MATHEMATICAL INDUCTION

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 1.8.

Difficulty level: Easy to moderate, depending on your past familiarity with induction.

What students should definitely get: To prove a statement using mathematical induction, it is important to prove the base case, and to show the induction step, which is the *conditional implication*. Also, the induction step needs to be shown for all natural numbers.

What students should hopefully get: The reason why mathematical induction is true, and the reason why encapsulating it as a principle allows us to show in finite time something that would otherwise take infinite time. Induction is most useful in problems where the statement for k is closely related to the statement for $k+1$. Even when we use induction, we have to prove a new statement for all positive integers (namely, the inductive step). The hope is that this new statement is easier than the old statement. The concept of induction for sufficiently large integers.

Executive summary

Words...

(1) Induction is a powerful tool that allows us to prove a statement for all positive integers (sometimes, for all positive integers $\geq$ some given positive integer) by proving it in just two special cases. These are the *base case* (proving it for the smallest positive integer in the set, usually 1) and the *induction step*. The induction step is a *conditional implication* that shows that if the statement is true for the positive integer k, then it is true for $k+1$.
(2) *Statement* here could be some equality or inequality depending on the positive integer. Usually, it is something like a sum of n terms or a product of n terms being equal to some nice polynomial or rational function in n. Sometimes, we have an inequality instead. There are other forms of statement too, such as divisibility statements, but we aren't dealing with them as of now.

Actions (try to recall problems on induction)...

(1) Proving the base case is straightforward, as long as you remember to do it.
(2) To prove the induction step, write what it means for the statement to be true for k, and write what it means for the statement to be true for $k+1$. Try to figure out a way to prove the *conditional implication*: assuming true for k, prove true for $k+1$.
(3) With summations, we usually start with the expression for k and add the $(k+1)^{th}$ term to both sides. Then, we do some algebraic manipulation and we're done. With products, we multiply instead of add.
(4) When dealing with inequality instead of equality, it is usually required to prove an *auxiliary inequality*. Basically, the right side that you get from the k assumption needs to be shown to be related to the right side you need to get for the $k+1$ conclusion.
(5) Sometimes, you may want to make an educated guess about what you should prove before proving it by induction. We saw some examples involving $(1-1/n)$ and $(1-1/n^2)$. These are all nice tricks, and this kind of cancellation of successive terms is called *telescoping*. But you will not be expected to guess what to prove – you'll be told. Proving it by induction is largely procedural

Caution ...

(1) Always clearly indicate that statements that you want to show and have not yet established are statements that you want to show.
(2) Please make sure that you show the base case correctly.

Frills ...

(1) Induction is a bit like differentiation/integration. Specifically, the inductive step is an analogue of the derivative, and the base case is an analogue of a specific value of the $+C$ that we see in indefinite integration.
(2) To prove the inductive step in an induction problem, we could try using induction again. This is analogous to differentiating/integrating twice.
(3) There is a concept of induction for sufficiently large integers, where we try to establish a statement only for natural numbers $n \geq n_0$. Both the base case and inductive step need to be suitably modified (the base case is n_0 and the inductive step can assume $k \geq n_0$).
(4) In some variants on induction, we show that $P(k)$ and $P(k-1)$ implies $P(k+1)$. If using such a variant, we need to make the base case correspondingly thicker, i.e., we need to show $P(1)$ and $P(2)$. In yet another variant of induction, we assume the truth of P for *all* smaller natural numbers.
(5) It is possible to induct on several parameters, either simultaneously or sequentially. This is a bit like differentiating a function of multiple variables in terms of each of the variables one by one.
(6) Induction can also be used to prove statements that are qualitatively different for different congruence classes modulo d. For such statements, we can either do the usual induction $k \rightsquigarrow k+1$, making cases based on congruence class, or a jump induction $k \rightsquigarrow k+d$, again making cases based on congruence class. In the latter case, we need to establish the first d natural numbers as base cases.

1. Basic ideas and motivation

1.1. **Basic idea: preview.** The main aim of proofs by induction is to show that some statement holds for all natural numbers. A *natural number* here means one of the numbers $1, 2, 3, \dots$. The set of natural numbers will be noted $\mathbb{N}$. Now, some people have the convention that they call zero a natural number. We don't follow that convention, and neither does the book, but this is just to warn you. By the way, a *natural number* is also called a *positive integer*. When I want to include zero as well, I'll use the term *nonnegative integer*.

The axiom of induction states that if S is a set of positive integers, with (A) $1 \in S$, and (B) $k \in S \implies k+1 \in S$, then S is the set of all positive integers.

In terms of properties: if (A) 1 satisfies a property, and (B) whenever k satisfies the property, so does $k+1$, then the property is satisfied by all natural numbers.

Now, this is a somewhat tricky statement so I just want to emphasize one thing before we delve into examples. In daily language, the word *induction* is typically used for some form of heuristic reasoning – it's supposed to be induction as opposed to deduction. So, for instance some of those "think-outside-the-box" type of people will tell you to use the creative, inductive, open-ended part of your brain as opposed to the formal, deductive, closed part of your brain. That kind of induction basically says something like: a statement is true for a few cases, so it's probably true for more. It's kind of a loose heuristic way of generalizing.

But that is *not* the concept of *mathematical* induction. This isn't to say that the loose heuristics of generalization aren't useful in mathematics. They really are, but they are useful only as guesses – they don't give clear proofs and they don't establish facts with certainty. *Mathematical induction*, on the other hand, *does* establish a fact with certainty.

1.2. **The way things work.** The book uses dominos to illustrate the notion of induction. And that is sort of how induction works. Think of all the positive integers as dominos lined up, with 1 on the front, 2 next, then 3, and so on. The condition (A) for induction says that you topple 1, and the condition (B) says that when k gets toppled, it also topples $k+1$. So the ripple effect goes on and eventually everything gets toppled.

Let's get away from domino language and take an example.

Suppose S is a set that contains 1 and, whenever k is in S, $k+1$ is in S. How do we check, for instance, that 5 is in S? Well, let's try. 1 is in S. And if k is in S, then $k+1$ is in S. So putting $k=1$, we have that 2 is in S.

And now putting $k=2$, we get that $2+1=3$ is in S. And putting $k=3$, we get that $3+1=4$ is in S. And putting $k=4$, we get that $4+1=5$ is in S.

So you see how this works. Now if I wanted to explicitly write a proof that 2009 is in S, that should take 2008 steps. That's a lot, but you see that it can be done. What the concept of mathematical induction does is to allow us to not have to go through all these steps for every number and just directly arrive at the conclusion that every positive integer is in S.

1.3. **A review of the idea of proof.** The concept of proof may seem scary, but it is basically just the idea of showing that something is true beyond doubt, by exhaustively covering all cases.

So suppose you have to prove that some statement is true for all numbers from 1 to 10. One way of proving the statement is to check the statement for every number from 1 to 10. If that statement looks very complicated, then you basically have to check a complicated statement ten times.

So that would be a proof, but we can sometimes prove statements without checking each and every case. For example if you wanted to prove that $(2a)^2 = 4a^2$ for all $a \in \{1, 2, 3, \ldots, 10\}$, one proof of the statement would check it for each number specifically. But you could also prove the statement simply by using algebra, which would in fact prove the statement not just for $a \in \{1, 2, 3, \ldots, 10\}$ but for *every* real number a.

Now, the *brute-force*, *check-every-case* approach, that may work when the number of cases is small, doesn't work when the number of cases is large. So, when proving statements for all natural numbers n, we have an infinite number of cases to check. On the other hand, a *direct algebraic attack* may not work for statements specific to natural numbers. So we need to do something smarter than check the statement for every natural number. Induction offers one such tool.

If P, then Q. I might discuss logical implications in proofs at some later stage, but for now, I'll state one very important fact.

To prove a statement of the form $P \implies Q$, also written as "if P then Q," here's what you do: you assume that P is true, and derive, or prove, from that that Q is true. Now, this part often confuses people. How do we know that P is true? Well, $P \implies Q$ is what is called a *conditional implication*. We are not stating that P is true, but we're saying that if we assume that P is true, then Q is true.

For instance, suppose P is the statement $x + y = 2$, and Q is the statement $x^2 + y^2 = 4 - 2xy$. Well, how do you prove Q using P. You prove it roughly as follows:

$$\begin{aligned} x + y &= 2 \\ \implies (x + y)^2 &= 2^2 \\ \implies x^2 + 2xy + y^2 &= 4 \\ \implies x^2 + y^2 &= 4 - 2xy \end{aligned}$$

So notice that P need not be true for every x and y, but if P is true, then Q is true as well.

1.4. **Understanding the two main components of proof by induction.** The part (A) in the proof, which is showing that 1 satisfies the condition, is typically called the *base case* or *base step* for induction or *basis for induction*. You really do need a base case for an induction because without the base case, things don't really get started. If something isn't even true for 1, how can it be true for *every* positive integer? Please remember this: *it is very important to show the base case.*

Next, part (B) of the proof. This is the part that gives the *conditional implication*. This is sometimes called the *induction step* or *inductive step*. It says that if a statement is true for a positive integer k, then it is also true for the positive integer $k + 1$. The important thing to remember is that the induction step has to be proved for *all* positive integers k. It isn't enough to prove it for small values of k. So ultimately, instead of proving the original statement for all positive integer k, we are trying to prove a conditional implication – the induction step – for all positive integers k.

So why should this be any simpler? Well, it isn't always simpler. But there are some problems, like the ones we will see, where the induction step is particularly simple to prove whereas proving the original statement directly requires some ingenuity. And it is those kinds of problems for which we use mathematical induction.

2. Induction to prove identities involving natural numbers

2.1. **Review of Example 1 from the book.** This is a classic example of the use of induction. We want to prove that, for all positive integers n, the following holds:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Before getting started on the statement, notice a few things. The expression on the right side is a polynomial, which, as such, makes sense for all real numbers. The expression on the left side, in contrast, is a *summation*, and the nature of the summation makes it clear that it makes sense only for positive integers.

Now, the kind of sum written on the left side can be a little confusing to some people, because some people may look at the expression and assume that n has to be at least 3. No, the left-hand side does not mean that. What it is code for is: *the sum of all the positive integers from* 1 *to* n. So when $n = 1$, the sum is 1, and when $n = 2$, the sum is $1 + 2$. When $n = 3$, the sum is $1 + 2 + 3$.

So let's begin the proof. We have to first establish the base case. The base case would say that we need to establish the result in the particular case that $n = 1$. So let's evaluate the left and right sides in the special case that $n = 1$.

When $n = 1$, the left side has just one term, namely 1. The right side is $\frac{1(1+1)}{2} = 1$. So, the left and right sides are equal. Thus, the statement is true for 1.

Now, there are a lot of 1s appearing in the above description, so it is worth emphasizing that what is important for the base case is not that the left side equals 1 or the right side equals 1, but that the left and the right side are equal.

So that's the base case. Now we come to the tricky part, namely the induction step.

For the induction step, what we need to assume is that if the statement is true for k, then the statement is true for $k + 1$. Now, you may ask: *how can we assume that the statement is true for k?* Well, just assume it for now. The point of the induction step is to show precisely that if a statement is true for k, then it is true for $k + 1$. [See the discussion in the earlier section about conditional implications].

So, we assume:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \tag{*}$$

And we want to prove:

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2} \tag{**}$$

Now, there are lots of ways of proving this, but it is important to understand what we did so far and what we need to do. What we did was assumed the statement is true for k, and that gives $(*)$, which is assumed as given. What we hope to do is show that the statement is true for $k+1$, and $(**)$ is just the mathematical formulation of that.

So, we need to derive $(**)$ from $(*)$. There are many different ways of presenting this. I'll choose one way.

Add $(k + 1)$ to both sides of $(*)$:

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
\implies 1 + 2 + \cdots + k + (k+1) &= (k+1)\left(\frac{k}{2} + 1\right) \\
\implies 1 + 2 + \cdots + k + (k+1) &= \frac{(k+1)(k+2)}{2} \\
\implies 1 + 2 + \cdots + k + (k+1) &= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}$$

And this proves $(**)$.

2.2. **Executing the induction step.** The base case usually isn't very tricky – the main difficulty that people usually have with the base case is that they forget to execute it. I very much hope you will not. The induction step, on the other hand, could be pretty tricky. So how do we execute the induction step? Also, how do we figure out that a problem needs or is helped by mathematical induction?

The main feature of problems that are helped by mathematical induction are problems where the statement for k is closely related to the statement for $k+1$. For instance, in Example 1, the statement for k was trying to compute $1+2+\cdots+k$, and the statement for $k+1$, was trying to compute $1+2+\cdots+(k+1)$. Now, computationally, if you've already added the first k positive integers, adding one more doesn't take much time. So the left sides here are pretty close. What about the right sides? They're pretty close too – a k gets replaced by a $(k+2)$.

So a problem involving cumulative summation/product of the first k terms of a sequence is a problem that would probably benefit from induction. But there are some problems where induction helps unexpectedly. These are problems where it is not immediately obvious that we should choose induction as the strategy – you may think at first there is some other method. But after a little while, you see induction as a possible strategy.

2.3. **Proof writing red flag.** It is *very important* that, when writing proofs by induction, and proofs in general, you clearly indicate any statement that you are trying to prove and that you have not already established. By default, when you write a sentence, or an equation, you are implicitly making an assertion that the statement *is already established to be true from what has been written so far*. Thus, for statements that you *want to show as true* but have *not yet shown as true*, please indicate clearly that the statement is still in the realm of desires rather than achievements.

In particular, it is *wrong* to prove the base case as follows:

$$\begin{aligned} 1 &= \frac{1(1+1)}{2} \\ 1 &= \frac{(1)(2)}{2} \\ 1 &= 1 \qquad \text{Proved} \end{aligned}$$

Rather, you should *work with the left and right sides separately*, get 1 on both sides, and then note that this settles the base case. Starting off the way shown above would be interpreted as *already assuming* the base case is true rather than *showing it*.

2.4. **Some additional caveats with proofs by induction.** Here are some additional things to keep in mind for proofs by induction:

(1) Sometimes, a problem may have multiple variables, some of which are real-valued, some are integer-valued, etc. For these problems, try inducting on a variable that takes values among the positive integers. Further, choose the variable to induct on in such a way that the induction step promises to work out well. (cf. previous subsection).
(2) A slight modification of the idea is to prove that a statement is true for all *sufficiently large* integers. For instance, we may want to prove that a statement holds for all integers $n \geq 4$. In this case, we show the induction step as usual, but for the base case, we take the base case $n = 4$. This is just induction, shifted over.
(3) Fancy induction techniques that we will see in a little while: Simultaneous induction on two parameters, induction on absolute values, induction after dividing into congruence classes.

2.5. **Strengths and weaknesses of induction.** The main weakness of mathematical induction is that it doesn't give a clear idea of how to formulate the statement in the first place. For instance, once you have a formula for the sum of the first n positive integers, you can verify it using induction. But how do you arrive at the formula in the first place?

As you can see, dreaming up complicated formulas is not best done through mathematical induction. So in some sense most of the examples we have done here are not examples of mathematical discovery but of *post facto* proof. However, there are cases where qualitative statements are easy to guess, and the easiest proofs are by induction.

Also, while inductive proofs are sometimes nice, they are not always the most insightful. Sometimes, a direct proof gives a much clearer idea of *why* a certain statement is true.

2.6. **Summation notation: a brief introduction.** Suppose we want to write:

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

The "..." (called the ellipsis or ellipses) in betwee in somewhat ambiguous. Since we're good mind readers, we know what is meant. However, it would be better to have a notation that allows us to compactify this while removing the ambiguity. More generally, for a function f defined on $\{1, 2, 3, \ldots, n\}$, we want a shorthand notation for:

$$f(1) + f(2) + \cdots + f(n)$$

The shorthand notation is:

$$\sum_{k=1}^{n} f(k)$$

Here, k is a *dummy variable* called the *index of summation*. The expression $k = 1$ written under the $\sum$ symbol tells us where we start k off. The n on top of the $\sum$ symbol tells us the *last* value of k that we use. The default increment is 1.

Similarly, the summation:

$$\sum_{k=5}^{8} 2^k$$

is shorthand for the summation:

$$2^5 + 2^6 + 2^7 + 2^8$$

The $k =$ is sometimes eliminated, when there is clearly only one dummy variable and there is no scope for confusion. So, we can write the above summation as:

$$\sum_{5}^{8} 2^k$$

We can also start the summation from 0; for instance:

$$\sum_{k=0}^{6} k^3$$

2.7. **More induction problems involving summation.** Let's consider the result:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(This is Exercise 5 of the book).

In the summation notation, the result would read:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case: For $n = 1$, the left-hand side is $1^2 = 1$, while the right-hand side is $1(2)(3)/6 = 1$. The left-hand side and right-hand side are equal, so the result is true for $n = 1$.

Induction step: Suppose the result is true for a given value of k. In other words, we have:

$$(*) \qquad 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k+1)^2$ to both sides, we get:

$$\begin{aligned}
1^2+2^2+\cdots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= (k+1)\left(\frac{k(2k+1)}{6}+(k+1)\right) \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= (k+1)\left(\frac{k(2k+1)+6(k+1)}{6}\right) \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= (k+1)\left(\frac{(k+2)(2k+3)}{6}\right) \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\implies 1^2+2^2+\cdots+k^2+(k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

Thus, we have proved the statement for $k+1$.

Here's another one. We'll try to prove that:

$$1+3+5+\cdots+(2n-1)=n^2$$

Thus is Exercise 4 from the book.

This is basically the sum of the first n terms where the k^{th} term is $2k-1$. In the summation notation, this is written as:

$$\sum_{k=1}^{n}(2k-1)=n^2$$

Base case: For $n=1$, both the left and right side are equal to 1.

Induction step: Suppose the result is true for k. In other words:

(*) $$1+3+5+\cdots+(2k-1)=k^2$$

We add $2k+1=2(k+1)-1$ to both sides:

$$\begin{aligned}
1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \\
\implies 1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= (k+1)^2
\end{aligned}$$

This proves that the result is true for $n=k+1$.

Note that for this example, we skipped the step of writing what we *need to show* at the outset of the induction step. If you are confident of your approach, then you can skip writing this at the beginning of the induction step. However, if you're not sure of how to tackle the problem, it is best to explicitly write the statement for $k+1$ that you hope to prove using the inductive step, since it helps set your sight on the goal.

2.8. **The product notation.** Similar to the summation notation, there is also a notation for products. You are unlikely to see this notation very often, and are not expected to know it, but it may be worthwhile seeing it at least once.

Suppose f is a function defined on $\{1,2,\ldots,n\}$. The *product* $f(1)f(2)\ldots f(n)$ can be expressed by the following notation:

$$\prod_{k=1}^{n} f(k)$$

As before, k is a *dummy variable*. Further, the limits of the dummy variable (in this case 1 and n) could be any pair of integers $a \leq b$, and the product would be interpreted as the product of $f(k)$ for all k between (and inclusive of) the two integers. For instance:

$$\prod_{k=5}^{8} \sin(k\pi/23)$$

is the product $\sin(5\pi/23)\sin(6\pi/23)\sin(7\pi/23)\sin(8\pi/23)$.

2.9. **A trickier example involving telescoping.** This is Exercise 13 from the book. In the product notation, it would be written as:

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \frac{1}{n}$$

We need to find a simplifying expression for:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{n}\right)$$

So, this is a little different from the previous problems in the sense that we first have to figure out what to prove, and then use induction to prove it. The main trick here is to note that:

$$1 - \frac{1}{k} = \frac{k-1}{k}$$

Thus, the original expression can be written as:

$$\frac{1}{2} \cdot \frac{2}{3} \dots \frac{n-1}{n}$$

Now, notice what happens. The 2 in the denominator of the first fraction cancels the 2 in the numerator of the second fraction. The 3 in the denominator of the second fraction cancels the 3 in the numerator of the third fraction. And so on. So what finally gets left is a 1 in the numerator and a n in the denominator. So what we expect to get is $1/n$. This kind of cancellation is called *telescoping*. More specifically, this is *multiplicative telescoping* – there is a similar notion of *additive telescoping*. We will see telescoping in more detail in the distant future.

So let's now prove by induction the claim that:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Note that in this case, the smallest n for which we can talk about this result is $n = 2$. So, the base case for induction is the case $n = 2$ rather than $n = 1$.

Base case: The base case is $n = 2$, and in this case, the left side is $1 - (1/2) = 1/2$, which is the right side.

Induction step: Suppose the result is true for k:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{k}\right) = \frac{1}{k}$$

We multiply both sides by $1 - 1/(k+1)$:

$$\begin{aligned}
\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) &= \frac{1}{k}\left(1 - \frac{1}{k+1}\right) \\
\implies \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) &= \frac{1}{k} \cdot \frac{k}{k+1} \\
\implies \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) &= \frac{1}{k+1}
\end{aligned}$$

This shows that the result is true for $k+1$.

We next look at a trickier example:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right)$$

Let's first try to simplify by taking the common denominator in each term. We have:

$$\left(\frac{2^2 - 1}{2^2}\right)\left(\frac{3^2 - 1}{3^2}\right)\dots\left(\frac{n^2 - 1}{n^2}\right)$$

Now, there isn't any cancellation of the kind that we saw earlier, so what we need to do is a little trick. We write $k^2 - 1 = (k-1)(k+1)$, and we get:

$$\left(\frac{(2-1)(2+1)}{2^2}\right)\left(\frac{(3-1)(3+1)}{3^2}\right)\dots\left(\frac{(n-1)(n+1)}{n^2}\right)$$

Now, we see cancellations, and in fact, two kinds of cancellation. One kind of cancellation happens because the $k-1$ in the numerator of one fraction cancels with the $k-1$ in the denominator of the fraction to its left. The other kind of cancellation happens between the $k+1$ in the numerator and the $k+1$ in the denominator of the fraction to the right.

What finally survives the carnage is:

$$\frac{(2-1)(n+1)}{(2)(n)} = \frac{n+1}{2n}$$

This example also demonstrates why induction is useful, because when you do all this cancellation, you may be a little unsure of whether there is something you are not keeping track of well. So the answer we have got, namely $(n+1)/(2n)$, may not be correct. To confirm it, we should use induction.

Base case for induction: For $n = 2$, the left-hand side is $1-(1/2^2) = 3/4$ and the right side is $(2+1)/(2\cdot 2) = 3/4$. Thus, the left-hand side and right-hand side are equal for $n = 2$, so the statement holds for $n = 2$.

Induction step: Suppose the statement holds for k. In other words:

$$(*) \qquad \left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

We want to prove the statement for $k+1$. In other words, we want to prove:

$$(**) \qquad \left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$$

Multiplying both sides of (*) by $1 - (1/(k+1)^2)$, we obtain:

$$\begin{aligned}
&\left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &=& \frac{k+1}{2k} \cdot \left(1 - \frac{1}{(k+1)^2}\right) \\
\implies &\left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &=& \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2} \\
\implies &\left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &=& \frac{k+1}{2k} \cdot \frac{k(k+2)}{(k+1)^2} \\
\implies &\left(1 - \frac{1}{2^2}\right)\dots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &=& \frac{k+2}{2(k+1)}
\end{aligned}$$

The last step gives (**), completing the proof.

If you weren't able to guess the correct answer, you may have guessed something like $1/n^2$ (pattern-matching with the previous example) and then tried to prove it by induction. And when trying to do such a proof, you may have run into trouble. *That's a good thing*, because it shows that you are not able to prove something wrong by induction.

Incidentally, when we have a nice rational function or exponential function or that kind of compact expression for a sum or product, that is called a *closed-form expression*. Most summations and products do not have closed-form expressions. Some of those that do (and that you may have seen in high school) are arithmetic progressions and geometric progressions. Generally, proving closed-form expressions can be done using induction, but obtaining them requires some deeper thinking.

For example, the product:

$$\left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{3^3}\right)\dots\left(1 - \frac{1}{n^3}\right)$$

does *not* have a closed-form expression. In other words, we cannot write it down using a simple formula, let alone prove the formula by induction.

3. Induction and inequalities

3.1. **Induction and inequalities: why we need them.** So far, we have looked at the principle of mathematical induction and its applicability to proving *equalities*, or *identities* that hold for all positive integers n, or for all large enough positive integers n. We looked at a range of examples:

$$\begin{aligned}
1 + 2 + \dots + n &= \frac{n(n+1)}{2} \text{ (Example 1 of book)} \\
1 + 3 + \dots + (2n-1) &= n^2 \text{ (Exercise 4 of book)} \\
1^2 + 2^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \text{ (Exercise 5 of book)} \\
1^3 + 2^3 + \dots + n^3 &= (1 + 2 + \dots + n)^2 \text{ (Exercise 6 of book)} \\
\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\dots\left(1 - \frac{1}{n}\right) &= \frac{1}{n} \text{ (Exercise 13 of book)} \\
\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right) &= \frac{n+1}{2n} \text{ (Exercise 14 of book)}
\end{aligned}$$

In all these cases, we were first lucky enough to come across a good closed form expression (a polynomial in the first four cases, and a rational function in the last two) which we were then easily able to establish by induction. However, in the vast majority of cases, when we are trying to determine the sums or products of series, closed-form expressions, such as polynomials or rational functions, do not exist. In other words, a simple formula doesn't even exist, so it is hopeless to try and prove it by induction.

In these situations, we try the next best thing: get a handle on what the expression can be, by bounding it from above and below. For instance, suppose you are looking at:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

There is no rational function or nice closed-form expression for this summation, but it would at least be helpful if we could get some easy-to-understand function that bounds this from below and an easy-to-understand function that bounds this from above. That would allow us to *understand the qualitative nature* of the function without having an actual expression for it.

This brings us to the realm of inequalities and the proof of inequalities by induction. At some more advanced stage of mathematics, people are expected not just to prove the inequalities by induction but also to come up with likely candidates for what should be proved by induction. But as of now, all you're expected to do is prove by induction an inequality that is already stated explicitly.

3.2. **Inequalities and auxiliary inequalities.** The way we try to prove inequalities is a *little* different from the way we try to prove equalities when using proof by induction.

When proving equality, we execute the induction step as follows: we write the statement for k, we write the statement for $k+1$, and then we determine how we can manipulate the existing statement for k (typically by adding or multiplying something on both sides) in order to obtain the statement for $k+1$. Since we are working with equality throughout, most of the manipulation work consists in (i) figuring out what to add/multiply, which is usually just the $(k+1)^{th}$ term of the summation/product expression that forms the left side, and (ii) simplifying some algebraic expression using rules learned in middle school.

With inequalities, things are a little trickier. Again, we start with the inequality for k and we write down the inequality for $k+1$ that we need to prove. But now, even after we add stuff and manipulate, we need not get precisely the expression we have with $k+1$.

In this case, what we try to do is *guess an auxiliary inequality* such that, if that inequality were true, that would complete the problem. Then we prove that auxiliary inequality.

The typical situation is that we have:

(*) Long expression (summation, product) involving first k terms $>$ Short expression in k

And we want to prove:

(**) Long expression (summation, product) involving first $k+1$ terms $>$ Short expression in $k+1$

What we do is add/multiply the $(k+1)^{th}$ term on both sides of (*), and we obtain:

(***) Long expression (summation, product) involving first $k+1$ terms $>$ Some new expression

We now try to show that:

(****) Some new expression $\geq$ Short expression in $k+1$

Because once we show that, then combining (***) and (****) gives (**).

How do we show (****)? This basically boils down to the old tricks of proving inequalities.

Detailed discussion of example 2 from the book. Example 2 from the book (Page 50) asks you to prove that, if $x \geq -1$, then, for all positive integers n, we have:

$$(1+x)^n \geq 1 + nx$$

This problem is interesting for many reasons. First, notice that there are two variables, x and n, in the problem. We need to choose which variable to do the induction on. For this, notice that the variable x takes arbitrary real values, while n takes values in the positive integers. Hence, it makes sense to induct on n.

Second, this is an example of an inequality problem that is helped through proof by induction.

We handle the base case and the induction step.

Base case: Consider the case $n = 1$. In this case, the left-hand side is $(1+x)^1 = 1 + x$ and the right-hand side is $1 + 1 \cdot x = 1 + x$. Thus, the left-hand side and the right-hand side are equal, so the base case has been proved.

Induction step: Let's assume the result for $n = k$. This says:

(*) $$(1+x)^k \geq 1 + kx$$

What we need to prove is the result for $n = k+1$. In other words, we need to prove:

(**) $$(1+x)^{k+1} \geq 1 + (k+1)x$$

We need to derive (**) from (*). We begin by multiplying both sides of (*) by $1+x$, to get:

(***) $$(1+x)^{k+1} \geq (1+kx)(1+x)$$

Note that this is valid because, since $x \geq -1$, $1 + x \geq 0$ and hence multiplying by $1+x$ does not change the sign of the inequality.

Thus, what we need to prove is:

$$(1+kx)(1+x) \geq 1 + (k+1)x$$

Let's do this. Note that:

$$(1 + kx)(1 + x) = 1 + (k + 1)x + x^2$$

Since $x^2 \geq 0$, we get:

(****) $$(1 + kx)(1 + x) \geq 1 + (k + 1)x$$

Combining (***) and (****), we obtain (**), as desired, and this completes the proof of the induction step.

3.3. **Confusing: marching forwards and bending over backwards.** These applications of induction are a little confusing because they involve a combination of working forwards and working backwards and jumping between them. So here's a little philosophical explanation of what is happening.

In general, for most of the problems you've done in mathematics, you start with whatever you have and then work, step by step, to where you want to go. This is like you know you're at one end of the street, and you want to get to this coffee shop which is located at the other end of the street, and you need to walk from where you are to where the coffee shop is.

But sometimes, you are not sure where the coffee shop is. So you call them up, or may be Google them, and you get to know that it is next to the gas station. So now, instead of looking for the coffee shop, you can look for a gas station, that might be a little more prominent and hence harder to miss. And that might be within sight, or may be it will be if you walk a little bit. And so this backward-forward thing is what we're doing with the induction inequality business as well. We start with what we have (the statement for k), where we need to get (the statement for $k + 1$), then we try to walk a little bit forward from where we are and a little bit backward from where we need to be until where we are matches where we need to be.

When doing this backward-forward thing, it is very important that for all statements that you want to show but have not yet shown, you clearly indicate this in words. By default, any statement you make comes with an implicit assertion that the statement has already been established from previous assertions.

3.4. **Discussion of Exercise 9 from the book (homework problem).** This exercises asks you to prove that, for n a positive integer with $n \geq 2$:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots \frac{1}{\sqrt{n}} > \sqrt{n}$$

We consider the base case and induction step.

Base case: The base case is $n = 2$. In this case, the left side is $1 + 1/\sqrt{2}$ and the right side is $\sqrt{2}$. Thus, to show the base case, we need to show that $1 + 1/\sqrt{2} - \sqrt{2} > 0$. This can be shown as follows: $1 + 1/\sqrt{2} - \sqrt{2} = (\sqrt{2} - 1)/\sqrt{2}$. Since $\sqrt{2} > 1$, both the numerator and denominator are positive, so the expression is positive. This completes the proof for the base case.

Induction step: Suppose the result is true for k. In other words, we have:

(*) $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}$$

We need to prove:

(**) $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

How do we get from (*) to (**)? This is the challenge. The first step to do might be to get the left sides to match. This can be done by adding $1/\sqrt{k+1}$ to both sides of (*), giving:

(***) $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

In order to get from (***) to (**), we should try showing that:

(****) $$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

So, we have *reduced the original problem* to proving (****). In other words, if we somehow manage to prove (****), then we would have completed the proof of the induction step. To prove this, try to prove that the difference of the left-hand side and the right-hand side is positive for $k \geq 2$.

3.5. **Detailed discussion of an example not in the book or exercises.** Suppose you are given that the following holds for all $x > 0$

(A) $$x > \ln(1+x)$$

$e = 2.718\ldots.$

You want to show that for positive integers n:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$$

Let's see how we can do this. First, the base case.

Base case: Set $n = 1$. The left side is 1 while the right side is $\ln(2)$. Since $2 < e$, $\ln(2) < 1$ so the left side is greater than the right side. This completes the proof of the base case.

Induction step: Suppose the statement is true for k. In other words, we have that:

(*) $$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} > \ln(k+1)$$

We want to prove that the statement is true for $k+1$. In other words, we want to prove that:

(**) $$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} > \ln(k+2)$$

Let's try to prove this. We try the usual recipe: we add stuff to (*), and see how things pan out.

Adding $1/(k+1)$ to both sides of (*), we obtain:

(***) $$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} > \ln(k+1) + \frac{1}{k+1}$$

Let's look at (***) and pause, because we need to remember what exactly we are trying to prove. We are trying to prove (**), whose left side looks exactly the same as that of (*), but whose right side looks a little different. The right side of (**) is $\ln(k+2)$, while the right side of (***) is $\ln(k+1) + 1/(k+1)$.

Thus, what we would like to prove is:

$$\ln(k+1) + \frac{1}{k+1} > \ln(k+2)$$

We need to use (A), which says that for any $x > 0$, we have $x > \ln(1+x)$.

What x should we pick? A reasonable guess here would be $x = 1/(k+1)$. Let us see what happens when we put $x = 1/(k+1)$ in (A). We get:

$$\begin{aligned} \frac{1}{k+1} &> \ln\left(1 + \frac{1}{k+1}\right) \\ \implies \frac{1}{k+1} &> \ln\left(\frac{k+2}{k+1}\right) \\ \implies \frac{1}{k+1} &> \ln(k+2) - \ln(k+1) \end{aligned}$$

Rearranging this, we obtain:

$$\text{(****)} \qquad \ln(k+1) + \frac{1}{k+1} > \ln(k+2)$$

Combining (***) and (****), we obtain (**), as desired. This completes the induction step.

4. Fancy forms and interpretations of induction

These techniques are mentioned briefly for completeness. You will not get questions in the test that rely on such techniques (and if you do, you will be provided a sufficiently detailed solution template that you can fill in the details based on your basic knowledge of induction. Thus, the discussion here is kept at a qualitative level. The main purpose is to prepare you conceptually for encountering these ideas at a later stage.

4.1. **Induction is conceptually like differentiation/integration.**

This seems preposterous at first but it's true in the following analogical sense: the inductive step is a lot like the "derivative" of the statement that we are trying to prove by induction. Checking the induction step is a bit like checking that the "derivatives" match, and the base case of induction serves as an analogue of the "+C" that we see in integration.

The conceptual analogy becomes more precise in specific situations – for instance, when dealing with summations $\sum_{k=1}^{n} f(k)$, the operation of summation is a *discrete* version of integration, and the inductive step, which involves considering the value $f(k+1)$ added, is basically taking the integrand, which is the derivative of the sum (fundamental theorem of calculus of sorts). We will flesh out this analogy in more detail when we move to the topic of series.

One conceptual corollary is as follows: in cases where the inductive step is itself hard to prove, we might benefit by trying to prove the inductive step *itself* by induction, which is analogous to differentiating twice.

4.2. **Induction for sufficiently large integers.**

Sometimes we want to prove that a statement is true for all natural numbers $n \geq n_0$, i.e., it is true for all *sufficiently large integers*. The approach in this case is as follows:

- The base case now becomes n_0.
- The inductive step $P(k) \implies P(k+1)$ needs to be shown only for $k \geq n_0$. In some cases, we may not need the additional assmuption $k \geq n_0$, while in others, this additional assumption may be critical to executing the induction step.

4.3. **Induction where a statement is assumed for all smaller values.**

In all the cases we have seen so far, the inductive step was of the form $P(k)$ implies $P(k+1)$. Often, however, for the statement as originally formulated, $P(k)$ is not strong enough to easily give $P(k+1)$. We may need to use not just $P(k)$ but $P(k-1)$ and the truth of P for other smaller values. In other words, this modified variant of the induction step uses the truth of P for all smaller values to deduce its truth for $k+1$.

In cases where we need the truth of $P(k)$ and $P(k-1)$ to deduce the truth of $P(k+1)$, the *base case* needs to be correspondingly thicker: it needs to cover $P(1)$ and $P(2)$, because the inductive step can only kick in from 3 onward.

Pushing the analogy with differentiation further, this is a but like differentating/integrating twice – there are *two* arbitrary constants that arise when integrating twice, and pinning them down requires checking/verifying two conditions. We will return to this analogy and its precise meaning after a dose of differential equations and while covering series.

4.4. **Induction on multiple parameters: simultaneous and separate.**

Sometimes, a statement may involve more than one parameter that is a natural number, and to prove the statement, we may need to apply induction separately to both parameters.

For instance, suppose $P(n, m)$ is a statement that takes in two natural number parameters n and m. We could try this inductively as follows:

- We first assume m fixed, and we try proving the statement by induction on n. This involves proving a base case of $P(1, m)$ for all m (let's call this statement $B(m)$) and an inductive step which says that $P(k, m)$ implies $P(k+1, m)$ (let's call this $Q(k, m)$).

- In some cases, both $B(m)$ and $Q(k, m)$ are easy to show directly. In others, proving these may itself require induction – this time on m.

Using the analogy between induction and differentiation/integration, we first "differentiate" P with respect to n, and then try "differentiating" with respect to m. Perhaps we may need to "differentiate" with respect to n yet again.

These is another approach to induction for statements involving more than one variable, which is an approach of *simultaneous* induction. The idea here is, roughly, to prove $P(k+1, l+1)$ assuming that *both* $P(k+1, l)$ and $P(k, l+1)$ are true. This has a nice graphical interpretation if we represent pairs of natural numbers as a lattice. [Explanation to be delivered in class, if there is time, otherwise ignore it].

4.5. **Induction on congruence classes.** Often, statements involving the natural numbers split into cases based on the congruence class modulo some small number like 2, 3, or 4.

The congruence class of a natural number n mod d is characterized the remainder obtained on dividing n by d. For instance, when $d = 2$, there are two congruence classes: the even numbers, which leave a remainder of 0, and the odd numbers, which leave a remainder of 1. The trait with values even and odd is termed "parity." We are familiar by now with the way many aspects of behavior depend on parity – for instance, many aspects of the asymptotic behavior of a polynomial or rational function depend upon its parity.

In a similar fashion, there many be cases where a result depends on the congruence class of n mod 3, or perhaps mod 4. Note that in each congruence class mod d, successive members are a distance d apart.

There are two possible approaches to such problems using induction. The first is to use induction the usual way, but be wary that adjacent numbers are in different congruence classes. The second is to use induction by "jumps" of d. We use the even/odd situation to illustrate.

Suppose I want to prove a statement $P(n)$ for all natural numbers n, but P has different flavors for even n and odd n, which we call $P_e(n)$ and $P_o(n)$ respectively. We consider both possible strategies:

Strategy one: We prove $P(1)$ (the base case) and show that $P(k)$ implies $P(k+1)$ for all natural numbers (the inductive step). For the inductive step, we split into cases: when k is even, we show that $P_e(k)$ implies $P_o(k+1)$, and when k is odd, we show that $P_o(k)$ implies $P_e(k+1)$.

Strategy two: We prove $P(1)$ and $P(2)$ (the base cases) and show that $P(k)$ implies $P(k+2)$ for all natural numbers k (the inductive step). For the inductive step, we split int ocases: when k is even, we show that $P_e(k)$ implies $P_e(k+2)$, and when k is odd, we show that $P_o(k)$ implies $P_o(k+2)$.