

INTEGRATION BY PARTS AND RELATED MISCELLANEA

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 8.2, part of Section 8.3 (some of the material of Section 8.3 is a repeat of earlier ideas; that part is not discussed in these lecture notes).

Difficulty level: For the key ideas, easy to moderate, depending on the extent to which you have seen integration by parts before. For some of the subtler notions, moderate to hard.

What students should already know: The product rule for differentiation, graphical interpretation of integral as area.

What students should definitely get: The formal procedure/formula for integration by parts. Applying it to products of polynomials and trigonometric functions, products of polynomial and exponential functions, products of trigonometric and exponential functions, expressions involving logarithmic and inverse trigonometric functions. Repeated application. The key idea for using it in circular or self-referential situations, i.e., the original integral reappearing as a new integral.

What students should hopefully get: The interpretation of integration by parts as two ways of calculating an area. The rationale for integration by parts. The notion of complexity that is critical to deciding how to choose parts correctly.

Note on difference with text: The text explains integration by parts with the letters u and v for what I have called F and G . You may be more familiar with the letters u and v and the corresponding way of thinking. However, some of the added discussion of complexity and choosing parts becomes easier (in my view) with the notation and context set in the lecture notes. Rest assured, there is no substantive difference.

EXECUTIVE SUMMARY

Words ...

- (1) Integration by parts is a technique that uses the product rule to integrate a product of two terms. If F and g are the two functions, and G is an antiderivative of g , we obtain:

$$\int F(x)g(x) dx = F(x)G(x) - \int F'(x)G(x) dx$$

This basically follows from the product rule, which states that:

$$\frac{d}{dx}[F(x)G(x)] = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + F'(x)G(x)$$

In particular, we *integrate* one function and *differentiate* the other.

- (2) Applying integration by parts twice stupidly tells us nothing. In particular, if we choose to re-integrate the piece that we just obtained from differentiation, we get nowhere.
- (3) The definite integral version of this is:

$$\int_a^b F(x)g(x) dx = [F(x)G(x)]_a^b - \int_a^b F'(x)G(x) dx$$

In particular, note that the part outside the integral sign is simply evaluated between limits.

- (4) We can use integration by parts to show that integrating a function f twice is equivalent to integrating f and the function $xf(x)$. More generally, integrating a function f k times is equivalent to integrating $f(x)$, $xf(x)$, $x^2f(x)$, and so on up till $x^{k-1}f(x)$.
- (5) To integrate $e^xg(x)$, we can use integration by parts, typically taking e^x as the second part. We could also do this integral by finding a function f such that $f + f' = g$, and then writing the answer as $e^xf(x) + C$. The latter approach is feasible and sometimes quicker in case g is a polynomial function.

Actions ...

- (1) Integration by parts is *not* the first or best technique to consider upon seeing a product. The first thing to attempt is the u -substitution/chain rule. In cases where such a thing fails, we move to integration by parts.
- (2) For products of trigonometric functions, it is usually more fruitful to apply the trigonometric identities, such as $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$, than to use integration by parts.
- (3) To apply integration by parts, we need to express the function as a product. The *part to integrate* should always be chosen as something that we *know* how to integrate. Beyond this, we should try to make sure that: (i) the part to differentiate gets *simpler in some sense* after differentiating, and (ii) the part to integrate does not get too much more complicated upon integration.
- (4) Beware of the circular trap when doing integration by parts. In particular, when using integration by parts twice, you should always make sure that the part to integrate is *not* chosen as the thing you just got by differentiating.
- (5) For polynomial times trigonometric or exponential, always take the polynomial as the first part (the part to differentiate). The trigonometric or exponential thing is the thing to integrate. After enough steps, the polynomial is reduced to a constant, and the trigonometric part (hopefully) does not become any more complex.
- (6) The ILATE/LIATE rule is a reasonable precedence rule for doing integration by parts.
- (7) In some cases, we may use integration by parts once or twice and then relate the integral we get at the end to the original integral in some other way (for instance, using a trigonometric identity) to solve the problem. Examples include $e^x \cos x$ and $\sec^3 x$.
- (8) For functions such as $\ln(x)$, we typically take 1 as the part to integrate and the given function as the part to differentiate.
- (9) In general, for functions of the form $f(\ln x)$ or $f(x^{1/n})$, we can first do a u -substitution (setting $u = \ln x$ or $u = x^{1/n}$ respectively). This converts it to a product, on which we can apply integration by parts. *We can also apply integration by parts directly, but this tends to get messy.*

1. THE PRODUCT RULE AND THE STATEMENT OF INTEGRATION BY PARTS

1.1. **The product rule for differentiation, nicely stated.** Suppose F and G are functions with $F' = f$ and $G' = g$. The product rule states that:

$$(F \cdot G)' = f \cdot G + F \cdot g$$

Stating this in terms of antiderivatives, we have:

$$\int f \cdot G + F \cdot g = F \cdot G$$

More concretely:

$$\int f(t)G(t) + F(t)g(t) dt = F(t)G(t)$$

This can be rearranged as:

$$\int F(t)g(t) dt = F(t)G(t) - \int f(t)G(t) dt$$

Writing G as $\int g(t) dt$ and f as F' , we get:

$$\int F(t)g(t) dt = F(t) \int g(t) dt - \int F'(t) \left(\int g(t) dt \right) dt$$

The above formulation is termed *integration by parts*. But before getting into the explanation, let us understand the product rule a little more closely.

1.2. **Parametric curves.** There have been three ways we have created curves in the past:

- (1) A curve obtained by writing y as a function of x .
- (2) A curve obtained by writing x as a function of y .
- (3) A curve obtained as the set of (x, y) such that $F(x, y) = 0$, where F is an expression in two variables.

This came up in the context of implicit differentiation.

There is a fourth way of describing curves, where we write both x and y as functions of a variable t . The standard example of this is the circle, where we can describe the circle as the set $(\cos t, \sin t)$, with $t \in \mathbb{R}$. In this context, x is given by the function \cos and y is given by the function \sin , with the parameter being t .

Suppose F and G are two functions, with $F' = f$ and $G' = g$. First, consider the curve given by $(F(t), G(t))$. In other words, it is the set of points whose x -coordinate is $F(t)$ and whose y -coordinate is $G(t)$. Suppose $a < b$ are points such that $[a, b]$ is in the domain of both F and G . Then $[F(t)G(t)]_a^b = F(b)G(b) - F(a)G(a)$.

This can be interpreted graphically as follows. Consider the rectangle with two sides along the axes and one vertex at the point $(F(a), G(a))$. The area of this rectangle is $F(a)G(a)$. Similarly, consider the rectangle with two sides along the axes and one vertex at the point $(F(b), G(b))$. The area of this rectangle is $F(b)G(b)$. The difference of these areas is also the area of the region between the rectangles, which is like a corridor with one turn.

Now, the curve $(F(t), G(t))$ (in the simple case) divides this corridor region into two parts. One part is the region between the curve and the x -axis. The other part is the region between the curve and the y -axis. The area of the first part is given by:

$$\int y \, dx = \int_a^b G(t)F'(t) \, dt$$

The other part is given by:

$$\int x \, dy = \int_a^b F(t)G'(t) \, dt$$

We thus obtain:

$$[F(t)G(t)]_a^b = \int_a^b F(t)G'(t) + F'(t)G(t) \, dt$$

which is just a reformulation of the product rule for differentiation. This idea that the product rule in integral form is just *two ways of calculating areas* is an important idea.

1.3. **Using integration by parts: the key idea.** To begin integration by parts, we need to do the following: express the integrand as the product of a function F (that we know how to differentiate) and a function g (that we know how to integrate). We first find the derivative f of F and an antiderivative G of g . We then use integration by parts:

$$\int F(t)g(t) \, dt = F(t)G(t) - \int f(t)G(t) \, dt$$

We can do everything inline:

$$\int F(t)g(t) \, dt = F(t) \int g(t) \, dt - \int F'(t) \left[\int g(t) \, dt \right] dt$$

The following things should be kept in mind, though:

- (1) Although we use the symbol $\int g(t) \, dt$, what we are interested in is picking *one* antiderivative for g . We are not interested in the general expression for the antiderivative (i.e., we don't need the $+C$). Moreover, the expression for the antiderivative must be the *same* in both parts of the formula.
- (2) When using this formula for definite integration, the evaluation between limits is done *after* the final expression is calculated. Specifically, the inner integrals, such as $\int g$, are *not* the places where we evaluate between limits – we only evaluate between limits the entirety of the expression.

The part F that we differentiate is typically termed the *first part*, or Part I, or the *part to differentiate*, while the part g that we integrate is termed the *second part*, or Part II, or the *part to integrate*.

2. USING INTEGRATION BY PARTS

2.1. The burden of choice. To use integration by parts, we need to write the integrand as a product of two terms, such that the first part is easy to differentiate and the second part is easy to integrate. There are many elements of choice here:

- (1) We first need to think of the integrand as a product. In some cases, the integrand may not look like a product at all. In other cases, it may be a product of a large number of pieces.
- (2) We next need to decide which of the parts is Part I and which of the parts is Part II.

2.2. Some reflections. We divide the elementary functions that we have found into five easy classes:

- (1) *Inverse trigonometric functions:* The derivatives of these are algebraic functions (rational functions or functions involving rational functions, radicals). Further differentiation keeps us in the algebraic domain.
- (2) *Logarithmic functions:* The derivatives of these are algebraic functions (specifically, rational functions). Further differentiation keeps us in the algebraic domain.
- (3) *Algebraic functions:* These include polynomials, rational functions, and functions created using rational functions and radicals (squareroots, etc.). Differentiation keeps us in the algebraic domain. Further, for polynomials, differentiation eventually gets us down to constants, while integrating again and again causes the degree to go to infinity. Integration of polynomials keeps us in the algebraic domain, but for other rational functions, it may land us in the inverse trigonometric or logarithmic domains.
- (4) *Trigonometric functions:* These include \sin , \cos , and rational functions of these. Differentiation keeps us in the trigonometric domain, and integrating once gets us into the domain of linear plus trigonometric functions. Integrating k times gets us into the polynomial plus trigonometric domain. Note also another interesting thing: for the most part, neither differentiation nor integration increases the complexity of the trigonometric part. This is particularly the case for \sin and \cos , whose sequence of derivatives is periodic with period 4.
- (5) *Exponential functions:* This includes \exp and various functions constructed from it such as \cosh and \sinh . These functions are remarkably stable under differentiation and integration, and the complexity does not change much under either operation.

2.3. Products of polynomials and trigonometric/exponential functions. Suppose we have a function of the form $F(x) \cos x$, where F is a polynomial. How do we integrate it?

It turns out that it is ideal to choose F , the polynomial, as the first part, since differentiating a polynomial gives a polynomial of lower degree. On the other hand, integrating \cos gives \sin , which has the same amount of complexity.

Consider, for instance:

$$\int x \cos x \, dx$$

We take x as the first part and $\cos x$ as the second part. An antiderivative for $\cos x$ is $\sin x$, so we get:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x + \cos x + C$$

Now, consider:

$$\int x^2 \cos x \, dx$$

We take x^2 as the first part and $\cos x$ as the second part. An antiderivative of $\cos x$ is $\sin x$, so we get:

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

We now use integration by parts for the $2x \sin x$ integrand, first pulling the 2 out and then taking x and $\sin x$ as the first and second part respectively:

$$\int x^2 \cos x \, dx = x^2 \sin x - 2x(-\cos x) + 2 \int (-\cos x) \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

You can convince yourself that this technique can be used to handle the product of any polynomial with $\sin x$ or $\cos x$. Note, however, that this technique fails when we have a fractional or negative exponent, because what we really want is that, after a finite number of iterations of integration by parts, the polynomial gets down to a constant. If we start with something like $x^{3/2} \sin x$ then repeated differentiation never gets to a constant because the exponent jumps past 0 from $1/2$ to $-1/2$. Similarly, when trying to integrate $(\sin x)/x$, we never hit an exponent of 0 on the x because the exponent, starting from -1 , keeps going down further and further.

The technique does not *always* work for *everything* that is a product of a polynomial and a trigonometric function. \sin and \cos are particularly nice because they are (up to sign) integrals and derivatives of each other. The situation gets more complicated with \tan . For instance, suppose we are trying to integrate:

$$\int x \tan x \, dx$$

We start off by taking x as the first part and $\tan x$ as the second part. We get:

$$\int x \tan x \, dx = -x \ln |\cos x| - \int (-\ln |\cos x|) \, dx$$

But the inner integral is no easier than what we started with. More generally, the problem with other trigonometric functions is that we may be unable to repeatedly integrate the trigonometric function the requisite number of times. But the technique does work occasionally. For instance, consider:

$$\int x \tan^2 x \, dx$$

We know that $\tan^2 x$ is the derivative of $\tan x - x$, and we thus get:

$$\int x \tan^2 x \, dx = x(\tan x - x) - \int (\tan x - x) \, dx = x(\tan x - x) + \ln |\cos x| + \frac{x^2}{2} + C = x \tan x + \ln |\cos x| - \frac{x^2}{2} + C$$

2.4. Integration by parts could be a red herring. The fact that the integrand is a product is *not* a compelling case for integration by parts. In fact, integration by parts should be treated as a last resort approach. Generally a product suggests a u -substitution or chain rule approach. So how do we distinguish between the chain rule-suggesting products and integration by parts?

Situations that are good for the chain rule are where one of the factors of the product is the derivative of something in terms of which the second factor can be written. Situations that are good for the product rule are those where the two factors seem unrelated.

For instance, the integral $\int x \sin(x^2) \, dx$ suggests the chain rule, setting $u = x^2$. We also see that in this case, integration by parts cannot really be applied because $\sin(x^2)$ doesn't have an antiderivative expressible in elementary terms. On the other hand, $x^2 \sin x$ is suitable for integration by parts.

Some situations are more mixed. For instance, consider an integrand of the form $x^3 \sin(x^2)$. Here, we first make a u -substitution, setting $u = x^2$, to get an integrand of the form $u \sin u$. Then, we use integration by parts. There are ways of combining this all into one step, but as beginners, you are encouraged to do it in two steps.

2.5. Products involving a polynomial and an exponential. These obey very similar rules to those outlined for products of polynomials and sine or cosine. Basically, the same nice feature is present: the exponential function, like sine and cosine, remains remarkably oblivious to any attempts to integrate and differentiate it. Thus, for its product with a polynomial, we can steadily bring down the degree of the polynomial to zero.

Recall that to integrate a function of the form $g(x)e^x$, one approach was to find a function f such that $f + f' = g$. Integration by parts gives an alternative, more hidden way of achieving the same goal.

More on this is given in section 4.1.

2.6. Logarithmic and inverse trigonometric functions. Logarithmic and inverse trigonometric functions have the behavior of being hard to integrate but easy to differentiate, and the derivatives land in the algebraic domain. This means that they can be differentiated as many times as desired, giving answers in the algebraic domain.

Thus, when dealing with the product of a logarithmic or inverse trigonometric function and a polynomial or rational function, we take the polynomial or rational function as Part II.

We first begin by integrating the logarithmic function itself. This is just a single function, but we use integration by parts by choosing it as Part I and the constant function 1 as Part II.

Thus:

$$\int \ln x \, dx = \int \ln x \cdot 1 \, dx = \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x + C$$

Similarly, we obtain that:

$$\int \arctan x \, dx = x \arctan x - \int \frac{x \, dx}{1+x^2} = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$$

where, for the latter integration, we use the form g'/g . We can use a similar approach to integrate arcsin. This approach also extends to more tricky computations. For instance, consider the integral:

$$\int x^r (\ln x)^s \, dx$$

where $r \neq -1$ and s is a nonnegative integer. We take $(\ln x)^s$ as part I and x^r as part II and integrate. We can then reduce this to integrating $x^r (\ln x)^{s-1}$. Specifically, the power of $\ln x$ goes down by 1. Repeating the procedure s times, we finally get down to simply integrating x^r .

As observed earlier, you should first be on the lookout for u -substitutions before attempting integration by parts. Thus, it is a bad idea to integrate $\int (\ln x)/x \, dx$ using integration by parts: that integration is easily settled using a u -substitution.

2.7. Integration by parts and recursive relationships. In some cases, we attempt to do an integration by parts and end up with a new integral that just looks like a copy of the old integral. We consider a familiar example.

$$\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

We now consider another approach for this integral: integration by parts. We take $\sin x$ as the first part and $\sin x$ as the second part. An antiderivative for $\sin x$ is $-\cos x$, so we get:

$$\int \sin^2 x \, dx = -\sin x \cos x - \int \cos x (-\cos x) \, dx = -\sin x \cos x + \int \cos^2 x \, dx$$

Now, we need to integrate $\cos^2 x$. We observe that $\cos^2 x = 1 - \sin^2 x$. We thus get:

$$\int \sin^2 x \, dx = -\sin x \cos x + \int 1 - \sin^2 x \, dx = x - \sin x \cos x - \int \sin^2 x \, dx$$

We now move terms around and obtain:

$$2 \int \sin^2 x \, dx = x - \sin x \cos x$$

Thus, we obtain:

$$\int \sin^2 x \, dx = \frac{x - \sin x \cos x}{2}$$

Note that this is the same as the previous answer, using the fact that $\sin(2x) = 2 \sin x \cos x$.

A similar approach works for $\int \cos^2 x dx$. So, it turns out that, by using integration by parts, we can avoid remembering the identities about $\cos(2x)$.

Note that the key idea here is to identify the copy of the original integrand in the new integrand we get when we apply integration by parts.

Here is another example:

$$\int \sec^3 x dx$$

We write $\sec^3 x = \sec x \cdot \sec^2 x$, taking $\sec x$ as the first part and $\sec^2 x$ as the second part. We thus obtain that:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$$

Since $\tan^2 x = \sec^2 x - 1$, the new integrand can be written as $\sec^3 x - \sec x$, and we get:

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

Thus, we obtain:

$$\int \sec^3 x dx = \frac{1}{2}[\sec x \tan x + \ln |\sec x + \tan x|] + C$$

Note that the $+C$ is added as an afterthought.

This idea, combined with induction, can be used to integrate odd powers of secant, as well as even powers of sine and cosine. You see some of this in your homework problems.

2.8. Products of trigonometric and exponential functions. Consider the integration problem:

$$\int e^x \cos x dx$$

We can try resolving this using integration by parts, taking e^x as the second part:

$$\int e^x \cos x dx = (\cos x)e^x - \int (-\sin x)e^x dx$$

We again take e^x as the second part, and obtain:

$$\int e^x \cos x dx = (\cos x + \sin x)e^x - \int e^x \cos x dx$$

We can now rearrange and obtain:

$$\int e^x \cos x dx = \frac{\cos x + \sin x}{2} e^x + C$$

2.9. Converting compositions to products. *Note:* I might not cover this in class, but you are encouraged to read it.

In some cases, tricky composites can be integrated using integration by parts. The general rule is that the inner function of the composition should have an inverse that is nice. We take two kinds of examples.

First, consider:

$$\int f(x^{1/n}) dx$$

The idea here is to set $u = x^{1/n}$. This gives:

$$\int f(u) \cdot nu^{n-1} du$$

Pulling the n out, we are left with a polynomial times f . In particular, this technique can be used to integrate functions of the form $\sin(x^{1/n})$, $\cos(x^{1/n})$, $e^{x^{1/n}}$.

We can directly use integration by parts on the original integrand and avoid this intermediate step by taking part II as 1. This is essentially equivalent but perhaps less intuitive since it is not clear what is going on.

A similar approach works for expressions of the form:

$$\int f(\ln x) dx$$

Here, we set $u = \ln x$, and obtain:

$$\int e^u f(u) du$$

after which we apply the usual techniques of integration by parts.

2.10. Circular traps. If applying integration by parts twice, one may end up with a tautology. This happens if the second application of integration by parts simply *reverses* the first application, in which case, you will just end up precisely with the original integrand. You need to be careful about this.

What this means is that if you write your original function as $F \cdot g$, and reduce the computation of $\int F \cdot g$ to the computation of $\int f \cdot G$, then taking G as the first part and f as the second part will simply retrace your steps. To actually do something new, you need to take f as the first part or G as the second part.

2.11. The ILATE/LIATE rule: a thumb rule for integration. A reasonable thumb rule for integration by parts is the ILATE/LIATE rule. This says that we give preference for part I in the integration by parts to the following functions in decreasing order:

- (1) *Logarithmic and inverse trigonometric functions:* These deserve first preference in integration by parts. In general, lump together all of these into part I of the integration by parts and whatever is left into part II (if nothing is left, take part II as 1).
- (2) *Algebraic functions:* This includes polynomial and rational functions, as well as things with radicals. These are taken as part II when placed along with a logarithmic or inverse trigonometric function, because by taking the logarithmic or inverse trigonometric function as part I, we can differentiate it and get completely in the algebraic domain (or so we hope).
- (3) *Trigonometric and exponential functions:* These are taken as the second part, because integrating them does not increase their complexity (unlike the case of polynomials, where every integration increases the degree). Among these, exponential functions are preferred as part II when opposed to trigonometric functions.

The ILATE/LIATE mnemonic arises by taking the first letters of these classes of functions (Inverse trigonometric, Logarithmic, Algebraic, Trigonometric, Exponential).

3. MORE ON COLLECTIONS OF FUNCTIONS CLOSED UNDER INTEGRATION AND DIFFERENTIATION

3.1. Polynomial functions. The collection of polynomial functions is closed under the following operations:

- (1) Addition and subtraction: The sum of two polynomial functions is a polynomial function. The difference of two polynomial functions is a polynomial function.
- (2) Scalar multiplication: Any scalar times a polynomial function is a polynomial function.
- (3) Multiplication: The product of two polynomial functions is a polynomial function.
- (4) Composition: A composition of two polynomial functions is a polynomial function.
- (5) Differentiation: The derivative of a polynomial function is a polynomial function. Another important thing to note is that polynomials are the only functions for which the higher derivatives eventually become identically zero.
- (6) Integration: Every antiderivative of a polynomial function is a polynomial function.

This collection is not closed under the following two operations:

- (1) Multiplicative inverses (reciprocals) and division: Applying this operation to polynomial functions gets us into the larger class of rational functions.
- (2) Inverse functions: Even in the cases where a polynomial function is one-to-one, its inverse is not a polynomial function unless the polynomial we started out is linear.

3.2. Rational functions. Rational functions are closed under addition, subtraction, scalar multiplication, multiplication, division, and differentiation. With respect to multiplicative inverses, they are more closed (i.e., better) than polynomials. On the other hand, with respect to integration, they are much worse behaved. Recall that we have so far seen two types of rational functions whose antiderivatives shatter the algebraic glass ceiling:

- (1) The rational function $1/x$, and more generally, rational functions with linear denominators, have antiderivatives that involve natural logarithms.
- (2) Rational functions of the form $1/(x^2 + a^2)$, and more generally, rational functions with quadratic denominators that have negative discriminant, have antiderivatives that involve arctangents.

It turns out that we can use the above two facts to obtain a method for integrating *any* rational function. This uses a technique called *partial fractions*, which essentially writes the original rational function as a sum of rational functions such that the denominator of each is either linear or a quadratic with negative discriminant. However, the question of interest for now is: can we integrate *repeatedly*? It is not enough to have a method for integrating the original rational function – we must have a method for integrating the antiderivatives obtained.

Indeed, the process of integration by parts gives a general approach for this. Note that:

- (1) Integrating $1/x$ gives $\ln x$. Integrating $\ln x$ gives $x \ln x - x + C$. Each further round of integration gives a constant times $x^k \ln x$ plus a polynomial in x . To execute each step, we use integration by parts with $\ln x$ as the first part.
- (2) Integrating $1/(x^2 + 1)$ gives $\arctan x$. Integrating $\arctan x$ gives $x \arctan x - (1/2) \ln(1 + x^2) + C$. We can integrate this again, but notice that it gets tricky: *both* the expressions $x \arctan x$ and $\ln(1 + x^2)$ require integration by parts to resolve. At any stage, the expression will involve a polynomial times $\arctan x$, a polynomial times $\ln(1 + x^2)$, and a free-floating polynomial. (See the discussion in the last subsection on why a rational function can always be repeatedly integrated).

3.3. Some functions that we cannot integrate. Integration by parts is the last of the classical methods of integration. (There remain a few tricks here and there, and some tying-together ideas, but no major further breakthrough). So, we are now in a position to start appreciating, at least at a basic level, why some kinds of functions do not have antiderivatives that can be expressed in elementary terms.

Roughly, integration by parts is the *only* viable strategy for handling non-obvious heterogeneous products, i.e., products of functions of very different kinds. Thus, for such heterogeneous products, if integration by parts does not work, nothing can.

We describe here some important classes of functions that are not easy to integrate:

- (1) *Product of a fractional or negative power and a trigonometric or exponential function:* The famous examples are $(\sin x)/x$ (the sinc function) and e^x/x (integrating this is equivalent, via a u -substitution, to integrating e^x , which arises in the Gompertz function, and is also equivalent to integrating $1/\ln x$, which is called the logarithmic integral).
- (2) *A composition $f \circ g$ where f is trigonometric or exponential and g is anything more complicated than a linear or constant function:* Examples are $\sin(x^2)$ (whose integral is termed the Fresnel integral) and e^{-x^2} (which arises in the normal distribution). Also, things like $\sin(\sin x)$.
- (3) *Composites of logarithmic and trigonometric functions, composites of inverse trigonometric and exponential functions:* This includes functions such as $\ln(\sin x)$ and $\arctan(e^x)$.
- (4) *Squareroots of polynomials of degree greater than two, and expressions involving them:* In most cases, these do not have antiderivatives in the elementary domain. However, it is possible to invent analogues of trigonometry to integrate some of these functions.

3.4. Equivalences between integrating functions that we cannot integrate. Even for the functions for which there is no elementary integral, it is useful to understand the rules of integration, because these allow us to relate the antiderivatives of different functions. For instance, we can use u -substitutions to move back and forth between integrating e^{e^x} , e^x/x and $1/\ln x$. A u -substitution allows us to go back and forth between integrating $\sin(e^x)$ and $(\sin x)/x$.

Integration by parts is also useful in relating different integrals to each other, even when none of them is elementary. For instance, we can try to integrate $x^2 e^{-x^2}$ using integration by parts, taking the first part

as x and the second part as xe^{-x^2} . This allows us to write an antiderivative for $x^2e^{-x^2}$ in terms of an antiderivative for e^{-x^2} .

4. THREE ADDITIONAL IDEAS/ELABORATIONS

These were alluded to in the previous parts of the lecture notes, and are covered in more detail here.

4.1. **A deeper understanding of integrating $e^xg(x)$.** We have the following fact:

$$f + f' = g \iff \int e^xg(x) dx = e^xf(x)$$

In other words, finding, for a given function g , a function f such that $f + f' = g$ is equivalent to solving $\int e^xg(x) dx$.

Both directions of implication are useful. Sometimes, it is easy to find the function f using the fact that $f + f' = g$. Sometimes, it is easier to use integration by parts to solve $\int e^xg(x) dx$.

We first consider the case where g is a polynomial. In this case, we guess (rightly) that f is also a polynomial. Since f' has smaller degree than f , f and $f + f' = g$ have the same degree and same leading coefficient. We thus know the degree and leading coefficient of f . We next need to find the other coefficients.

We give variable letter names to each of the coefficients of f and then compare coefficients to get a system of linear equations. This system of linear equations is particularly nice – it is *triangular*, so we can read off the solutions. In particular, we can compute the coefficients of f quite easily.

For instance, consider the case $g(x) = x^2 + x + 1$.

In that case, we see that $f(x) = x^2 + Ax + B$, where A, B are real numbers. Further, we have:

$$f(x) + f'(x) = (x^2 + Ax + B) + (2x + A) = x^2 + (A + 2)x + (B + A)$$

We thus have that, as polynomials:

$$x^2 + (A + 2)x + (B + A) \equiv x^2 + x + 1$$

If two polynomials are equal as polynomials, they are equal coefficient-wise, and we obtain:

$$A + 2 = 1, \quad B + A = 1$$

This solves to give $A = -1$, $B = 2$, and we get that $f(x) = x^2 - x + 2$.

We can also use integration by parts to obtain the same answer. Although integration by parts looks like a very different procedure, it is actually the same procedure in disguise.

Note for those who have seen matrices: Integration by parts can literally be thought of as multiplying by the “inverse matrix” to the matrix of the (triangular) linear operator that sends f to $f + f'$. There is some pretty mathematics going on here that is not completely accessible to you. (End note)

We see that in order to solve $f + f' = g$, our approach was:

- (1) First make a guess about the *general form* of f (in our case, we guessed that f is a polynomial).
- (2) Next, use the constraints to reduce the general form to a form with a finite number of unknown parameters (in our case, we noted that f has the same degree and leading coefficient as g , and the other coefficients were the unknown parameters).
- (3) Next, use the constraint $f + f' = g$ to obtain a system of (typically linear) equations in the unknown parameters. (In our case, the system was triangular).
- (4) Finally, solve that system of linear equations.
- (5) Check that the final solution you get is correct (we didn't need to do this in our case, but it is advisable in general).

We see that for things trickier than polynomials, step (1) is the most intimidating step, which is why this approach often flounders, and we decide to try integration by parts on $\int e^xg(x) dx$ instead.

On the other hand, we are sometimes able to make inspired guesses even outside the polynomial guesses. For instance, for the integral $\int e^x \cos x dx$, we need to find f such that $f + f' = \cos$. A nice guess is that $f = A \cos + B \sin$ for constants A and B . We thus get:

$$f'(x) = -A \sin x + B \cos x$$

and:

$$f(x) + f'(x) = (A + B) \cos x + (B - A) \sin x$$

Thus:

$$\cos x \equiv (A + B) \cos x + (B - A) \sin x$$

Comparing coefficients, we see that $A + B = 1$ and $B - A = 0$, yielding that $A = B = 1/2$. We thus get $f(x) = (\cos x + \sin x)/2$. This is the same as the answer we got using integration by parts.

4.2. Integration by parts and dealing with unsimplified stuff. Suppose that, after integration by parts, we get an equality of the form:

$$\int A(x) dx = P(x) + \int Q(x) dx$$

Suppose that we now need to evaluate this between limits a and b . The way we do this is:

$$\int_a^b A(x) dx = [P(x)]_a^b + \int_a^b Q(x) dx = P(b) - P(a) + \int_a^b Q(x) dx$$

In other words, for the parts that are still under the integral sign, we put the upper and lower limits on the integral sign. For the parts that have already been simplified, we do the evaluation between limits.

4.3. Repeated integration and integration of products with polynomials. We note the following equivalence:

We know how to integrate f twice \iff We know how to integrate f and $xf(x)$.

This is essentially a direct application of integration by parts. This is also related to the problem about Taylor series in advanced homework 2. For instance, if we start by integrating $xf(x)$ taking f as the second part, we see that this reduces to integrating f once and twice.

More generally:

We know how to integrate f n times \iff We know how to integrate $f, xf(x), x^2f(x), \dots, x^{n-1}f(x)$

We note two applications:

- (1) Using repeated integrability to deduce integrability of products with polynomials: Using the fact that \cos and \sin can be integrated repeatedly, we obtain that any polynomial times \cos or \sin can be integrated. On the other hand, since \tan cannot be integrated twice, we cannot integrate $x \tan x$.
- (2) Using integrability of products with polynomials to deduce repeated integrability: We begin with the "fact" (that we have not yet established) that every rational function can be integrated once. We want to argue that every rational function can be integrated any number of times. To do this, we note that in order to integrate a rational function f n times, we need to know how to integrate $f, xf(x), x^2f(x)$, and so on, up till $x^{n-1}f(x)$. But all these are again rational functions, so we can integrate all of them. Thus, a rational function can be repeatedly integrated.