EXPONENTIAL GROWTH AND DECAY

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 7.6.

What students should definitely get: The definition of exponential growth and exponential decay, how this is used.

What students should hopefully get: The intuition of exponential growth and decay, how it is used in trend prediction, the discrete versus continuous aspects of exponential growth and decay.

Note to newcomers

The lecture notes cover the material that was intended for the lecture, but may not correspond precisely with what is covered during the lecture. In some cases, the lecture notes may have more details about an example that was sketched quickly during lecture due to time constraints.

Each lecture notes file begins with an "executive summary" which is a useful way of reviewing the lecture *after* you have already understood the material, and is not designed for people learning the material the first time.

Executive summary

0.1. **Basics of exponential growth and decay.** Words ...

(1) A function f is said to have exponential growth if $f'(t) = kf(t)$ for all t. Such a function must be of the form $f(t) := Ce^{kt}$. Here, $C = f(0)$, and can be thought of as the initial value. k is a parameter controlling the *rate of growth*. When $k > 0$, we have growth, and when $k < 0$, we have *decay*. When $k = 0$, there's no growth or decay.
(2) For a function with exponential growth and growth rate k, the time taken for the function value to multiply by a number $q > 0$ depends only on q and k. Specifically, the time interval is $(\ln q)/k$. In particular, for growth, the doubling time is $(\ln 2)/k$. Note that if $\ln q$ and k have opposite signs, the time taken is negative – which means that we need to go *back* in time to multiply by a factor of q.
(3) If a function takes a time interval t_{d1} to multiply by a factor of q_1 and a time interval t_{d2} to multiply by a factor of q_2, we have the relation: $\ln(q_1)/t_{d1} = \ln(q_2)/t_{d2}$. Thus, given three of these quantities, we can calculate the fourth.
(4) Exponential functions grow faster than all positive power functions (and hence all polynomial functions) while logarithmic functions grow slower than all positive power functions.
(5) Exponential functions *decay* slower than linear functions. In other words, the time taken for the "first" $1/n$ of a material to decay is less than the time taken for the "second" $1/n$ to decay, and so on.

Actions ...

(1) Suppose we know that f is a function of the form $f(t) := Ce^{kt}$, but we do not know the values of the constants C and k. One way of determining these values is to determine the value of f and f' at some point t_0. We can then get k as $f'(t_0)/f(t_0)$ and then solve to get $C = f(t_0)/e^{kt_0}$. This type of specification is termed an *initial value specification* and a problem with such a specification is termed an *initial value problem.*
(2) Another way we can determine C and k is if we are given the value of f at two points t_1 and t_2. In this case, we solve to obtain that $k = \frac{1}{t_2 - t_1} \ln[f(t_2)/f(t_1)]$, and we can plug back to get $C = f(t_1)/e^{kt_1}$. Note that this is a reformulation of the formula that the time taken to multiply by a factor of q is $(\ln(q))/k$. This kind of specification is somewhat related to the notion of *boundary value specification.*

(3) But in many real-world situations, we do not need to actually determine the constants C and k. Rather, we use the fact that $\ln(q_1)/t_{d1} = \ln(q_2)/t_{d2}$ to compare the rates of growth in two intervals.
(4) Even nicer, in many cases, we do not need to know the actual values of $\ln(q_1)$ and $\ln(q_2)$, because the only thing that matters is their *quotient* $\ln(q_2)/\ln(q_1)$, which can also be viewed as the relative logarithm $\log_{q_1}(q_2)$. Thus, for instance, if q_2 is a rational power of q_1, we know the quotient precisely even though we may not know $\ln(q_1)$ and $\ln(q_2)$. For instance, $\ln(8)/\ln(4) = 3/2$.

0.2. **Compound interest.**

(1) Compound interest: This is written as $A(t) = A_0 e^{rt}$ where r is the *continuously compounded interest rate*, A_0 is the initial principal or initial amount, and $A(t)$ is the amount at time t. The corresponding differential equation is $A'(t) = rA(t)$. Continuously compounded interest differs from simple interest, where $A(t) = A_0(1+rt)$ and from discretely compounded interest, where the interest earned is added to the principal at periodic intervals.
(2) The time taken for the amount to double under continuously compounded interest with rate r is $(\ln 2)/r$, which is approximately $0.7/r$. When r is expressed as a percentage, we need to divide 70 by that percentage to get the doubling time. (Times here are typically measured in years). This is called the *rule of 70*. *Note*: The rule of 70 also applies to discretely compounded interest rates when r is very small, but that is a topic for next quarter.

0.3. **Radioactive decay.**

(1) A radioactive material undergoes *decay*, i.e., its quantity goes *down* with time. The constant k in the expression Ce^{kt} is thus a negative number.
(2) The fraction that remains is 1 minus the fraction that decays. Thus, if 1/3 of the material decays, then the relevant q to plug into formulas is $q = 2/3$, *not* 1/3.
(3) The rate of decay of radioactive materials is typically measured by their half-life, which is the time taken for half the material to decay and half to remain. (1/2 is the only number that is equal to 1 minus itself). We have the formula $k = (-\ln 2)/\text{half-life}$.

1. Discrete exponential growth

1.1. **Multiplication by division.** Suppose you know that a given unicellular organism divides into two organisms precisely one hour after its "birth". This kind of reproduction is called *binary fission*, and is observed, for instance, in the amoeba. Suppose that, at time 0, God (or Craig Venter!) creates N such unicellular organisms. Then, one hour down the line, there will be $2N$ unicellular organisms (because each one splits into two at that time). One more hour down the line, there will be $4N$ organisms. More generally, k hours from the time God/Venter created the first bunch of organisms, the number of organisms will be $2^k N$.

This is a kind of exponential growth, and it is an example of *discrete* exponential growth. Incidentally, we see here that the base of the exponent is 2, because the discrete action is a division into two organisms. Similarly, if instead we had *ternary fission*, where an organism splits into three distinct organisms, the natural base of the exponent would be 3.

As a more complicated example, suppose, in a society, every pair of parents chooses to have three children and then dies. Then, with every passage from one generation to the next, the population increases by a factor of 3/2. Thus, after k generations, the new total population is the original population times a factor of $(3/2)^k$. [1]

The hallmark of such discrete processes, we see, is exponentiation, where the base of exponentiation is a positive integer or, at worst, a rational number (usually something simple such as 2, 3, or 3/2), and the exponent is also a positive integer.

[1]We are assuming here that in each generation, there is an exact balance between males and females, that they all get paired into couples, and that they are all able to fulfil their desired fertility, and that there are no such complications as divorces, out-of-wedlock children, etc.

1.2. **Approximating by continuous processes.** Even though a given unicellular organism may split in two after a precise one hour, a huge collection of unicellular organisms may be increasing continuously in population. Why? Because their births may not all have been synchronized. Similarly, when dealing with human populations, even though no single woman continuously gives birth to children, births are happening across the world to women. There are multiple issues:

(1) Lack of synchronizations of starting times.
(2) Tempo and quantity difference: Some amoeba may tend to split slightly faster than others. Some women may choose to have kids earlier than others, and some may choose to have more.
(3) Other factors: Some amoeba may burst and die and hence not be able to undergo binary fission. Some women may die before giving birth to children, or may choose not to have children, or may be unable to have children.

When all these factors are taken into account, it may turn out that using discrete rates and natural numbers creates a model that is too complicated for suitable analyses. A simple *continuous model* may thus be more tractable. Note that *the use of continuous models to track human or amoeba populations is only an approximation, and a shaky one*. It works only when we are working with large and uniform samples. But before discussing the drawbacks of continuous approximations, let us appreciate their virtue.

2. Continuous growth: an introduction

2.1. **Proportional growth and exponential growth.** Recall that if $f(t) := Ce^{kt}$, then $f'(t) = Cke^{kt} = kf(t)$. In other words, the derivative of an exponential function is the coefficient in the exponent times the function itself. The fascinating thing is that the *converse also holds*, i.e., the only solutions to the functional equation $f'(t) = kf(t)$ are function of the form $f(t) = Ce^{kt}$. For this lecture, we use the letter t for the domain variable because we are studying situations where the input parameter is a time parameter.

This is an example of a *linear differential equation* (in fact, it's a very special case). We will skip over the proof for now. It is there in the book. In any case, we will return to this idea later in greater generality when covering linear differential equations.

The value C can be determined as the value $f(0)$. In other words, given the values $f(0)$ and the factor of proportionality k, the function f is uniquely determined. Specifying f in such a way corresponds to an *initial value problem* – i.e., a problem where a law governing the derivative is combined with the value of the function at one point.

2.2. **Proportional growth laws.** We now explore the possible justifications for proportional growth laws, i.e., laws where the growth rate is proportional to the current quantity. One justification was encountered from the discrete situation of reproduction rates: reproduction with a fixed number of offspring after a fixed time period gives discrete exponential laws, but various averaging effects may allow us to approximate them by continuous proportional growth laws. In other words, it may be reasonable to assume, at least in the short run, that if P denotes the population function, then P is of the form:

$$P'(t) = kP(t)$$

which gives:

$$P(t) := P(0)e^{kt}$$

where k is an exogenous parameter[2] that measures the ratio of the instantaneous rate of change in the population to the total population. Note that the assumption here is that k itself does not change with time. Is the assumption reasonable?

For a colony of bacteria or amoebae growing in a large and sustaining environment with no binding resource constraints, there may be so few premature deaths and so little fighting for resources that the growth rate remains constant even as the population grows. However, once the population becomes very large, and the available resources in the region start getting strained, the growth rate may fall – or even become negative. More generally, the growth rate may be constant or near constant for a long time and then start a decline.

[2] "exogenous" is jargon for a parameter whose value is determined by things that are outside of the current model

In the case of humans, the greater degree of human agency and the existence of complex societal influences complicate matters. For instance, the two main factors that affect world population growth rates are *birth rates* and *death rates*. Within a subregion of the world that is not closed to the outside world, *migration rates* also play a role. Also, birth rates are determined by such factors as *desired fertility for females*, *proportion of the population comprising females of birth-giving ages*, while death rates are also affected by population composition, access to nutrition, sanitation, and health care, incidence of wars and tribal in-fighting, and natural disasters. All the aforementioned factors exhibit not only random fluctuations but systemic changes from one generation to the next. To take an example, until the nineteenth century, European nations lost significant fractions of their male populations in wars – and these fractions varied from generation to generation in each country depending on the war/peace situation. So the idea of a constant k is nothing more than a polite fiction.

Nonetheless, one idea remains important: whatever role exogeneous parameters play is played out through their effect on k, and the effect is a proportional effect. If country A and country B are similar in all relevant demographic characteristics (hence have the same k) except that country A has twice the population of country B, the rate of population growth in country A will be twice that of country B.

Exponential growth is also sometimes called *geometric growth* because if you measure population at discrete intervals and look at the time series, you get a *geometric progression*. Geometric or exponential growth is faster than *linear growth* or *arithmetic growth*, where the rate of growth is constant and does not depend on the current level. This is basically the fact that e^x eventually grows faster and way way faster than any linear function.

Malthus was worried about the idea that population grows geometrically while resource supplies will grow at a linear rate (what he called *arithmetic growth*) ultimately leading to a severely resource-constrained world with mass starvation and the end of human civlization as we know it. Although Malthus's pessimistic predictions did not come to bear in the human realm, the interplay between the natural tendency for exponential growth and the finiteness of various kinds of resource constraints is a theme that recurs in understanding many ecological and biological phenomena. We may return to some of these topics at a later stage.

3. More arithmetic and computation with exponential growth

3.1. **Using two observations to determine growth rates.** Suppose we have strong reason to believe that a particular growth is exponential (based on our theoretical model for how such growth occurs) but we do not have any theoretical way of determining the *constant of proportionality k*. Then, what we do is to use *two observation points*. Basically, we know that the function f is of the form:

$$f(t) := Ce^{kt}$$

where both C and k are unknown constants. Given the value of f at two points, we can find both C and k. Specifically, if $f(t_1) = a_1$ and $f(t_2) = a_2$, then:

$$\frac{f(t_2)}{f(t_1)} = e^{k(t_2 - t_1)}$$

Taking logarithms both sides, we obtain:

$$k = \frac{1}{t_2 - t_1} \ln \left[\frac{f(t_2)}{f(t_1)}\right]$$

Once we know k we can determine C.

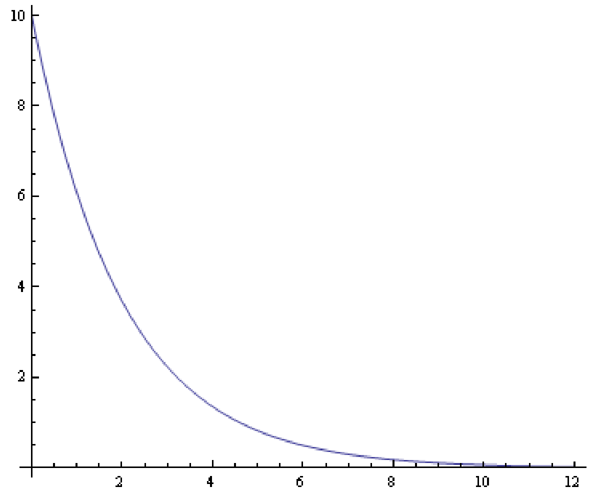
Sometimes, there is no known reference point from which to measure times. In this case, we can simply pick the first observation as the time point $t = 0$ and measure times forward from there.

3.2. **Time taken to multiply by a fixed proportion.** For a growth function $f(t) := Ce^{kt}$, the time taken to multiply by a factor of q is given by:

$$\frac{\ln(q)}{k}$$

This is just a reformulation of the result derived previously.

In particular, the time taken to multiply by a fixed proportion is independent of the original value. Thus, if the quantity doubles from time $t = 1$ to time $t = 6$, it also doubles from time $t = 13$ to $t = 18$. What matters is the *length of time interval*.



3.3. **Exponential decay.**

Mathematically, exponential decay is just like exponential growth. The main difference is that in the case of growth, the constant of proportionality k is positive, so the function grows. With decay, the constant of proportionality is negative. We now call k the *decay constant* and it is negative. A function undergoing exponential decay has the t-axis (what we usually call the x-axis, but we're dealing with functions of time) as a horizontal asymtptote, i.e., it goes to zero as $t \to \infty$.

With exponential growth, the limit $\lim_{t \to \infty} f(t) = \infty$. In a finite resource-constrained world, exponential growth must stop at some stage simply because we run out of resources. With exponential decay, the limit is zero. Thus, exponential decay will not stop due to resource constraints.

3.4. **Taking logarithms.**

A useful way of studying exponential growth with time is to plot the graph of the *logarithm* of the function in terms of time. Some observations:

- The logarithm of an exponential function is a linear function. Specifically, the logarithm of Ce^{kt} is $kt + \ln C$.
- In particular, the *slope* of the linear function thus obtained is the growth rate k and the *intercept* is the logarithm of the initial value.
- If $k > 0$, we have growth, which corresponds to a positive slope or increasing linear function.
- If $k < 0$, we have decay, which corresponds to a negative slope or linear decreasing function.
- The fact that we can use two observations to find C and k corresponds to the fact that after taking logarithms, knowing two points on a line determines the line.
- In particular, given *more than two* observations, one way to *test* whether they do fit a pattern of exponential growth/decay is to take logarithms and test whether the observation points thus obtained are collinear.

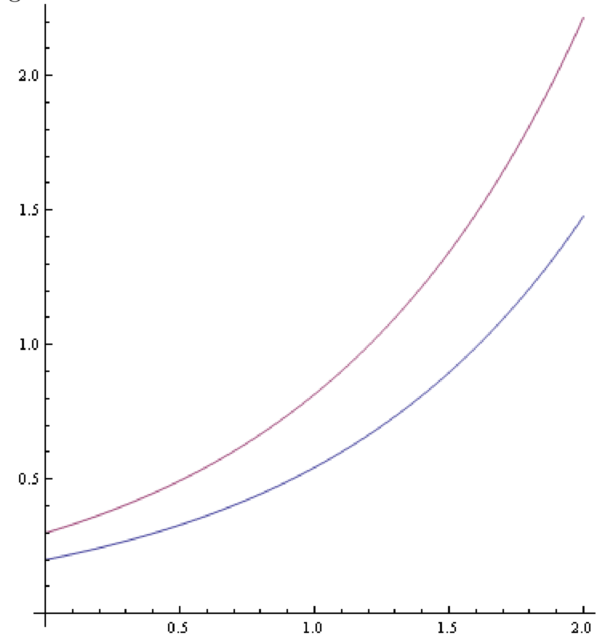
Some of you may recall the concept of *logarithmic differentiation*. The logarithmic derivative of a function f is defined as the derivative of $\ln |f|$, and is also defined as:

$$\frac{f'(x)}{f(x)}$$

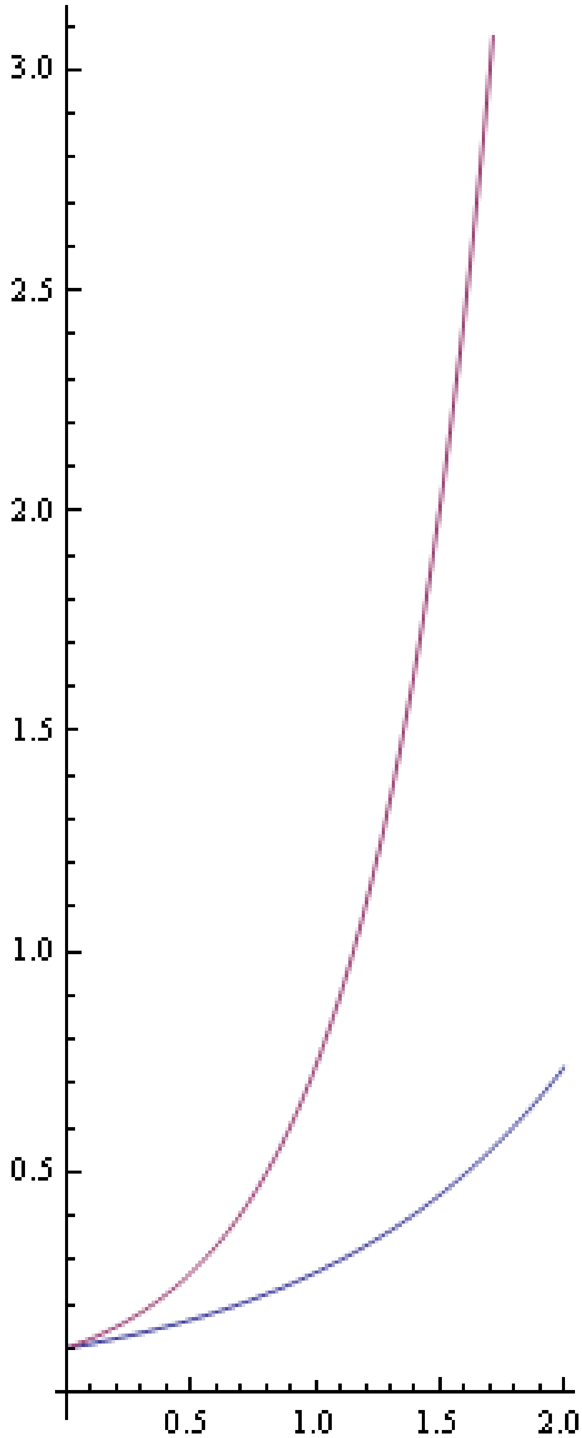
In fact, exponential growth corresponds *precisely* to the situation where the logarithmic derivative is a *constant function*, or equivalently, the derivative is *proportional* to the original function.

3.5. **Comparative analysis with different growth rates.**

- If two quantities are growing exponentially with the *same exponential growth rate*, the quantity that starts out bigger stays bigger. In fact, the ratio of the two quantities remains constant as a function of time. Taking logarithms, we get a pair of parallel lines. Here's how the picture looks before taking logarithms:



- If two quantities are growing exponentially, the quantity with the faster growth rate, if *originally bigger, stays bigger*, and if *originally smaller, overtakes the other quantity just once and after that stays bigger*. Below is a pictorial example where they start at the same point initially and the one with the faster growth rate becomes bigger:



4. Real world examples

4.1. **Radioactive decay.** The standard example of exponential decay is radioactive decay. With radioactive decay, the rate at which decay occurs is proportional to the amount of undecayed stuff. Here, instead of doubling, we foresee halving. A common measure of how quickly radioactive stuff decays is given by its *half-life*, i.e., the amount of time taken for it to become half of its original value. By the formula above, the half-life is related to k by the formula:

$$k = \frac{-\ln 2}{\text{Half-life}}$$

Again, the amount of time taken to decay by a factor of q is independent of the original mass.

A little further note on radioactive decay. At heart, radioactive decay is a probabilistic process. The correct model to keep in mind for radioactive decay is that any given nucleus has a fixed probability of decaying per unit time. *This probability does not depend on the number of other nuclei or how many nuclei have decayed so far.* It is *not the case* that as less and less of the substance is left, individual nuclei choose to decay slower and slower. Rather, since the *number of nuclei left* is fewer, the overall rate of decay declines.

Although radioactive decay is probabilistic, the *very large* number of nuclei in a given sample makes the macroscopic measurements qusai-deterministic. This is again a phenomenon that will appear repeatedly: *randomness at the individual level appears deterministic at the aggregate level.*

4.2. **Slowing down.** This is an important feature of exponential decay. With exponential decay, the rate of decay is proportional to the amount that is decaying. This exponential decay is *slower* than linear decay. Thus, the time taken for 2/3 of a material to decay is *more than twice* the time taken for 1/3 of the material taken to decay. In other words, the time taken for the first 1/3 of the material to decay is less than the time taken for the next 1/3 to decay, because the next 1/3 forms a larger fraction of the material left over after 1/3 has already decayed.

4.3. **Dating fossils.** Here, *dating* refers to determining how old a fossil is. The most common tool for dating is carbon-14 dating. The idea is as follows: the ratio of the unstable carbon-14 isotopes (6 protons, 8 neutrons) to the stable carbon-12 isotope (6 protons, 6 neutrons) in most living organisms is almost fixed. We assume that this ratio was the same in the prehistoric time when the fossil lived. After death and fossilization, there was no exchange of carbon with the surroundings, so the carbon-12 remained the same, but carbon-14 underwent radioactive decay, turning into the stable nitrogen-14. By determining the current ratio in the fossil, it is possible to determine (under all these assumptions) when the corresponding living organism lived.

The half-life of C-14 is about 5730 years, and carbon-14 dating has been used to claim that some fossils are millions of years old.[3]

4.4. **Continuously compounded interest rate.** *Compound interest* is a form of interest earned where past interest earned is added to the principal for future interest computations. Compound interest as given by banks is usually done on a discrete basis: the interest may be added to the principal every month, in which case we say that the interest is *compounded monthly*. Note that the same annual interest rate gives a higher *effective* annual interest rate if the compounding is done more frequently.

For instance, an annual interest rate of 100% compounded every six months effectively gives an annual interest rate of 125%, because after the first six months, the principal becomes 1.5 times its original value, and in the next six months, it becomes 1.5 times that value again, thus becoming 2.25 times its original value.

Continuously compounded interest means that the interest is compounded continuously. if the interest rate is r (expressed as a raw number, not a percentage), this means that the rate of change of the principal is r times the principal. We thus get exponential growth:

$$A = A_0 e^{rt}$$

where A_0 is the initial principal and A is the total amomunt accumulated after time t.

Also, note that the doubling time for continuously compounded interest is given by:

$$\frac{\ln 2}{r} \approx \frac{0.7}{r}$$

When the interest rate is expressed as a percentage, we get that the doubling time is 70 divided by the interest rate. This is the famous rule of 70. Note that our derivation of this formula assumes *continuously compounded interest*, which is not usually the way interest rates are specified or calculated. To show that

[3]If the Young Earth Creationist claim that the earth is only a few thousand years old is true, then at least one of the assumptions/claims made in the preceding two paragraphs must be false.

this formula is also reasonably valid for interest rates that are compounded annually, we need a further approximation result that we shall see later in the course.

4.5. **Interest rates, present value, and discount rates.** [May not get time to cover this in class.]

When we say that "the present value of such-and-such three years from now is this much" what we mean is, roughly, that in order to have such-and-such three years from now, we need to invest this much. For instance, at an interest rate of 10% per annum (compounded annually), the present value of 1100 dollars a year from now.

The concept of present value is used in more complicated contexts as well. For instance, what is the present value of 1200 dollars next year *plus* 1100 dollars two years from now? With an interest rate of 10% per annum compounded annually, the answer is 2000 dollars. If we start with 2000 dollars, this becomes 2200 dollars after a year. Withdrawing 1200 of these dollars leaves 1000 dollars, which in turn becomes 1100 dollars the year after that. This kind of concept is useful if you are planning on drawing upon savings to pay for your living after retirement.

Interest rates occur in a different guise as *discount rates*. For instance, a discount rate of 1% per annum means that we value the same thing a year from now at only 99% of how much we value it right now. With a discount rate of 1%, you would be indifferent between receiving a hundred dollars a year from now and 99 dollars today. A discount rate is related to an interest rate, because if your money can earn 1% interest in a year by being placed in the bank, then that explains your 1% discount rate.[4]

The concept of discounting is also used in policy analysis. When comparing different "costs" and "benefits" to society that occur over a long period of time, it is customary to specify a discount rate used for making an overall judgment. A zero discount rate would mean that a poicy that saves one life today is no more or less preferable than an identical policy that saves one life five years from now. A higher discount rate would mean that the former policy is preferable.

4.6. **Predicting the future: noise.** [May not get time to cover this in class.]

In situations such as radioactive decay, there are strong theoretical grounds for the claim that the decay is, at the macroscopic level, an exponential decay. The situation is less clear in other cases. However, in many cases, there are reasonable grounds for a claim that growth is approximately exponential.

However, the *secular exponential trend* needs to often be sorted out from *periodic seasonal fluctuation* and *random fluctuation*. In the context of US retail sales, if sales at a store go up by a factor of 3 from October to December, that does not mean that they will triple every two months to reach 729 times their original volume by next October. Part of the explanation for the jump may be seasonal patterns in consumer shopping (spurred on by discount sales).

There are a number of ways to tease out the secular trends from the seasonal trends and random fluctuations. We will not go into these in detail, but here are some obvious things:

- Compare apples to apples: Choose the same day of the week and the same time of the year as far as possible. This helps overcome seasonal trends.
- Use *moving averages* (for instance, average over the last 364 days) rather than single data points. Take the base of the moving average as the "period" for any periodic trend, e.g., 364 days works well because it is almost a year and is also a multiple of 7. This handles both seasonal trend issues and random fluctuations.
- When taking two data points to determine the rate of growth, do not take points that are spaced too closely, because even a small fluctuation can lead to a spurious high growth rate.

A deeper understanding of the methods used would require us to go into statistical methods, which we are not equipped for.

[4]There is technical issue here, which is that discount rates are measured negatively and interest rates are measured positively. For small discount rates/interest rates, this is not a big issue in the discrete situation. In the continuous version, it is a complete non-issue. This subtlety will be discussed later.