DIFFERENTIAL EQUATIONS

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Sections 9.1, 9.2 (though covered in a different ordering within these notes).

What students should already know: The prime and Leibniz notation for derivatives, the meaning of differentiation, implicit differentiation, and integration.

What students should definitely get: What a differential equation means, what a solution to a differential equation means, how to solve a multiplicatively separable first-order differential equation, how to solve an initial value problem. The specific choice of integrating factor used to solve a linear differential equation, and how to execute the solution.

What students should hopefully get: The notion of parameters as degrees of freedom, the notion of constraints as pinning these down, the basic concerns in differential equation manipulation. How first-order and higher order differential equations can be used to understand real world phenomena.

Executive summary

0.1. **Solving differential equations at large.** Words ...

(1) A differential equation with dependent variable y and independent variable x is something of the form $F(x, y, y', y'', \dots) = 0$.

(2) The *order* of a differential equation is the *largest* k for which the k^{th} derivative appears in the differential equation. In particular, a *first-order differential equation* only involves x, y, and y', and does not involve y'' or higher derivatives. A *second-order differential equation* only involves x, y, y', and y''.

(3) A *polynomial differential equation* is one where F looks like a polynomial in y and its derivatives. A *linear differential equation* is a differential equation of the form:

$$f_k(x)y^{(k)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$

We can clear out the coefficient of $y^{(k)}$ by dividing throughout by $f_k(x)$. The *homogeneous* case is where the right side is zero.

(4) A *particular solution* is a relation $R(x, y) = 0$ that, when plugged into the differential equation, satisfies it. (Here, higher derivatives are computed using *implicit differentiation*). A particular solution in *functional form* is one where we explicitly find a function f with $y = f(x)$ that satisfies the differential equation.

(5) A *solution family* is a family with one or more parameters such that for every permissible value of the parameter, we obtain a particular solution.

(6) The *general solution* is a solution family that contains all particular solutions.

(7) A general principle is that the number of freely varying parameters in the general solution, also described as the number of *degrees of freedom*, equals the order of the original differential equation. The reason for this is roughly that the number of integrations we do that introduce new degrees of freedom equals the order of the differential equation.

(8) An *autonomous* differential equation is a differential equation where the independent variable does not appear explicitly (except as the thing in terms of which differentiation is carried out). The independent variable can be thought of as *time*. Autonomous differential equations have the property that any time translate of a solution is also a solution. This property is found in most physical laws, and essentially states that the formulation of the physical law does not depend on when we started measuring time, i.e., there is no natural time origin.

(9) To solve a second-order differential equation we usually do a substitution to break it up into solving two first-order differential equations.
(10) Of first-order differential equations, there are two broad classes that we know how to solve: *separable* differential equations and *linear* differential equations. For the latter case, the solution method isolates y as a function of x. In the former case, we can get a mixed bag situation.

Actions ...

(1) The separable case is where we have $y' = f(x)g(y)$. In this case, we rearrange to obtain $\int dy/g(y) = \int f(x)\,dx$, and integrate both sides. We need to put the $+C$ on only one side, because additive constants emanating from both integrals can be combined into one additive constant.
(2) In the autonomous separable case, we have $dy/dt = g(y)$, and we integrate to obtain $\int dy/g(y) = \int dt$. This is the case that arises when we look at the logistic equation and its many variants.
(3) In the linear case, we have $y' + p(x)y = q(x)$ (after dividing out by any coefficient of y'). Let $H(x) = \int p(x)\,dx$. The integrating factor that we choose is $e^{H(x)}$. When we multiply by this integrating factor the left side becomes the derivative of $ye^{H(x)}$. Thus, we obtain:

$$y = e^{-H(x)} \int q(x)e^{H(x)}\,dx$$

Note that the $+C$ arises in the *inner* integral, so the general solution is a particular solution plus $Ce^{-H(x)}$.
(4) When solving differential equations (particularly the separable case) we often get a solution involving logarithms. In some cases, it may be useful to *exponentiate* both sides. When we do so, the original additive constant C arising from indefinite integration becomes a multiplicative constant e^C. We can also absorb *sign uncertainty* into it and define a new constant $k = e^C \operatorname{sgn}(y)$ to get the answer in terms of a sign expression.
(5) In a similar vein to the above, if our answer involves an inverse trigonometric function, we can apply the trigonometric function to both sides. In this case, the additive constant *sticks inside*. For instance, if we get $\arctan y = x+C$, then applying tan to both sides yields $y = \tan(x+C)$. To simplify this further (if we so desire), we need to use the angle sum formula. The other major caveat that we need to bear in mind is that there is a *loss of information* when we apply the trigonometric substituion to both sides, because an inverse trigonometric function value is constrained to a particular range. This constraint needs to be kept track of separately.
(6) In some cases, before exponentiating or applying the inverse trigonometric function, it might help to use the initial value condition to pin down the freely varying parameters (see the next subsection).

0.2. **Graphical interpretation and initial value problems.** Words ...

(1) Any *particular solution* (whether expressed with y as an explicit function of x or in terms of a relation between x and y) can be plotted as a curve in the xy-plane. When it is an explicit function, this is the *graph of a function* – otherwise, it's just the set of points satisfying the relation. This picture in the plane is called an *integral curve* or a *solution curve* for the differential equation.
(2) The *general solution* is thus a picture which has all the particular solutions marked.
(3) Since solving a k^{th} order differential equation introduces k degrees of freedom, we expect that to pin down a unique solution, we need k pieces of information. In particular, to choose one particular solution for a first-order differential equation, we need (by and large) one piece of information. In an *initial value problem*, this is provided by specifying an initial value, which is one point (x_0, y_0) on the particular solution curve.
(4) Geometrically, we expect that the solution family to a first-order differential equation has one real parameter and that, except in some degenerate cases, knowing one point on the curve determines the curve. In other words we expect that by and large, the solution curves do not intersect.
(5) For higher-order differential equations, on the other hand, we expect that even after knowing one point on the curve, we have pinned down only one of many degrees of freedom, and we still have solution families to deal with rather than isolated solutions. More information, such as information about higher derivatives, or information about the curve passing through other points, is desirable.

Actions ... Nothing really, except that we plug in the initial value condition to pin down the constants.

1. Understanding differential equations and solutions

1.1. **Differentiation: the two interpretations.** We have dealt with two interpretations of differentiation that it would be useful to recall at this stage. One interpretation is in terms of functions. Here, we think of a function f as a black box that takes as input a variable x and outputs a variable $f(x)$, that we may choose to call y. f' is a new function, i.e., a new black box, that takes as input x and gives an output called $f'(x)$, that we may also call y'.

In this interpretation, it is the function, rather than the inputs and outputs to it, that takes on primal importance. The disadvantage of this approach is that it does not allow us to go beyond functions.

The second interpretation is to view a function as a *relation* between two *quantities* – the *input quantity* and the *output quantity*. The function describes the nature of the dependence of the output quantity upon the input quantity. Under this approach, we denote the derivative as dy/dx, the Leibniz notation.

The Leibniz notation dy/dx arises from the fact that the derivative is the *limit* of the *difference quotient*:

$$\frac{dy}{dx} = \lim \frac{\Delta y}{\Delta x}$$

The focus here is *not* on the function that relates x to y, but on the variables x and y. The advantage of this approach is that we can apply this approach even when neither of the variables is a function of the other. For instance, we could do something called *implicit differentiation*, which allows us to find dy/dx when x and y are entangled. For instance, given:

$$y^2 + \sin(xy) = x^3 \cos(x+y)$$

We differentiate and get:

$$2y\frac{dy}{dx} + \cos(xy)\left[x\frac{dy}{dx} + y\right] = 3x^2\cos(x+y) - x^3\sin(x+y)\left[1 + \frac{dy}{dx}\right]$$

We can collect terms and obtain an expression for dy/dx in terms of x and y.

1.2. **A differential equation.** Consider two variables x (the so-called independent variable) and y (the so-called dependent variable). A *differential equation* is an equation involving the variables x, y, and first and higher derivatives of y with respect to x. For instance, here's a differential equation.

$$x + yy' + xy\sin(y') = 0$$

Here, y' is shorthand for dy/dx. Thus, this differential equation can also be written as:

$$x + y\frac{dy}{dx} + xy\sin\left(\frac{dy}{dx}\right) = 0$$

If we want to get $y = f(x)$, the above can be rewritten as:

$$x + f(x)f'(x) + xf(x)\sin(f'(x)) = 0$$

Another way of putting this is that a differential equation is something of the form $F(x, y, y', y'', \dots) = 0$ where F is some expression in many variables.

Before proceeding further, however, we must understand what a differential equation *means*, and how it differs from an ordinary equation.

(1) A *functional solution* or *function solution* is a function $y = f(x)$ such that, taking derivatives the usual way, we find that the differential equation is satisfied for *all* x. More specifically, a *function* $y = f(x)$ solves the differential equation if $F(x, f(x), f'(x), f''(x), \dots) = 0$ for all x.

(2) A *solution* to a differential equation is a relation $R(x, y)$ between x and y such that if we consider the set $R(x, y) = 0$, and use implicit differentiation to find the higher derivatives, these satisfy the condition $F \equiv 0$. A solution may differ from a functional solution in the sense that y may not be written *explicitly* as a function of x, and, in fact, globally, may not give a unique function of x.

(3) More pictorially, a solution to a differential equation is a curve in the plane $\mathbb{R}^2$ with the property that the differential equation holds at all points on the curve. What this means is that at any point on the curve, if we calculate the higher derivatives based on their geometric interpretations, we obtain a bunch of stuff that satisfies the differential equation.

What does this mean and how does this differ from an ordinary equation? Some important differences:

(1) Each solution to an ordinary equation (such as a polynomial equation) is a *number*. In contrast, each solution to a differential equation is a *function* or *relation* between two variables.
(2) When we are looking at an ordinary equation, such as $x^2 + x + 2 = 0$, we are looking at points where this equation holds. When we are looking at a differential equation, we are looking at curves such that the equation holds at all points on the curve.
(3) To check that an ordinary equation holds at a point, we evaluate at the point. However, to check whether a differential equation holds, we need to understand behavior locally, on a neighborhood. In other words, *it makes no sense to ask whether a differential equation holds for a given point* (x_0, y_0)*; it only makes sense to ask whether it holds on a given curve.*

Basically, a differential equation seeks to *find a function* that exhibits certain *local behavior* (in contrast with *pointwise behavior*) as described by an *expression involving the function and its derivatives.*

Aside: Differential equations as functional equations. A *functional equation* is an equation that asks for a function satisfying certain conditions. Specifically, a functional equation is an equation in terms of a function that we require to be true for *every* choice of value for all the letters in the equation, i.e., we require it to be an identity in all letter variables.[1]

For instance, the equation:

$$f(x) = f(-x) \ \forall \ x \in \mathbb{R}$$

has solution set precisely the set of all *even* functions. Similarly, the equation:

$$f(ax) = af(x) \ \forall a, x \in \mathbb{R}$$

has solution set precisely the set of all functions f of the form $f(x) = \lambda x$, where λ is a constant.

The desired solutions to functional equations are *functions*, and is does not make sense to ask whether a particular input-output pair satisfies a functional equation.

Differential equations are a *particular kind* of functional equation. Specifically, differential equations are functional equations involving derivative behavior all considered *at a single point.* [2]

1.3. **Some examples of differential equations and solutions.** We begin with a simple differential equation:

$$\frac{dy}{dx} = 1$$

We claim that $y = x + 13$ is a solution to this differential equation. In the various jargon that we have introduced:

(1) The function $f(x) = x + 13$ satisfies the condition that $f'(x) = 1$.
(2) The curve $y = x + 13$ satisfies the condition $y' = 1$.

Both these statements are clearly true. Graphically, the curve $y = x + 13$ is a straight line with slope 1 and intercept 13. Since its slope is 1, $dy/dx = 1$ everywhere on the line.

However, this is not the only solution. Astute observers would have noted that there was nothing particularly auspicious about the number 13. In fact, for any constant C, $y = x + C$, or the function $f(x) = x + C$, solves this differential equation.

Thus, any line with slope 1 is a *solution curve* to this differential equation.

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[1] In mathematical jargon, the letter variables for numbers are typically quantified over all reals.

[2] There are more complicated functional equations involving derivatives that are *not* differential equations in the sense that we have talked about. Examples include *delay differential equations*, which relate the values of the function and its derivative at far-off points.

Note that every value of C gives one solution, called a *particular solution*. Each such solution corresponds to a line with slope 1. Pictorially, we get a bunch of parallel lines that cover the entire plane.

Are these the only ones? Indeed, which brings us to the next topic.

1.4. **De ja vu.** When we first learn algebra in middle or high school, we are given examples such as:

> How many more apples need to be added to 3 apples to obtain 5 apples? Algebra version: Solve $3 + t = 5$.

At first, these seem like silly examples, because anybody who has grasped the concept of *subtraction* (the inverse operation to addition) can probably solve the problem without any knowledge of algebra. It is only after seeing harder examples of equations that people begin to appreciate the power of algebraic manipulation in solving problems that are too hard to manipulate with basic arithmetic.

In the same way, our first example of a differential equation is in fact an integration problem. Specifically, solving:

$$\frac{dy}{dx} = 1$$

is equivalent to performing the indefinite integration:

$$y = \int 1 \, dx$$

which gives the answer $y = x + C$ where C is an arbitrary constant. Each value of C gives a particular solution.

Let us elaborate the process a little further:

$$\frac{dy}{dx} = 1$$

Moving the dx to the numerator on the other side, we obtain:

$$dy = dx$$

Note that this is just formal manipulation, akin to the way you learned formal algebra when you got started. Don't ponder about the intrinsic meaning of dy and dx.

Integrating both sides, we get:

$$\int dy = \int dx$$

Which gives:

$$y = x + C$$

(Note that we can actually get positive constants with *both* indefinite integrals, but we can absorb the two constants into one).

This drawn-out process seems pointless for this specific example, *just like* algebra seemed pointless when you were solving $3 + t = 5$. Unlike middle school, however, where we waded through a lot of these silly examples before getting to more substantive examples of the use of algebra, we can now jump straight to harder situations where we see the machinery of differential equations and how it gets used.

Aside: Yesterday's problem is today's solution. In the elementary grades, addition, subtraction, multiplication and division were the problems, and numbers were the answers. In the middle grades, simple algebraic equations were the problems, and reducing them to a clearly stated arithmetic computation was the crux of the solution (the rest was trivial). When we learned differentiation, differentiating functions was the problem, but when we studied graphing and integration, differentiation was one of the many tools that was used in coming up with solutions.

This is the fundamental nature of mathematics: *yesterday's focus problems become the taken-for-granted tools of solution to today's focus problems.* Reducing the solution to today's focus problem to solving a bunch of yesterday's focus problems is almost as good at solving yesterday's problem.[3]

Incidentally, this is also related to the continually shifting boundaries between concrete and abstract. For five-year olds, a bunch of three apples is concrete, and the number three is abstract. For middle-schoolers first learning algebra, numbers are concrete and variables and algebraic expressions are abstract. For people just learning precalculus and calculus, functions given by explicit expressions are concrete but the definition of a function, limit, or derivative, is abstract. Needless to say, if you continue studying mathematics, you'll later view everything you learned in calculus as "concrete" as opposed to the abstract things you may learn later.

Until very recently in this course, integration was the *problem.* Now, integration is part of the *solution.* Once a differential equation is reduced to calculating a bunch of integrals, we're home. *Our goal in finding solution functions to differential equations is to reduce differential equations to (one or more) integration problems.*

2. Understanding and solving separable equations

2.1. **Separable equations: a crash course.** A differential equation of the form:

$$\frac{dy}{dx} = f(x)$$

has as solution:

$$y = \int f(x)\,dx$$

where the right side can be computed by finding one antiderivative and then tacking on a $+C$.

This is a form of differential equation that *directly reduces to a single integration problem* – the calculus analogue to $3 + t = 5$.

Let's look at a slightly more complicated example:

$$\frac{dy}{dx} = f(x)g(y)$$

also written as:

$$y' = f(x)g(y)$$

In words, we say that y' is a *multiplicatively separable function* of x and y – it is the product of a function that depends *only* on x and a function that depends *only* on y.

We do some algebra-like manipulations whose aim is to *bring together on one side* all terms involving y and *bring together on the other side* all terms involving x:

$$\frac{dy}{g(y)} = f(x)\,dx$$

We now put integral signs and carry out indefinite integration:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Once we find antiderivatives, we put the $+C$ on just one side (because two additive constants can be absorbed into one).

For instance, consider:

$$\frac{dy}{dx} = (x^2 + 1)(y^2 + 4)$$

We proceed to get:

[3]Assuming you remember how to solve yesterday's problems. Mathematical knowledge is cumulative, but an individual's mathematical knowledge is cumulative only if that individual actually accumulates knowledge.

$$\int \frac{dy}{y^2+4} = \int (x^2+1)\,dx$$

This becomes:

$$\frac{1}{2}\arctan\left(\frac{y}{2}\right) = \frac{x^3}{3} + x + C$$

What the $+C$ means is that every *particular value* of C gives a *particular solution*. For instance, when $C = 0$, we get the solution:

$$\frac{1}{2}\arctan\left(\frac{y}{2}\right) = \frac{x^3}{3} + x$$

The curve in the plane given by this solution (that we don't need to imagine) satisfies this differential equation.

Note that although the expression is not in the form of y as a function of x, we can bring it in that form with some algebraic manipulation, to get:

$$y = 2\tan\left[\frac{2x^3}{3} + 2x\right]$$

However, this kind of separation and writing things as functions is not always possible (I am glossing over *many* details here).

2.2. **Separable equations from the other side.** Suppose we start with a family of curves of the form:

$$F(x) + G(y) = C$$

where C varies over $\mathbb{R}$. We want to find a differential equation that is satisfied by all curves in the family. We use implicit differentiation to get:

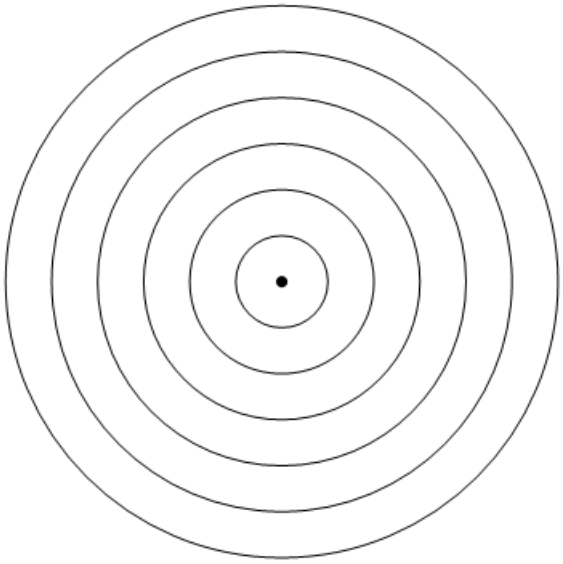
$$F'(x) + G'(y)y' = 0$$

which can also be written as:

$$y' = \frac{-F'(x)}{G'(y)}$$

Which is a (slightly differently written) version of the original thing we stated out with.

Basically, an expression where the derivative y' is *multiplicatively separable* in x and y solves to get a situation where an *additively separable function* of x and y takes constant values, where each possible constant value gives a particular solution.



2.3. **Of circles.**

Consider, for instance, the family of circles centered at the origin (a *concentric family*):

$$x^2 + y^2 = a^2$$

Differentiating and rearranging terms, we obtain the differential equation:

$$ydy = -xdx$$

In fractions, this becomes:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Conversely, solving this differential equation yields:

$$x^2 + y^2 = C$$

Note, however, something peculiar here. The claim in general is that each value of C gives a particular solution. However, from the way sums of squares behave, we know that:

(1) For $C > 0$, we do get particular solutions – the circles we began with.
(2) For $C = 0$, we get a single point circle, which cannot be called much of a solution to the differential equation, because there isn't room to move around within the point.
(3) For $C < 0$, we get the empty set, which again cannot be called much of a solution.

In other words, many of the values of C give empty sets as their curves, which could not be called solutions. Thus, in general, it is *not* correct to say that each value of C gives a legitimate and nontrivial solution. On the other hand, the general theory we have developed so far is not strong enough to predict precisely which values of C give legitimate solutions and which ones give degenerate (as in the case of the single point circle) or empty solution curves.

3. Initial value problem

3.1. **Terminology recall and improvement.** Every function or relation that *solves* a particular differential equation is called a *particular solution* and the corresponding curve in $\mathbb{R}^2$ is termed an *integral curve* or *solution curve*. We have seen that, in general, there could be more than one solution curve – in fact, there could be entire families of solution curves parametrized by constants $C \in \mathbb{R}$. A general expression that describes the entire family of solutions is termed a/the *general solution* to the differential equation.

Some other terminology: We say that the *order* of a differential equation is the largest n such that the n^{th} derivative of the dependent variable appears in the differential equation. A *polynomial differential equation* is a differential equation of the form $F(x, y, y', \dots)$ where F looks like a polynomial in y and in each of the

derivatives. A *linear differential equation* is a differential equation of the form $F(x, y, y', \dots)$ where F is a *linear* function of the variables y, y', ... The *degree* of a (usually polynomial) differential equation is the power to which the highest order derivative is raised.

3.2. **The constants as parameters.** Remember that when we integrate a function once, we get a $+C$ in the solution. The $+C$ indicates that the integral is not a single unique function but rather a family of functions, all of which can be obtained by taking a *particular* antiderivative and adding any constant function to it. We can think of the set of possible antiderivatives as being parametrized by the real numbers.

When we integrate a function k times, the general solution is of the form of (particular solution) plus (an arbitrary polynomial of degree less than k, i.e., degree at most $k-1$). The *coefficients* of this polynomial are the k constants, one arising from each integration. The set of solutions is thus parametrized by the set of possible k-tuples of real numbers.

In physics and chemistry (for instance, in statistical mechanics and thermodynamics), each *freely varying real parameter* is termed a *degree of freedom*.

When solving a *first-order differential equation* (i.e., a differential equation where second or higher derivatives do not appear) what we hope to do is *separate* x and y (something that can be done in the case y' is multiplicatively separable) and then integrate both pieces. We get constants at both places, but these constants can be merged into one constant. *The general idea is that when solving a first-order differential equation, we expect to have one free real parameter, or one degree of freedom, in the solution.* Similarly, when solving a differential equation of order k, we expect to have k free parameters, or k degrees of freedom, in the solution.

3.3. **Initial value specification for first-order differential equations.** As noted above, the general solution to a first-order differential equation contains a degree of freedom, typically described by a freely varying real parameter C. Geometrically, there is a family of solution curves, and this family is parametrized by a real number.

To find a *particular solution*, we need some piece of information that helps us narrow down to a particular choice of curve.

Information that tells us the location of *one point* on the desired solution curve is termed an *initial value condition* or an *initial value specification*. A problem that consists of a differential equation along with an initial value condition is termed an *initial value problem*.

Typically, an initial value specification helps us determine a specific value of C, i.e., it helps us pin down and destroy the one degree of freedom.

For instance, consider:

$$\frac{dy}{dx} = xy$$

Rearranging, we have:

$$\int \frac{dy}{y} = \int x\, dx$$

We obtain:

$$\ln|y| = \frac{x^2}{2} + C$$

Exponentiating both sides, we obtain:

$$|y| = e^{x^2/2}e^C$$

Note that we can pick a new constant $k = e^C \operatorname{sgn}(y)$ and obtain:

$$y = ke^{x^2/2}$$

This is the *general solution*. If, however, we are given that $y(1) = 1$ (in other words, the solution curve passes through $(1, 1)$), we get:

$$1 = ke^{1/2}$$

Thus, we obtain $k = 1/\sqrt{e}$. Plugging this back in, we get the particular solution that we are interested in:

$$y = \frac{1}{\sqrt{e}} e^{x^2/2}$$

or equivalently:

$$y = e^{(x^2-1)/2}$$

3.4. **Brief note on initial value specifications for higher orders.** As noted above, when solving a general differential equation of order k, we expect to obtain a general solution with k degrees of freedom. To constrain these, we need k pieces of information (in a rough sense). An *initial value condition* would provide these k pieces of information by providing the information of a point and the first $k-1$ derivative values at the point.

3.5. **An example of a second-order differential equation and a geometric interpretation.** Consider the second-order differential equation:

$$y'' = 0$$

In the Leibniz notation, this is:

$$\frac{d^2y}{dx^2} = 0$$

This basically involves *integrating twice*. We let $z = y'$, and get:

$$z' = 0$$

Solving, we obtain:

$$z = C_0$$

where C_0 is an arbitrary real constant. We now need to solve:

$$y' = C_0$$

which becomes:

$$\frac{dy}{dx} = C_0$$

Solving this yields:

$$y = C_0 x + C_1$$

Thus, the *general solution* involves two arbitrary real constants. Looking at the *solution curves* in the plane, we see that these are precisely all the non-vertical lines.

With first order differential equations, the integral curves typically do not intersect each other. In other words, in the case of first order differential equations, every point is on a unique integral curve (there are sometimes exceptions, such as some special points that lie on lots of curves, and there are sometimes situations where every point lies on two curves). On the other hand, for this second order differential equation, each point lies on an entire infinite family of curves.

Thus, simply specifying *one point* is not enough to determine an integral curve. On the other hand, if we specify one point on the curve *and* the value of the derivative at that point, that information together is enough to determine the integral curve. Thus, *higher order derivative information* is necessary to obtain a strong enough initial value condition to uniquely solve the problem.

The fact that we need *two* pieces of information to pinpoint a curve in this family is not surprising considering that there are two degrees of freedom, or two parameters.

3.6. **Other ways of pinning down degrees of freedom.** An initial value specification constrains degrees of freedom by providing a point and derivative information at that point. There are other ways of specifying a unique integral curve among the set of all possible integral curves, and one of these is to specify the value of the function at *multiple* points. For instance, in the case of the family of all non-vertical straight lines given by the second-order differential equation $y'' = 0$, specifying *two* points is enough to specify a line – a fact which we already learned in high school geometry.

4. Linear differential equations

4.1. **What is a linear differential equation?** A linear differential equation of order k with independent variable x and dependent variable y is a differential equation of the form:

$$f_k(x)y^{(k)} + f_{k-1}(x)y^{(k-1)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$

where the f_i are all functions given to us explicitly. In case $g \equiv 0$, we say that we have a *homogeneous linear differential equation* of order k.

We are interested in solving linear differential equations of order 1. These look like:

$$f_1(x)y' + f_0(x)y = g(x)$$

Dividing throughout by $f_1(x)$, we obtain:

$$y' + p(x)y = q(x)$$

where $p(x) := f_0(x)/f_1(x)$ and $q(x) = g(x)/f_1(x)$.

This is the prototype of the first-order linear differential equation that we describe here how to solve.

4.2. **Flashback.** We first consider the special case where $p(x) \equiv 1$, and we get:

$$y' + y = q(x)$$

Recall that $f(x) + f'(x) = q(x)$ if and only if the derivative of $e^x f(x)$ is $e^x q(x)$. In other words, solving this problem is equivalent to integration the function $e^x q(x)$, and then dividing the answer we obtain by e^x.

In the past, we used this approach to *convert the integration problem* to a problem of finding a function f such that $f + f' = q$. We were able to do this in cases where q was a polynomial function and we could make a reasonable starting guess as to what f must be.

Now, we employ the procedure in reverse, i.e., we start with the differential equation, and convert it to an integration problem. Let's write out what's happening in this particular case.

We start with:

$$y' + y = q(x)$$

We multiply both sides by e^x and obtain:

$$e^x(y' + y) = e^x q(x)$$

The left side is the derivative of ye^x, so we obtain:

$$\frac{d}{dx}(ye^x) = e^x q(x)$$

Integrating, we obtain:

$$ye^x = \int e^x q(x)\, dx$$

or:

$$y = e^{-x}\left[\int e^x q(x)\, dx\right]$$

Note that the $+C$ comes *inside* the parentheses, so the general solution is some particular solution plus Ce^{-x}.

Here now is the more general case:

$$y' + p(x)y = q(x)$$

Define:

$$H(x) = \int p(x)\,dx$$

In other words, H is an antiderivative of p. In particular, $H'(x) = p(x)$.
Multiply both sides by $e^{H(x)}$. We obtain;

$$e^{H(x)}[y' + p(x)y] = e^{H(x)}q(x)$$

We can rewrite this as:

$$e^{H(x)}[y' + H'(x)y] = e^{H(x)}q(x)$$

The left side is now the derivative of $e^{H(x)}y$, so we obtain:

$$e^{H(x)}y = \int e^{H(x)}q(x)\,dx$$

This simplifies to:

$$y = e^{-H(x)}\left[\int e^{H(x)}q(x)\,dx\right]$$

Note that the $+C$ emerging from integration arises inside the parentheses, so the general solution is a particular solution plus $Ce^{-H(x)}$.

The function $e^{H(x)}$ is termed an *integrating factor*, because multiplication by it converts the differential equation to a form where we can *integrate both sides*.

Here is one example:

$$y' - xy = 1$$

Here, $p(x) = -x$ and $q(x) = 1$. We obtain $H(x) = -x^2/2$ and the integrating factor is thus $e^{-x^2/2}$. We thus get:

$$y = e^{x^2/2}\int e^{-x^2/2}\,dx$$

Please see more examples from the book.

5. A little more in-depth understanding

5.1. **A "solvable" differential equation.** Let us return to the philosophical question of what it means to *solve* a differential equation. We arrange the different notions of solution in increasing order of preference:

(1) We reduce the problem to a bunch of integration problems. We have seen two methods of doing this: the *separable* case, where we simply collect the dx and dy terms and integrate, and the *linear* case, where we multiply by an integrating factor.
(2) We are able to obtain a relation $R(x, y) = 0$ that is free of derivatives, but may have one or more free parameters, such that *all* solutions to the differential equation arise by setting particular values of these parameters. This is typically achieved after we successfully carry out the integrations in step (1).
(3) We are able to write y as a function of x, *explicitly*. This is sometimes possible with some algebraic manipulation of the results obtained in step (2). Note that in the linear case, we get directly from step (1) to step (3) because y is already isolated on one side in the general expression for the solution.

We see that there are thus three steps that we would like to execute. People who work in differential equations *typically concentrate on the first step*, treating the second and third step as trivial, or as mere curiosities or afterthoughts. This may come as a surprise to those of you who have been struggling with indefinite integration for the past few months. However, it is still desirable to reduce differential equations (which could be really unshapely) to integration problems. Even if the integration cannot be carried out formally, we can still use other methods, such as numerical approximation, to understand the solutions qualitatively.[4]

The sad news is that there is no general procedure to take an *arbitrary* differential equation – even a first-order differential equation – and reduce it to an integration problem. There are a lot of special techniques that we could learn if we wanted to. However, these other techniques are not very important for our current goals, so we skip over them.

6. Autonomous differential equations and the Verhulst process

6.1. **What's autonomous?** In the context of the real world, a particularly important kind of differential equation is an *autonomous* differential equation. This is a differential equation where we are studying a quantity varying with time, and the differential equation does not involve the time variable *explicitly* except as the variable with respect to which we differentiate. An autonomous first-order differential equation is of the form:

$$\frac{dy}{dt} = g(y)$$

Note that an autonomous first-order first-degree differential equation is separable, and specifically, it can be integrated as:

$$\int \frac{dy}{g(y)} = t + C$$

6.2. **Physical significance of autonomous equations.** Most physical processes and physical laws satisfy the condition of *time translation invariance*. In other words, if we change the origin of time measurement, the physical law remains invariant. Another way of saying this is that if $f_1(t)$ is one solution function, the function $f_2(t) := f_1(t + C)$ is also a solution function. Thus, taking any solution function and translating the time coordinate would give another solution function. Geometrically, this means that the graph of a solution function can be translated horizontally to give the graph of another solution function.

Autonomous differential equations capture this property. Thus, differential equations describing physical or chemical processes in the day-to-day world are autonomous.

6.3. **Population growth.** Earlier in the course, we studied a process known as exponential growth. Exponential growth arises as the solution function to the autonomous differential equation:

$$\frac{dy}{dt} = ky$$

At the time we introduced this idea, I conjured up the solution out of thin air. We can now see exactly how we got the solution:

$$\frac{dy}{y} = k\,dt$$

Integrating, we obtain:

$$\ln|y| = kt + C$$

We thus get:

$$y = C_1 e^{kt}$$

[4]To borrow a colloquialism: the devil you know (indefinite integration) is better than the angel you don't (differential equations).

where $C_1 = e^C \operatorname{sgn}(y)$.

The autonomous differential equation describing exponential growth was based on the idea that the rate of growth at any given instant is in direct proportion to the quantity that is already around at that instant. We also know that in the case $k > 0$, this exponential growth goes on indefinitely at an ever increasing pace.

We now consider a variant of this that makes more sense in a *resource-constrained* world. The general variant is:

$$\frac{dy}{dt} = ay^m(b-y)^n$$

where a, b, m and n are all positive.

Note that a plays the role of the *growth rate parameter* as k did earlier. m and n, however, now modify the nature of growth. b can be thought of as a *ceiling* on y. Specifically, if the y value reaches b, the rate of growth becomes zero. Moreover, we see that, for $0 < y < b$, the growth rate (given by the right side) is low for very small y (because the y^m term is very small) but it is also small for y very close to b (because the $b - y$ terms is very small). In physical terms, the low growth rate for small y can be understood by the fact that a very small population cannot create new thins quickly, and the low growth rate as y gets close to b can be explained by *overcrowding* leading to a *struggle for resources* which thus slows down the growth rate.

Now, a little note of caution is in order here. We have given above a plausible mathematical model describing resource-constrained growth, with some parameters m and n. This does not mean that every example of resource-constrained growth fits into this model. Nor does it mean that any quantitative or qualitative conclusions we draw from this model necessarily hold for all examples of resource-constrained growth. If a particular real-world instance of resource-constrained growth is to be considered as an example of the model, some empirical or theoretical evidence needs to be provided.

In this model, we can predict the following – depending on the values of m and n:

(1) For any initial value of y strictly between 0 and b, we have $y(t) \to b$ as $t \to \infty$, but it does not reach there in finite time, i.e., the approach is asymptotic.
(2) For any initial value of y strictly between 0 and b, we have that $y(t)$ reaches b in finite time and stays there forever.

It is instructive to try to work out values of m and n for which these two cases arise. We shall consider the particular case $m = n = 1$, which is the standard *Verhulst process* or *logistic equation*.

6.4. **Verhulst process: working out.** The equation is:

$$\frac{dy}{dt} = ay(b-y)$$

In this simple model, the rate of growth of y is proportional to the product of y (the current value) and $b - y$ (the amount of unoccupied space).

Solving, we obtain:

$$\frac{1}{b} \ln \left| \frac{y}{b-y} \right| = at + C$$

This simplifies to:

$$\frac{y}{b-y} = ke^{abt}$$

for some constant k.

We can now simplify to obtain an explicit expression for y in terms of t. Note that time translating the t simpliy affects the corresponding value of k, as we might predict given that we're dealing wit han autonomous differential equation.

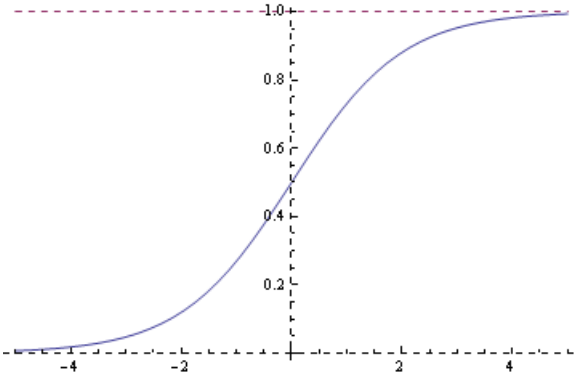
The explicit solution is:

$$y = b\frac{ke^{abt}}{1 + ke^{abt}} = b\frac{k}{e^{-abt} + k}$$

We see that in this case, for any starting value strictly between 0 and b, the limiting value of b is approached as $t \to \infty$, and the limiting value of 0 is approached as $t \to -\infty$. Some further observations:

- The value of y increases for all t, because y' is strictly positive.
- The growth rate is maximum when $y = b/2$.
- In fact, the graph has a half-turn symmetry about the pair $(t_0, b/2)$ where t_0 is the time it reaches $b/2$. The graph is concave up prior to reaching this point, and concave down beyond that.

Here is an example of a graph, where $a = b = k = 1$, and so the mid-value of $b/2$ is attained at time $t = 0$.



This can also be described in terms of the hyperbolic tangent function, which, alas, we did not discuss.

Note that the particular value of k can be determined from the initial value condition.

The book looks at the Verhulst process from a somewhat different perspective. It is recommended that you read this and try to work out how the constants in the book match up with what we have discussed here.

6.5. **Square roots, arc sine, and more.** Let's now consider the case $m = n = 1/2$:

$$y' = a\sqrt{y(b-y)}$$

Rearranging, we get:

$$\frac{dy}{a\sqrt{y(b-y)}} = dt$$

As usual, it is the left side that is a mess to integrate. The expression under the square root symbol has negative coefficient for y^2, so the correct way of thinking of it is as a square of a constant minus a square of a translate of y. Recalling quadratics, we see that the right choice is $(b/2)^2 - (y - b/2)^2$. We thus get an arc sine integration:

$$\frac{1}{a}\arcsin\left(\frac{y-(b/2)}{b/2}\right) = t + C$$

Let $\varphi = aC$, so we get:

$$\arcsin\left(\frac{y-(b/2)}{b/2}\right) = at + \varphi$$

Taking sin both sides, we get:

$$\frac{y-(b/2)}{b/2} = \sin(at+\varphi)$$

Rearranging, we get:

$$y = \frac{b}{2}\left(1 + \sin(at+\varphi)\right)$$

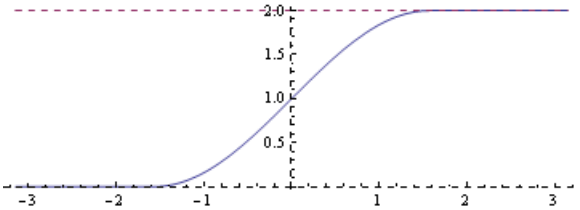
Further, since we are using the arcsin substitution, we know that $at + \varphi$ is between $-\pi/2$ and $\pi/2$. In particular, we get that:

$$\frac{-\pi}{2} \leq at + \varphi \leq \frac{\pi}{2}$$

Thus, we get that:

$$\frac{-\pi - 2\varphi}{2a} - \varphi \leq t \leq \frac{\pi - 2\varphi}{2a}$$

What does this mean? It means that the above differential equation is valid only in the interval indicated $\left[\frac{-\pi-2\varphi}{2a}, \frac{\pi-2\varphi}{2a}\right]$. For times *beyond* the right endpoint of the interval, the value of y is stable at b. For times *before* the left endpoint of the interval, the value of y is stable at 0.



Something is fishy here! The graph looks like 0 up to some point in time, then it suddenly springs to life, behaves like a sin curve for a while, then goes up to b (in finite time), and then becomes stable at b. There are also two other extreme solutions: that $y = 0$ *throughout*, and that $y = b$ *throughout*.

7. Vector spaces, linear algebra, and linear differential equations

Linear differential equations are a good pretext to introduce some extremely important ideas – namely, the ideas of vector space and linear algebra. We will not be concerned with these ideas in a computational sense, but they provide a good conceptual footing for much of what we're talking about here.

7.1. **Vector space basics.** A (real) vector space is a set of things where we can perform addition (the usual way) and scalar multiplication (where the scalars are real numbers), such that the usual roles of commutativity and associativity of addition, as well as the usual nice properties of scalar multiplication, hold.

A subspace of a real vector space is a subset that is closed under both the addition and the scalar multiplication operations.

The standard example of a vector space that we will be concerned about is the space of all real-valued functions on an interval I (usually, an open or closed interval, but we could also take I as all real numbers) of the reals. Addition and scalar multiplication are both done pointwise. Here are some things that are certainly true:

- The *continuous* functions on I form a subspace of the vector space of all functions. This subspace is denoted $C(I)$ or $C^0(I)$.
- Suppose I is an open interval. Denote by $C^k(I)$ the set of all real-valued functions on I that are k times continuously differentiable. Then, $C^k(I)$ is a subspace of the vector space of all functions. Further, $C^k(I)$ contains $C^{k+1}(I)$. The intersection of all these is the subspace of infinitely differentiable functions, denoted $C^\infty(I)$.

We define a mapping $T : V \to W$ of vector spaces to be *linear* if $T(f + g) = Tf + Tg$ and $T(\lambda f) = \lambda Tf$, i.e., T respects both the addition and scalar multiplication operation. The *kernel* of the mapping T is defined as the set of all f such that $Tf = 0$. The kernel of a linear mapping is always a subspace.

We note that:

- For an open interval I, *Differentiation* defines a linear mapping from $C^1(I)$ (the continuously differentiable functions) to $C^0(I)$ (the continuous functions).
- The range (image) of the mapping is the full $C^0(I)$ – every continuous function is the derivative of some continuously differentiable function.
- The kernel of the mapping is the set of *constant* functions on I, with each constant funcion described by its value (a real number). This is a (one-dimensional) subspace of the space of all functions.
- This differentiation mapping has the property that it sends each $C^k(I)$ to $C^{k-1}(I)$ and sends $C^\infty(I)$ to $C^\infty(I)$.

The significance of this as as follows. If T is a linear mapping, then $Tf = Tg$ if and only if $f - g$ is in the kernel of T. We already saw this with the differentiation mapping: two functions on an open interval have the same derivative iff their difference is a constant function.

What this means is that if T is a linear mapping, and we want to solve $Tf = q$ for known h and unknown f, it suffices to find (i) the kernel of T, and (ii) just one function g such that $Tg = q$ (a particular solution). The general solution is then functions of the form $g + \varphi$ where φ is a function in the kernel. Again, this is how we do indefinite integration. We first try to find a particular antiderivative, and then the general solution for the indefinite integral is that particular antiderivative plus an arbitrary constant function.

7.2. **Finite-dimensional vector spaces and span.** For a finite collection of functions $f_1, f_2, f_3, \ldots, f_n$, the vector space spanned by these functions is the space of all possible functions of the form $a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$, where the a_i are all real numbers. Typically, we call these *real linear combinations* of the f_is.

A vector space has dimension n if we can find a collection of n functions such that it is spanned by them, but we cannot span it using fewer than n functions. A one-dimensional vector space is a space spanned by a single function. For instance, the space of constant functions is a one-dimensional space, and it is spanned by the constant function 1.

7.3. **Application to differential equations.** Here's the relevance to linear differential equations: a linear differential equation can be thought of as trying to find a function f such that $Tf = q$ for a specified linear operator T and a specified function q. For instance, recall the first-order linear differential equation:

$$y' + p(x)y = q(x)$$

Here, the operator $Tf := f' + p \cdot f$ is a linear operator, and we want to find all functions f such that $Tf = 0$. We use the integrating factor method to do this: first we find an antiderivative H for f. The kernel of T is the one-dimensional vector space spanned by $e^{-H(x)}$. The general solution is a particular solution plus $Ce^{-H(x)}$.

8. A chemistry problem

Here is the homework problem that you need to solve:

It is known that m parts of chemical A combine with n parts of chemical B to produce a compound C. Suppose that the rate at which C is produced varies directly with the products of the amounts of A and B present at that instant. Find the amount of C produced in t minutes from an initial mixing of A_0 pounds of A with B_0 pounds of B, given that:

(1) $n = m$, $A_0 = B_0$, and A_0 pounds of C are produced in the first minute.
(2) $n = m$, $A_0 = \frac{1}{2}B_0$, and A_0 pounds of C are produced in the first minute.
(3) $n \neq m$, $A_0 = B_0$, and A_0 pounds of C are produced in the first minute.

If you just want to solve the problem, you can directly follow the hint in the book. However, it might be more helpful to understand the background, which is what we explore in this short note.

8.1. **Conservation of mass and constant proportions.** For now, let us forget how the reaction proceeds with time, and understand what exactly is happening in the reaction.

The reaction is of the form:

$$A + B \to C$$

We are also given that m parts of A combine with n parts of B to yield C. Although the question does not state this explicitly, the *parts* here are by mass. This means, for instance, that m pounds of A combine with n pounds of B. By the *law of conservation of mass*, the output of C is $m + n$ pounds.

For instance, consider the reaction:

$$2H_2 + O_2 \to 2H_2O$$

Here, the total weight of $2H_2$ is 4 units (atomic mass of 1, two atoms, giving molecular mass of 2, then multiplied by 2 to get 4) and the total weight of O_2 is 32 units (atomic mass of 16 with 8 neutrons and 8 protons, multiplied by 2 to get 32), so 4 units (By mass) of hydrogen combine with 32 units (by mass) of

oxygen to yield 36 units (by mass) of water. The proportions are thus $m : n = 4 : 32$, which can also be written as $1 : 8$. Thus, 1 part of hydrogen combines with 8 parts of oxygen to yield 9 parts of water. Note that it is the ratio of m to n that matters, not the values of m and n per se.

The fact that specific chemical reactions occur in specific proportions was once considered so important as to actually be given a name. This name is the *law of constant proportions* or the *law of definite proportions*. In fact, empirical observation of this law preceded acceptance of the atomic theory of matter and was one of the pieces of evidence that led to support for the atomic theory of matter. Some archaic chemistry textbooks and courses still begin with the statement of this law.

8.2. **Stoichiometric constraints.** Since a given chemical reaction occurs with constant proportions, this indicates that the *change in mass* of each of the reactants and products through the course of the reaction is in the same proportion. For instance, in the hydrogen-oxygen reaction to produce water, we know that 1 pound of hydrogen would combine with 8 pounds of oxygen to produce 9 pounds of water. This means that if x pound of hydrogen are lost, then $8x$ pounds of oxygen are lost, and $9x$ pounds of water are gained. Thus, if we know the initial amounts of hydrogen, oxygen, and water, and we know the final amount of hydrogen, we can measure the hydrogen lost, and use that to deduce the oxygen lost (and hence the final amount of oxygen) and the amount of water gained (and hence the final amount of water).

This can be viewed in terms of a three-dimensional configuration space. Consider a three-dimensional space with the three axes marked respectively by the current mass of hydrogen, the current mass of oxygen, and the current mass of water.

What the stoichiometric constraint (or the law of constant proportions) says is that, as the reaction proceeds, the configuration moves along a straight line whose direction is determined by the nature of the reaction. The straight line has the property that the oxygen axis coordinate changes at eight times the rate of change of the hydrogen axis coordinate, and the water axis coordinate changes in the opposite direction at nine times the rate.

In the more general case, with:

$$A + B \to C$$

the coordinate change ratios are $m : n : -(m + n)$, and the path of the reaction is along a straight line.

8.3. **Initial condition.** The path of the reaction is along a straight line whose direction is determined by the nature of the reaction (i.e., by the ratio $m : n$). However, there are many different straight lines with the same direction. The *particular* straight line that the reaction is confined to depends on the initial configuration.

Typically, we start out without any of the product (i.e., we only have the reactants A and B. In terms of the three-dimensional picture with coordinates A, B, and C, the initial configuration is in the AB-plane. *Where* it is in the AB-plane determines *which* line the reaction moves against. (This can be thought of as a three-dimensional analogue of the point-slope form).

8.4. **Keeping track of just one coordinate.** Suppose we want to know each coordinate at every point in time. We claim that it is enough to know:

(1) The initial value of each coordinate.
(2) The value, at any given time t, of just one coordinate.

In particular, if we know the initial amounts of A, B and C (which we assume to be zero), then knowing the quantity of C at time t allows us to determine the quantities of A and B at time t.

Here's how we can deduce what these quantities are. Let $C(t)$ denote the quantity of C at time t. We're assuming $C(0) = 0$, so $C(t)$ amount of C was gained. By constant proportions, we note that the amount of A lost is $mC(t)/(m + n)$ and the amount of B lost is $nC(t)/(m + n)$. Thus, the amount of A at time t is:

$$A(t) = A(0) - \frac{m}{m+n}C(t)$$

Similarly, the amount of B at time t is:

$$B(t) = B(0) - \frac{n}{m+n}C(t)$$

In the problem setup, we are given that $A(0) = A_0$ and $B(0) = B_0$.

8.5. **Bringing time and rates into the equation.** So far, we have largely focused on the path that the reaction takes. The next relevant question is the *rate* at which the reaction occurs. In the three-dimensional graphical representation, this is basically determining how fast we're moving along the line.

When trying to measure the rate of a reaction, there is a little ambiguity. Should we measure the rate at which A is being lost, the rate at which B is being lost, or the rate at which C is being gained? By the law of constant proportions, these rates are all related:

$$\frac{-1}{m}\frac{dA}{dt} = \frac{-1}{n}\frac{dB}{dt} = \frac{1}{m+n}\frac{dC}{dt}$$

We can keep track of just *one* of the three, and the one we choose to keep track of is C. Note that the proportionality in the rates of change is the *differential version* of the expressions $A(t) = A(0) - mC(t)/(m+n)$ and $B(t) = B(0) - nC(t)/(m+n)$.

Thus, when we say that the *rate of reaction* is proportional to some quantity, that statement would apply to all the three rates of change dA/dt dB/dt, and dC/dt, with the constants of proportionality themselves in the ratio $m : n : -(m+n)$.

8.6. **Formulating the differential equation.** For convenience, we take pounds (the mass measure) as our unit of measurement. Let A_0 and B_0 be the initial number of pounds of A and B respectively.

If the number of pounds of C at time t is $C(t)$, then $A(t) = A_0 - mC(t)/(m+n)$ and $B(t) = B_0 - nC(t)/(m+n)$.

We know that the rate dC/dt equals kAB, where k is some unknown constant. We thus have:

$$\frac{dC}{dt} = k\left[A_0 - \frac{m}{m+n}C(t)\right]\left[B_0 - \frac{n}{m+n}C(t)\right]$$

Note that this is an *autonomous* differential equation, which is to be expected since it arises from a physical law that is time translation invariant. Note: The *law* is time translation invariant, not the actual configuration.

Notice another feature of this problem. In general, when we solve a differential equation, we introduce one free parameter. For this differential equation, we *already* have a bunch of parameters, and we will introduce *one more* parameter when we solve the differential equation. However, the added parameter can easily be determined from the initial condition $C(0) = 0$.

8.7. **Solving the differential equation.** The equation is autonomous and separable, and we can rearrange terms to get:

$$\int \frac{dC}{\left[A_0 - \frac{m}{m+n}C\right]\left[B_0 - \frac{n}{m+n}C\right]} = \int k\,dt$$

The right side integrates to $kt + c_1$, and the left side can be integrated in one of the standard ways. Once we integrate the left side, we can choose c_1 in such a way that $C(0) = 0$.

There are two natural cases for the left side:

(1) The two linear factors in the denominator are equal, or proportional: In this case, the integrand is the reciprocal of the square of a linear function of C, and that is directly integrated. Note that this case occurs if $A_0/B_0 = m/n$, i.e., if the initial proportion of masses is the *stoichiometric proportion*. In the homework problem, part (a) is of this type.

(2) The two linear factors in the denominator are not proportional: In this case, we need to use the partial fractions approach to integrate. In the homework problem, parts (b) and (c) are of this type.

8.8. **Predicting answers: limiting reagent.** Looking at the answers, could we have predicted them to begin with? Qualitatively, yes:

(1) In the case of stoichiometric proportions, the equilibrium state of the reaction is where all of both A and B are used up so everything is converted to C. Note: This is assuming (as we're doing in this question) that the reverse reaction does not occur, otherwise an equilibrium will be attained before that.
(2) In the case of non-stoichiometric proportions, one of the reactants is what is called a *limiting reagent*. While the quantity of the limiting reagent approaches zero, the asymptotic quantity of the other reagent is strictly positive.

8.9. **An additional piece of information.** Suppose we are given m/n as well as A_0, B_0. Then, the only unknown in the original differential equation is k. Integration yields another unknown parameter (arising as the additive constant from integration) which can be determined using $C(0) = 0$. We thus get a general expression for $C(t)$ that features k. However, since k is not given, we would like to determine it.

To determine k, we need some other piece of information. In the question here, the additional piece of information is in the form of the value $C(1)$. Specifically, we *plug in* the value $C(1)$ in the general expression for $C(t)$ and *solve the corresponding equation* to determine the value of k. Having obtained this value of k, we *plug back* in the general expression for $C(t)$.

The use of a point-in-time measurement to determine k is analogous to many experimental approaches in physics and chemistry. In most of these, a general physical or chemical law tells us that the expression for a quantity is of some type, with one or more parameters appearing. These parameters are physical constants but we do not have the theoretical tools to determine them. So, we do an experiment that measures certain things, from which we deduce the value of the parameter.

For instance, in physics, there are some laws of friction between dry bodies, which explain how the magnitude of the friction force between two bodies depends on a certain dimensionless constant called the coefficient of friction, that depends on the surfaces in contact. However, the coefficient of friction cannot be determined theoretically. So, we do an experimental study that measures certain quantities, from which we can deduce the coefficient of friction.