not be equal to x – they are equal if and only if x is in the range of arcsin.
- (4) We often want to compute things like $\cos(2 \arctan x)$. To tackle these situations, we set $\theta = \arctan x$, so we obtain $\tan \theta = x$. The problem now reduces to determining $\cos(2\theta)$ in terms of $\tan \theta$, which is some elementary trigonometry. (It's elementary only if you know at least some of the double-angle formulas).
- (5) The integration formulas for $1/\sqrt{1-x^2}$ etc. give rise to many slightly more general integration formulas. A short list is given in the book; a longer list is given on the last page of the the lecture notes. Rather than *just* memorize the long list, try to make sure you can *see* how the more complicated formulas arise from the basic ones, and also how you would apply them to actual integration problems. Internalize it to the level that you *just know it*.

1. A BRIEF REVIEW OF TRIGONOMETRY

In this lecture, we will discuss the inverse trigonometric functions. But before doing that, we will quickly review our knowledge of trigonometric functions.

1.1. A review of sine and cosine. So far, we have seen two broad definitions of sine and cosine. We review those two definitions:

- (1) The definition in terms of right triangles. Here, if θ is one of the acute angles of a right-angled triangle, $\sin \theta$ is defined as the quotient of the side opposite to θ to the hypotenuse of the right-angled triangle. $\cos \theta$ is defined as the quotient of the leg adjacent to θ to the hypotenuse. This definition works for $\theta \in (0, \pi/2)$. That the definition makes sense relies on the notion of similar triangles: any two triangles with a common value of θ are *similar*, hence the quotients of corresponding side lengths are equal.
- (2) The definition in terms of the unit circle in the coordinate plane: Here, we start at the point $(1, 0)$ in the coordinate plane, and move a length of θ along the circle $x^2 + y^2 = 1$ in the counter-clockwise direction. The x -coordinate of the point we land at is defined as $\cos \theta$ and the y -coordinate is defined as $\sin \theta$. This definition works for all $\theta \in \mathbb{R}$.

We have seen the following key identities relating sin and cos:

- (1) $\sin^2 \theta + \cos^2 \theta = 1$.
- (2) $\sin(\pi/2 - \theta) = \cos \theta$.
- (3) $\cos(\pi/2 - \theta) = \sin \theta$.
- (4) $\sin(-\theta) = -\sin \theta$, i.e., sin is an odd function.
- (5) $\cos(-\theta) = \cos \theta$, i.e., cos is an even function.
- (6) $\sin(\pi/2 + \theta) = \cos \theta$.
- (7) $\cos(\pi/2 + \theta) = -\sin \theta$.
- (8) $\sin(\pi - \theta) = \sin \theta$.
- (9) $\cos(\pi - \theta) = -\cos \theta$.
- (10) $\sin(\pi + \theta) = -\sin \theta$.
- (11) $\cos(\pi + \theta) = -\cos \theta$.
- (12) $\sin(2\pi + \theta) = \sin \theta$.

Most of these can be deduced from the following four angle sum formulas:

- (1) $\sin(x + y) = \sin x \cos y + \cos x \sin y$.
- (2) $\cos(x + y) = \cos x \cos y - \sin x \sin y$.
- (3) $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

$$(4) \cos(x - y) = \cos x \cos y + \sin x \sin y.$$

From these, we have the following double-angle formulas:

$$(1) \sin(2x) = 2 \sin x \cos x.$$

$$(2) \cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x. \text{ This also translates to } \cos^2 x = (1 + \cos(2x))/2 \text{ and } \sin^2 x = (1 - \cos(2x))/2.$$

The integration and differentiation relationships are:

(1) The derivative of \sin is \cos .

(2) The derivative of \cos is $-\sin$.

(3) An antiderivative of \sin is $-\cos$.

(4) An antiderivative of \cos is \sin .

Aside: The tyranny of great expectations. Unlike the earlier, carefree days, you are now expected to be able to do all the following:

(1) Remember or be able to quickly reconstruct all the above formulas.

(2) Relate the above formulas to facts about the graphs. For instance, the fact that $\sin(\pi - \theta) = \sin \theta$ indicates that the graph of \sin exhibits mirror symmetry about the line $x = \pi/2$. Similarly, the fact that $\sin(2\pi - \theta) = -\sin \theta$ indicates a half-turn symmetry about π . If your graphical intuition is strong, you may be able to reconstruct the algebraic identities from their self-evident graphical interpretations.

(3) Derive the results from each other, and reconcile them. For instance, what is the derivative of $\sin(x + \pi)$? There are two ways of computing this. You should be able to do things in both ways and quickly reconcile your answers.

1.2. Finding solutions to $\sin x = a$. What are the solutions to $\sin x = a$? Obviously, the answer depends on a . We first consider four special cases:

(1) If $|a| > 1$, there are *no* solutions.

(2) If $a = 0$, then the solutions are the integer multiples of π .

(3) If $a = 1$, the solutions are of the form $2n\pi + \pi/2$, $n \in \mathbb{Z}$.

(4) If $a = -1$, the solutions are of the form $2n\pi - \pi/2$, $n \in \mathbb{Z}$.

What about the other values of a ? In these cases, we first find some angle θ such that $\sin \theta = a$. There are then two solution families:

$$\{2n\pi + \theta : n \in \mathbb{Z}\}$$

and

$$\{2n\pi + (\pi - \theta) : n \in \mathbb{Z}\}$$

Basically, these are the 2π -translates of θ and of $\pi - \theta$. In particular, in any interval of length 2π , there are two solutions.

Note that this general solution applies to the cases (2), (3) and (4) listed above. However, something special happens in each of the cases. In case (2), the two families are the even and odd multiples of π : they can be readily collapsed into a single family, namely, all multiples of π . In case (3), the two families coincide, because $\pi - \pi/2 = \pi/2$. For similar reasons, the two families coincide in case (4).

1.3. Finding solutions to $\cos x = a$. Here, if $|a| > 1$, there is no solution. Otherwise, we first find an angle θ such that $\cos \theta = a$. The two families are then:

$$\{2n\pi + \theta : n \in \mathbb{Z}\}$$

and

$$\{2n\pi - \theta : n \in \mathbb{Z}\}$$

The two families can be combined using the \pm symbol:

$$\{2n\pi \pm \theta : n \in \mathbb{Z}\}$$

Note that in the special case where $a = 1$, the two families coincide, and we simply obtain the multiples of 2π . In the case $a = -1$, the two families coincide, and we simply obtain the odd multiples of π . In the case where $a = 0$, the two families can be combined into one family: odd multiples of $\pi/2$.

1.4. The underclass of trigonometric functions. We recall a bunch of trigonometric functions that are usually under the radar, but recently surfaced in integration problems:

- (1) The tan function is defined as the quotient of the opposite side to the adjacent side. It is defined as \sin / \cos . Note that tan is undefined wherever cos is zero, which is to say, at odd multiples of $\pi/2$.
- (2) The cot function is defined as the quotient of the adjacent side to the opposite side. It is defined as \cos / \sin . Note that cot is undefined wherever sin is zero, which is to say, at all multiples of π .
- (3) The sec function is defined as the quotient of the hypotenuse to the adjacent side. It is defined as $1 / \cos$. Note that sec is undefined wherever cos is zero, which is to say, at odd multiples of $\pi/2$.
- (4) The csc function is defined as the quotient of the hypotenuse to the opposite side. It is defined as $1 / \sin$. Note that csc is undefined wherever sin is zero, which is to say, at all multiples of π .

tan and cot have periods of π each. tan is increasing on each interval where defined (but more than makes up for it at the isolated points where it is undefined) while cot is decreasing on each interval where defined (but more than makes up for it at the isolated points where it is undefined). You can think of these as phoenix functions – they start out as cute little creatures, grow to ugly monsters, then vanish and reappear as cute little creatures.

Here are the key relationships:

- (1) $\tan^2 x + 1 = \sec^2 x$.
- (2) $\cot^2 x + 1 = \csc^2 x$.
- (3) tan and cot are reciprocals of each other where both are defined.
- (4) sec and cos are reciprocals of each other where both are defined.
- (5) csc and sin are reciprocals of each other where both are defined.
- (6) $\tan(\pi/2 - x) = \cot x$.
- (7) $\sec(\pi/2 - x) = \csc x$.
- (8) The derivative of tan is \sec^2 .
- (9) The derivative of cot is $-\csc^2$.
- (10) The derivative of sec is $\sec \cdot \tan$.
- (11) The derivative of csc is $-\csc \cdot \cot$.
- (12) The antiderivative of tan is $-\ln |\cos|$.
- (13) The antiderivative of cot is $\ln |\sin|$.
- (14) The antiderivative of sec is $\ln |\sec + \tan|$.
- (15) The antiderivative of csc is $\ln |\csc - \cot|$.

1.5. Three forms of complementarity. There are six trigonometric functions, sin, cos, tan, cot, sec, and csc, and there are three ways we can pair them up.

- (1) *Complementary angle pairing:* Here, we pair f and g if $f(x) = g(\pi/2 - x)$ for all x where f is well-defined. Under this pairing $\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$, and $\sec \leftrightarrow \csc$. The prefix *co* basically refers to this pairing, hence *cosine*, *cotangent*, and *cosecant*.
- (2) *Reciprocal pairing:* Here, we pair f and g if $f(x) = 1/g(x)$ for all x where both sides make sense. Under this pairing, $\sin \leftrightarrow \csc$, $\cos \leftrightarrow \sec$, and $\tan \leftrightarrow \cot$.
- (3) *Square sum/difference relation pairing:* Here, we pair f and g if the sum or difference of their squares is constant. Under this pairing, $\sin \leftrightarrow \cos$, $\tan \leftrightarrow \sec$, and $\cot \leftrightarrow \csc$.

2. READY TO INVERT NOW

2.1. Looking before we leap. With this brief primer on trigonometric functions, we are almost ready to explore *inverse trigonometric functions*.

Recall that one of the conditions for inverting a function is that the function be one-to-one. However, even if a function is not one-to-one on its entire domain, it is possible that the function is one-to-one on some

subset of the domain. If the image of that subset equals the range of the function on the whole domain, then we can try to get an inverse by picking the representative in that subset.

For instance the squaring function is not one-to-one on \mathbb{R} . However, its restriction to the nonnegative reals is one-to-one. And we can define an inverse, otherwise known as the *square root* function, from the nonnegative reals to the nonnegative reals.

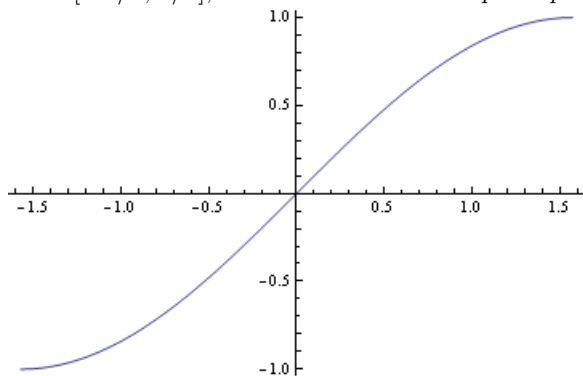
The trigonometric functions are far from one-to-one when viewed on their entire domain: after all, they're periodic. So the first step will be to find a subset of the domain that is small enough for the function to be one-to-one, but large enough for its image to be the whole range of the function. We will observe a lot of interesting things in the process. Enough looking, let's leap.

Aside: Photographic memories. When you hear a person's name, the person's face probably pops in your mind. When you hear the word "apple", the picture (and perhaps the taste/smell) of an apple probably pops into your mind. In the same way, when you hear the name of a familiar function, the picture of that function should immediately pop into your mind. Not a very refined picture with labels, just a crude picture. If the situation demands it, you should be able to refine the picture.

You should also be able to recall the graphs of multiple functions together, particularly functions that are commonly juxtaposed with each other. For instance, you should be able to imagine sin and cos graphed together, or x and x^2 graphed together, etc.

2.2. Inverting the sine function. We see that the sine function has a period of 2π , so definitely, if we take an interval of length 2π , the sine function takes all the values in its range, which is $[-1, 1]$. However, we observe that within an interval of length 2π , there are repetitions, and the function is not one-to-one.

In fact, all we need to do is look at an interval where the sin function is increasing from -1 to 1 or decreasing from 1 to -1 . Several such intervals come to mind. For instance, the interval $[-\pi/2, \pi/2]$ sees the sine function go from -1 to 1 , starting off at a sluggish pace, picking pace to max right in the middle (at 0), and then coming to a graceful halt at $\pi/2$. There are other such intervals, but we will stick to the interval $[-\pi/2, \pi/2]$, which we will call the *principal branch*.

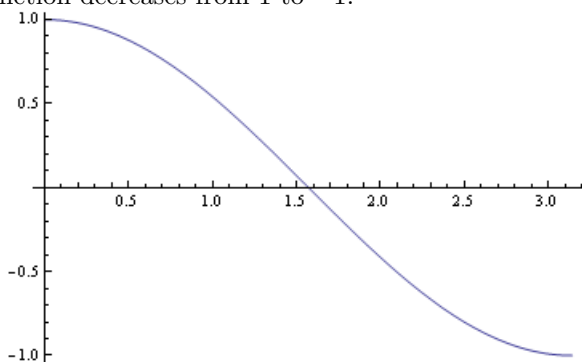


We now define the *inverse sine function* or the *arcsine function*, as the function from $[-1, 1]$ to $[-\pi/2, \pi/2]$ that sends a to the unique $\theta \in [-\pi/2, \pi/2]$ such that $\sin \theta = a$. This function is denoted \arcsin , and sometimes as \sin^{-1} . It is a one-to-one increasing function from $[-1, 1]$ to $[-\pi/2, \pi/2]$. Its graph can be obtained by reflecting about the $y = x$ line the graph of the sine function on the interval $[-\pi/2, \pi/2]$. Note that this has vertical tangents at the endpoints, and its sense of concavity is opposite to that of sin on the relevant intervals.

Now, consider the function cos on this interval. You see something very interesting: on the range of the arcsine function (which is the interval $[-\pi/2, \pi/2]$) the cosine function is ≥ 0 . This observation is very important for some of the trigonometric substitutions that we will see in integrations. Let's repeat: *cosine is nonnegative on the range of the arc sine function.*

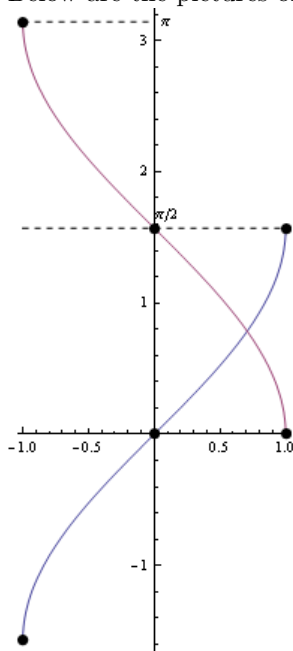
Aside: Something to feel good about next time. In the next class, we will study integrations involving inverse trigonometric functions. It would be great if, when we do that, one of you can point out this observation at the particular juncture in the reasoning where it becomes crucial.

2.3. Inverting the cosine function. For the cosine function, we choose the interval $[0, \pi]$, where the function decreases from 1 to -1 .



We define $\arccos(a)$ as the unique value $\theta \in [0, \pi]$ such that $\cos \theta = a$. \arccos is thus a decreasing function from $[-1, 1]$ to $[0, \pi]$. As in the previous example, we see that \sin is nonnegative on the range of \arccos .

Below are the pictures of the arc sine and arc cosine functions:



2.4. An important relationship. We have the following relationship:

$$\arcsin(x) + \arccos(x) = \pi/2$$

This basically follows from the fact that $\sin \theta = \cos(\pi/2 - \theta)$. We also need to use the fact that $\theta \in [-\pi/2, \pi/2] \iff \pi/2 - \theta \in [0, \pi]$.

2.5. Solving $\sin x = a$ and $\cos x = a$, again. Earlier, we talked about solving $\sin x = a$ (respectively, $\cos x = a$). The first step of the solution in the general case was finding a θ such that $\sin \theta = a$ (respectively, $\cos \theta = a$). Now, for that first step, we can simply write $\theta = \arcsin(a)$ (respectively, $\arccos(a)$).

Aside: No easy way out. Don't use \arcsin and \arccos mindlessly. For instance, when solving $\sin x = 2$, don't just write answers in terms of $\arcsin(2)$ – you should know that 2 lies outside the domain of \arcsin . Also, for values where you can easily find an angle θ , such as $a = 0, \pm 1, \pm 1/\sqrt{2}, \pm\sqrt{3}/2, \pm 1/2$, you must do so.

3. DIFFERENTIATING THE INVERSE SINE AND COSINE FUNCTIONS

3.1. A review of arc sine and arc cosine. We recall, in point form, some key facts about the arc sine and the arc cosine functions:

- (1) The arc sine function has domain $[-1, 1]$, range $[-\pi/2, \pi/2]$. It is an increasing function, and has vertical tangents at its ends. Further, by definition, $\sin(\arcsin x) = x$ for all $x \in [-1, 1]$.
- (2) The arc cosine function has domain $[-1, 1]$ and range $[0, \pi]$. It is a decreasing function, and has vertical tangents at both ends. Further, by definition, $\cos(\arccos x) = x$ for all $x \in [-1, 1]$.
- (3) $\arcsin(x) + \arccos(x) = \pi/2$ for all $x \in [-1, 1]$.
- (4) \cos is nonnegative on the range of \arcsin , and \sin is nonnegative on the range of \arccos .

Aside: odd and even. A little point of interest: if you take an odd one-to-one function, and consider the inverse function, that is also odd. Since \sin is odd, and the domain $[-\pi/2, \pi/2]$ to which we restrict it is symmetric about 0, the restriction of \sin to this domain is also an odd function – hence, its inverse function \arcsin is an odd function.

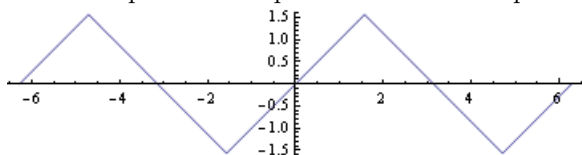
Is the inverse function to an even function also even? It turns out that the question does not make sense, because an even function cannot be one-to-one if it is defined anywhere other than 0. To get an inverse, we first need to restrict the domain – but the restricted domain cannot have any values a and $-a$ simultaneously. And then, since the inverse function is again one-to-one, it cannot be even.

3.2. Finding $\arcsin(\sin x)$. We next turn to problems of the form: find $\arcsin(\sin x)$.

Your first temptation may be to say that the answer is x . This is indeed true if x is in the principal branch, i.e., x is in the range of \arcsin . Otherwise, the answer cannot be x . What then? We know that if $t = \arcsin(\sin x)$, then $\sin t = \sin x$. Thus, t is the unique value in the principal branch $[-\pi/2, \pi/2]$ such that $\sin t = \sin x$. Finding this t essentially involves using the general solution method discussed last time or just an inspection based on the graph.

For instance, $\arcsin(\sin(7\pi/9))$ is an angle $\theta \in [-\pi/2, \pi/2]$ such that $\sin \theta = \sin(7\pi/9)$. Now, we know that $\sin(\pi - x) = \sin x$, so we have $\sin(\pi - 7\pi/9) = \sin(7\pi/9)$, giving that $\sin(2\pi/9) = \sin(7\pi/9)$. And $2\pi/9$ lies in the range of \arcsin , so the much coveted θ is $2\pi/9$.

The $\arcsin \circ \sin$ function is a continuous, periodic, and piecewise linear function. It has a different linear definition on each interval between successive odd multiples of $\pi/2$. The graph of the function overall has a sawtooth shape with sharp turns at odd multiples of $\pi/2$. Below is the picture:



3.3. Differentiating \arcsin . Recall a tidbit about inverse functions: if $f(a) = b$, then:

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Equivalently:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

In particular, we have that:

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))}$$

Since $\sin' = \cos$, we obtain that:

$$\arcsin'(x) = \frac{1}{\cos(\arcsin(x))}$$

The next key step involves trying to figure out if we can simplify $\cos(\arcsin x)$. So let's say $\arcsin x = \theta$. Then, $\sin \theta = x$ by definition. And we have the relationship:

$$\sin^2 \theta + \cos^2 \theta = 1$$

We need to find $\cos \theta$. The above relation gives:

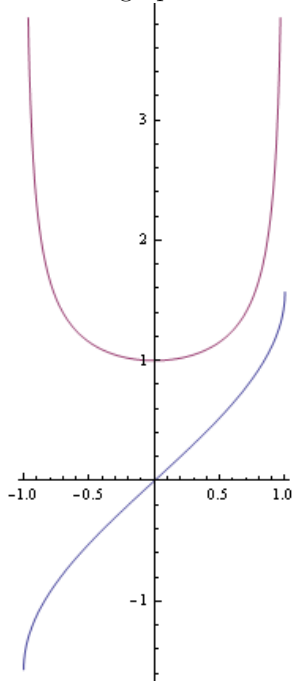
$$\cos^2 \theta = 1 - \sin^2 \theta = 1 - x^2$$

Thus, either $\cos \theta = \sqrt{1 - x^2}$ or $\cos \theta = -\sqrt{1 - x^2}$. how do we figure out which one it is?

It turns out that it is always the nonnegative square root, because, by the choice of principal branch, \cos is nonnegative on the range of \arcsin . Thus, we obtain that:

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}$$

Here is a graphical illustration of \arcsin and its derivative *on the same graph*:



3.4. Voodoo magic. Something amazing just happened. We defined trigonometric functions. First we did it for acute angles. Then we extended the definition to all real numbers. We developed the machinery of differentiation and integration. We developed the machinery of inverse functions. We used this machinery to define inverse trigonometric functions. And then we differentiated an inverse trigonometric function, and what did we get? We got something that looks *purely algebraic*! We got something that involves polynomials and squareroots, with nary a hint of trigonometry.

To see how magical this is, consider if, instead, I had presented the indefinite integration problem:

$$\int \frac{dx}{\sqrt{1 - x^2}}$$

From what we just saw above, the answer is $\arcsin(x) + C$. Thus, the entire machinery of trigonometry is necessary to unlock an innocuous indefinite integration problem arising from algebra.

Note that this is similar to the way we saw the natural logarithm pop up when integrating $(dx)/x$. However, there is a key difference. In that example, we used the integration of $(dx)/x$ to *define* the natural logarithm, and then noticed that the function relates quite a bit with our existing notions of exponentiation. In the trigonometric case, we obtained the arcsin function through other means, and then noticed that it works as an antiderivative for $(1 - x^2)^{-1/2}$.

3.5. **A general observation.** In the previous quarter, we saw that if $f'(x) = g(f(x))$, then f^{-1} is an antiderivative for $1/g$. The above is a special case, where $f = \sin$ (restricted to $[-\pi/2, \pi/2]$), and $g(t) = \sqrt{1-t^2}$.

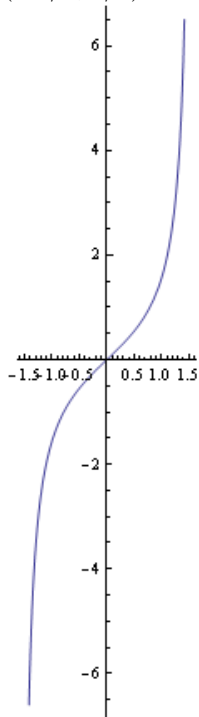
3.6. **The derivative of arc cosine.** By similar reasoning to the above:

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$$

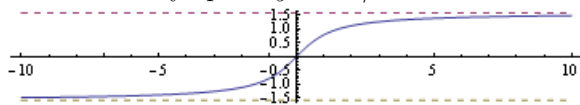
Note that this makes sense because $\arcsin + \arccos$ is a constant, hence its derivative should be zero, and hence, $\arccos' = -\arcsin'$.

4. THE OTHER INVERSE TRIGONOMETRIC FUNCTIONS

4.1. **The inverse of the tangent function.** Recall that \tan is not defined at odd multiples of $\pi/2$, and is increasing on each of the intervals on which it is defined. Thus, a natural choice of principal branch for \tan is $(-\pi/2, \pi/2)$.



The inverse function, denoted \arctan or sometimes \tan^{-1} , and called the arc tangent function, takes as input $x \in \mathbb{R}$ and outputs the unique $\theta \in (-\pi/2, \pi/2)$ such that $\tan \theta = x$. Thus, it is a one-to-one increasing function with domain \mathbb{R} and range $(-\pi/2, \pi/2)$. In particular, it has two horizontal asymptotes: the horizontal asymptote $y = +\pi/2$ for $x \rightarrow +\infty$ and the horizontal asymptote $y = -\pi/2$ for $x \rightarrow -\infty$.



The derivative of this is given by:

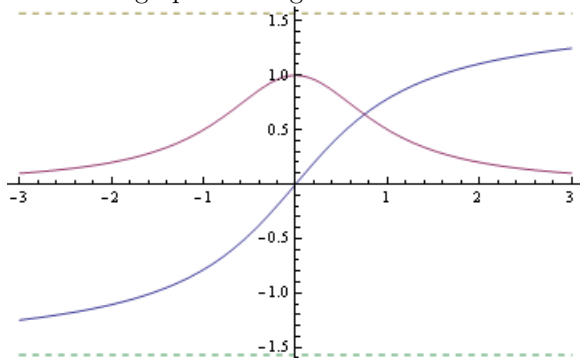
$$\arctan'(x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+x^2}$$

This also gives another indefinite integration formula:

$$\int \frac{dx}{1+x^2} = \arctan x + C$$

We saw a variant of this in Homework 9 of Math 152, Advanced Problem 1.

Below is a graph showing arctan and its derivative together:



4.2. The inverse of the cotangent function. Recall that cot is not defined at multiples of π , and is decreasing on each of the intervals where it is defined. Thus, a natural choice of principal branch for cot is $(0, \pi)$.

The inverse function, denoted arccot or sometimes \cot^{-1} , and called the arc cotangent function, is thus a one-to-one decreasing function from \mathbb{R} to $(0, \pi)$. Further, its derivative is:

$$\operatorname{arccot}'(x) = \frac{-1}{1+x^2}$$

Also, we have that:

$$\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$$

As was the case with arc sine and arc cosine, the sum of the derivatives of arctan and arccot is zero, as it should be.

4.3. The inverse of the secant and cosecant function. We do not talk much about these, but we formally define:

$$\operatorname{arcsec}(x) = \arccos(1/x)$$

with domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi] \setminus \{\pi/2\}$.

And we define:

$$\operatorname{arccsc}(x) = \arcsin(1/x)$$

with domain $(-\infty, -1] \cup [1, \infty)$ and range $[-\pi/2, \pi/2] \setminus \{0\}$. We have:

$$\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \frac{\pi}{2}$$

We also have the derivative formulas:

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

and:

$$\operatorname{arccsc}'(x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

The proof of these requires multiple cases, basically because tan is not nonnegative on the entire range of arcsec, and cot is not nonnegative on the entire range of arccsc. Look up the proof in the book if you are interested.

5. NOTE ON IMPROPER INTEGRALS

Consider the definite integral:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

This becomes:

$$[\arcsin x]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}$$

This integration is correct. However, the integrand itself ($1/\sqrt{1-x^2}$) is not defined at 1. In fact, \arcsin is defined at 1 but has a one-sided vertical tangent, with the derivative approaching $+\infty$.

This is an example of an *improper integration*. We will come back to improper integrations later in the course.

6. THE FORMULAS FOR INDEFINITE INTEGRATION

We have the following formulas for indefinite integration:

$$\begin{aligned} \int \frac{dx}{\sqrt{1-x^2}} &= \arcsin(x) + C \\ \int \frac{dx}{1+x^2} &= \arctan(x) + C \\ \int \frac{dx}{|x|\sqrt{x^2-1}} &= \operatorname{arcsec}(x) + C \end{aligned}$$

We now consider slight variants of these. In all the formulas below, $a > 0$ is a constant:

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2-x^2}} &= \arcsin\left(\frac{x}{a}\right) + C \\ \int \frac{dx}{a^2+x^2} &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \\ \int \frac{dx}{|x|\sqrt{x^2-a^2}} &= \frac{1}{a} \operatorname{arcsec}\left(\frac{x}{a}\right) + C \end{aligned}$$

We consider yet more variants of these. Here, $a > 0$ and b is arbitrary, but both are constants:

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2-(x-b)^2}} &= \arcsin\left(\frac{x-b}{a}\right) + C \\ \int \frac{dx}{a^2+(x-b)^2} &= \frac{1}{a} \arctan\left(\frac{x-b}{a}\right) + C \\ \int \frac{dx}{|x-b|\sqrt{(x-b)^2-a^2}} &= \frac{1}{a} \operatorname{arcsec}\left(\frac{x-b}{a}\right) + C \end{aligned}$$

Next, we see how these can be combined with other ideas. As before, $a > 0$ is a constant:

$$\begin{aligned}
\int \frac{f'(x) dx}{\sqrt{a^2 - (f(x))^2}} &= \arcsin\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f'(x) dx}{(f(x))^2 + a^2} &= \frac{1}{a} \arctan\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f'(x) dx}{|f(x)|\sqrt{(f(x))^2 - a^2}} &= \frac{1}{a} \operatorname{arcsec}\left(\frac{f(x)}{a}\right) + C \\
\int \frac{f(\arctan(x/a))}{a^2 + x^2} dx &= \frac{1}{a} \int f(u) du \text{ where } u = \arctan(x/a) \\
\int \frac{f(\arcsin(x/a))}{\sqrt{a^2 - x^2}} dx &= \int f(u) du \text{ where } u = \arcsin(x/a)
\end{aligned}$$