CONTINUOUS AND DISCRETE: THE INTERPLAY

MATH 153, SECTION 55 (VIPUL NAIK)

This is some additional material that does not directly correspond to material in the book but is helpful for perspective.

Executive summary

Words ...

(1) Given a function on $\mathbb{R}$, we can restrict the function to $\mathbb{N}$ and obtain a sequence. This restriction is unique.
(2) Conversely, given a sequence, i.e., a function on $\mathbb{N}$, we can extend it to a continuous function on $\mathbb{R}$. However, the extension is not unique, and there are a lot of different ways of extending. If the sequence is described by means of a nice closed form functional expression, we may be able to extend it by considering that functional expression for all real numbers.
(3) Usually, information about the function on the reals gives us corresponding information about the corresponding sequence, but we cannot get information in the reverse direction that easily. For instance, an increasing function gives an increasing sequence, but increasing sequences can arise from functions that are not increasing. A decreasing function gives a decreasing sequence, a monotonic function gives a monotonic sequence, and a bounded function gives a bounded sequence.
(4) A function with integer period gives a periodic sequence.
(5) The mean value theorem relates the derivative of a function to the discrete derivative (i.e., forward difference operator) of the corresponding sequence.
(6) We can define a notion of concave up and concave down for sequences based on the second discrete derivative. If a function is concave up, so is the corresponding sequence. If the function is concave down, so is the corresponding sequence.

1. Filling the holes: continuating the discrete

In the previous lecture or part of lecture, we developed the analogy between the calculus apparatus that we developed for continuous functions and the study of sequences. The analogy was illuminative, but is it more than just an analogy? Specifically, we know that:

(1) *Continuous to discrete*: Any function on the reals can be domain-restricted to give a function on the natural numbers, i.e., a sequence.
(2) *Discrete to continuous*: If a function on the natural numbers has a nice enough closed form expression (such as a polynomial) that closed form expression can be extended to more real numbers.

Some natural questions are: how does the discrete derivative (i.e., the forward difference operator) relate to its continuous counterpart for the same function? How does the discrete integral (which is basically just a summing up) relate to its continuous counterpart? The answers to these questions, while not completely satisfying, are useful.

1.1. **Filling the gaps.**

Recall that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Note that if a function is defined on a *bigger* set among these, it can always be restricted to a smaller set in a unique manner – just restrict inputs to that smaller set. *Extending* functions is a more interesting question: can a function on a smaller set be extended to a bigger set in a nice way? The answers are as follows:

(1) If f is a function on $\mathbb{Q}$, there is at most one way of extending f *continuously* to all of $\mathbb{R}$. The uniqueness of the extension arises from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, so where a particular real number goes is determined by where its rational neighbors go. However, it may not always be

possible to extend. For instance, $f(x) := 1/(x^2 - 2)$ is defined and continuous on $\mathbb{Q}$, but cannot be extended to the point $\sqrt{2} \in \mathbb{R}$ continuously.

(2) If f is a function on $\mathbb{Z}$ (or on $\mathbb{N}$), there are many ways of extending it to a continuous function on $\mathbb{R}$. Basically, all we have to do is join up the function values at consecutive integers by continuous curves. With a few kindergarten lessons in smooth drawing, we can even make sure that the function on $\mathbb{R}$ that we obtain is *continuously differentiable*, and with more practice yet, we can ensure that the function that we obtain is *infinitely differentiable*. Basically, there is a lot of freedom.

Since there are infinitely many ways of extending functions on $\mathbb{N}$ to continuous and even to infinitely differentiable functions on $\mathbb{R}$, we are faced with a paradox of choice. Nonetheless, in most cases, the nicest closed form expression for the function on $\mathbb{N}$ suggests an obvious extension to $\mathbb{R}$. Thus, for a polynomial sequence, the extension to a *polynomial* function on $\mathbb{R}$ is unique.

In some cases, we need to be more ingenuous in finding a natural extension to $\mathbb{R}$. Luckily, calculus has provided us with enough tools to find these functions in many cases. We discuss some examples.

1.2. **Exponential and superexponential sequences.** Consider the sequence:

$$1, 2, 4, 8, 16, 32, \ldots$$

The recurrence relation is clear:

$$a_n = 2a_{n-1}, \qquad a_1 = 1$$

We can also see that a closed form expression is:

$$a_n = 2^{n-1}$$

This kind of sequence is termed a *geometric sequence* or *geometric progression* because the quotient of successive terms is constant. It is a solution to the discrete differential equation $a_n = 2a_{n-1}$, whose continuous analogue is $y' = ky$. Both the continuous and the discrete versions result in exponential growth.

Note that in this case, the solution can be extended to all real numbers, via the function:

$$f(x) := 2^{x-1}$$

But we were able to do this *only because we had developed a prior theory of exponentiation* with arbitrary positive bases and arbitrary real exponents. Had we not developed such a theory, it would not have been clear how we could extend the function.

Consider instead the function:

$$-1, 1, -1, 1, \ldots$$

The n^{th} term is given by:

$$a_n = (-1)^n$$

Now, this particular expression *cannot* be extended to all real numbers, because $(-1)^x$ does not make sense for arbitrary x. However, there is another way of defining the n^{th} term, namely, as:

$$a_n = \cos(n\pi)$$

With this expression, we can extend it to all real numbers, as the function:

$$f(x) := \cos(\pi x)$$

This extension is not *natural* in any strong sense of the word. Other possible etensions to the reals include $\cos(3\pi x)$, $\cos(5\pi x)$ and $\cos^3(\pi x)$.

Consider another example:

$$1, 2, 6, 24, 120, 720, 5040, \ldots$$

The recurrence relation here is:

$$a_n = na_{n-1}, \qquad a_1 = 1$$

Note that this is *not* autonomous. We solve it to obtain that:

$$a_n = n \cdot (n-1) \cdot \ldots 2 \cdot 1 =: n!$$

We thus have a nice expression for a_n. However, unlike the previous case, it is not clear how this can be extended to a function on all real numbers, or even on *anything* that is not a positive integer. The reason is that *we have not pondered this question before*. However, a recent homework problem you did shows that:

$$\int_0^\infty x^n e^{-x} = n!$$

This actually gives a way of extending the function to all positive real numbers, and in fact, to all real numbers greater than -1. Basically, although the *right side* makes sense only for positive integers, the *left side* makes sense in a much broader context. This is closely related to the gamma function which is a mainstay of analysis and statistics, defined as $\Gamma(a) := (a-1)!$. We will not delve more into it, except to remark that $\Gamma(1/2) = (-1/2)! = \sqrt{\pi}$, and that this can be derived from the results about the integral of e^{-x^2}.

2. Back and forth between discrete and continuous

2.1. Continuous and discrete: relationship of the functions.

2.1. Continuous and discrete: relationship of the functions. When we restrict the domain to the natural numbers for a function defined on the reals, then what we are really doing is taking a very restrictive snapshot of the function. Some observations:

(1) If f is increasing on the reals, its restriction to the natural numbers is increasing. However, the converse does not hold. In other words, a function may be increasing on $\mathbb{N}$ but may be a lot more desultory on the real numbers. For instance, consider the function $f(x) := x - \sin(\pi x)$. Its restriction to the natural numbers gives the sequence $1, 2, 3, \ldots$ which is increasing. However, the function is not increasing throughout $\mathbb{R}$, because the derivative $f'(x) = 1 - \pi\cos(\pi x)$ takes both positive and negative values. An analogous statement holds for decreasing functions, and analogous examples work.

(2) If $\lim_{x\to\infty} f(x)$ exists, then the limit of the restriction to natural numbers, i.e., $\lim_{n\to\infty} f(n)$ exists and is equal to $\lim_{x\to\infty} f(x)$. However, the existence of the limit $\lim_{n\to\infty} f(n)$ for $n \in \mathbb{N}$ does *not* imply the existence of the limit for all x. For instance, consider the function $f(x) := \sin(\pi x)$. This is a *constant* function with value 0 when restricted to $\mathbb{N}$ and hence limits to 0. On the other hand, the function on $\mathbb{R}$ is periodic and oscillatory and has no limit.

2.2. Continuous and discrete: relationship of the derivatives. We now turn to the question of how the derivative of a continuous function is related to the discrete derivative (i.e., forward difference operator) of its restriction to $\mathbb{N}$ (or $\mathbb{Z}$).

The forward difference operator is a coarse measure of the average change over an interval of length 1 (from n to $n+1$). The derivative, on the other hand, is an *instantaneous* rate of change at a given point. The forward difference operator is thus the *average* value of the derivative over the interval from n to $n+1$. We can thus apply the mean value theorem, and conclude that, if f is continuous on $[n, n+1]$, and differentiable on $(n, n+1)$, then $(\Delta f)(n)$ equals the value $f'(c)$ for some $c \in (n, n+1)$.

Similar to the way we made observations in the previous subsection, we make some observations here:

(1) Suppose f has a fixed sense of concavity on $[n, n+1]$, i.e., f' is either increasing throughout the interval or decreasing throughout the interval. This forces that $(\Delta f)(n)$ lies between $f'(n)$ and $f'(n+1)$.

(2) If the continuous function f has a certain sense of concavity, then its discrete version also has the same sense of concavity. We say that a function on $\mathbb{N}$ is *concave up* if $\Delta^2 f(n) > 0$ for all n, and is *concave down* if $\Delta^2 f(n) < 0$ for all n.

(3) If f is a differentiable function and $\lim_{x\to\infty} f'(x) = L$ for some finite L, then $\lim_{n\to\infty}(\Delta f)(n) = L$ as well. In particular, if $\lim_{x\to\infty} f'(x) = 0$, then $\lim_{n\to\infty} f(n+1) - f(n) = 0$.

In all these cases, *universal constraints* on the continuous versions give, via some averaging procedures, corresponding constraints on the discrete versions. It is usually harder to go form universal constraints on the discrete version to corresponding constraints on the continuous version, because there is too much slack.

2.3. **Continuous and discrete: periodic functions.** Here are some easy facts:

(1) If a continuous function is periodic with period a positive integer, then the corresponding discrete function is also periodic with period at most that positive integer. (The discrete function could repeat at even shorter intervals).
(2) Given a discrete periodic function with period k, we can write it as a linear combination of a bunch of continuous periodic functions all arising from trigonometry. The details of this are beyond the current scope and have to do with Fourier analysis. An illustrative example is $k = 3$: any periodic sequence with period 3 can be expressed as a linear combination of the functions $\sin(2\pi n/3)$, $\cos(2\pi n/3)$ and the constant function 1. The coefficients for the linear combination can be determined by solving a system of linear equations.

2.4. **Continuous and discrete: integration.** [We may not get time to cover this in class right now, but will get back to it later anyway.]

We have already seen this, albeit without a lot of thoughtful reflection. Again, things are easiest to see when f is a monotonic function, though some of the observations carry over to functions of bounded variation.

Suppose we are looking at a continuous function f defined on $[1, n]$. We can now take a partition of the interval $[1, n]$: $1 < 2 < 3 < \cdots < n$. Let's assume that f is increasing on $[1, n]$. Then, the lower sum of f for the partition is $f(1) + f(2) + \dots f(n-1)$. The upper sum is $f(2) + f(3) + \cdots + f(n)$. The integral is between these, and we obtain:

$$f(1) + f(2) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + \cdots + f(n)$$

On the other hand, if f is decreasing, we get:

$$f(1) + f(2) + \cdots + f(n-1) > \int_1^n f(x)\,dx > f(2) + \cdots + f(n)$$

Now, suppse f is a decreasing function, and suppose further that $\int_1^\infty f(x)\,dx$ is finite. Then, taking the limit as $n \to \infty$ in the above, we get:

$$f(1) + f(2) + \cdots > \int_1^\infty f(x)\,dx > f(2) + f(3) + \dots$$

Rearranging, we obtain that the sum:

$$f(1) + f(2) + f(3) + \dots$$

is between $\int_1^\infty f(x)\,dx$ and $f(1) + \int_1^\infty f(x)\,dx$.

This procedure actually allows us to calculate sums using integrals. Specifically, in cases where an infinite integral is easy to compute but an infinite sum is not, the infinite sum can be computed approximately using the computation of the infinite integral. A modification of this procedure, that we will look at later, allows us to compute the infinite sum to a very high degree of precision using a combination of integral calculations and finite sums.