ABSOLUTE AND CONDITIONAL CONVERGENCE

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 12.5.

What students should definitely get: The distinction between absolute and conditional convergence for series with both positive and negative terms. The fact that for a conditionally convergent series, different rearrangements can produce different sums. The statement about convergence of series with terms of alternating signs whose magnitudes monotonically go to zero.

Executive summary

Words ...

(1) *Not for discussion*: When discussing the convergence of a series, we can throw out all the terms that are zero.
(2) *Not for discussion*: A series $\sum a_k$ is termed *absolutely convergent* if the series $\sum |a_k|$ converges. Note that for a series of nonnegative terms, being absolutely convergent is equivalent to being convergent.
(3) Suppose a series $\sum a_k$ is absolutely convergent. Then, the positive terms converge (say, to P) and the negative terms converge (say, to N). Moreover, $\sum a_k$ is the sum $P + N$, and $\sum |a_k| = |P| + |N|$.
(4) If a series is absolutely convergent, then it is convergent and *every rearrangement* of the series converges to the same sum.
(5) If a series is not absolutely convergent but is convergent, it is termed *conditionally convergent*, and the positive terms add up to $+\infty$, the negative terms add up to $-\infty$, and both the positive terms and the negative terms go to 0.
(6) The Riemann series rearrangement theorem states that a series that is conditionally convergent but not absolutely convergent can be rearranged to give any real number as its sum. It can also be rearranged to give a sum of $+\infty$ and it can also be rearranged to give a sum of $-\infty$. It can also be rearranged so that the sequence of partial sums oscillates between any two fixed locations. [Recall that you heard some non-symbolic, purely didactic reasoning for this in class. Please review this reasoning from the class notes.]
(7) The alternating series theorem states that a series whose terms have alternating signs, and have magnitudes *monotonically* decreasing to zero, must converge. Moreover, the point to which the series converges is the least upper bound of the decreasing sequence of even-numbered partial sums and the greatest lower bound of the increasing sequence of odd-numbered partial sums. [Recall the interpretation of this in terms of jumping along the number line.]
(8) (Questions: What happens if the magnitudes go to zero but not monotonically? What happens if the series is not alternating? What happens if the magnitudes decrease monotonically but not to zero?)

1. Absolute convergence and its consequences

1.1. **Absolutely convergent series.** A series $\sum a_k$ (where the terms may be positive, negative, or zero) is termed *absolutely convergent* if the series $\sum |a_k|$ converges.

1.2. **The number line and triangle inequalities.** Flash back to when you learned about addition on the number line. Let us use that procedure to understand how we sum up series and what happens when the terms have mixed signs.

Consider a series:

$$\sum_{k=1}^{\infty} a_k$$

We start off at 0. We now move a distance of a_1. More specifically, if a_1 is positive, we move a distance of a_1 to the right. If a_1 is negative, we move a distance of $|a_1| = -a_1$ to the left. if a_1 is zero, we stay put.

We are now at a_1. We look at a_2. If a_2 is positive, we move a distance of a_2 to the right. If a_2 is negative, we move a distance of $-a_2$ to the left. We are now at $a_1 + a_2$. We keep going this way. At the k^{th} stage, we are at the partial sum of the first k terms, and we then traverse a_{k+1} to reach the partial sum of the first $k + 1$ terms.

We now see that:

(1) If all the terms in the series are nonnegative, then the sequence of partial sums is monotonic increasing (i.e., non-decreasing). If the series converges, it converges to the least upper bound of the sequence of partial sums. If the series diverges, this just means that the sequence of partial sums monotonically hops off to ∞.
(2) If there are mixed signs among the terms in the series, then we keep going up and down (which, on the number line, means right and left). A special case is where the terms have alternating signs. In this case, the partial sum alternates between right moves and left moves. For the series to converge, the magnitude of these moves must keep getting smaller and smaller, so that we eventually get to some point in between. We can think of the partial sums as a swinging pendulum whose successive oscillations get smaller and smaller.
(3) In general, a series with mixed signs can go forward a few steps and back a few steps. It could converge if, in all this back and forth, it is still zeroing in on some particular point. It could be oscillating between finite limits if its magnitude of oscillations never gets smaller than a certain amount. It could be oscillatorily diverging to $\pm\infty$ if the oscillations get bigger and bigger and extend farther and farther in both directions.

1.3. **Remarkable results.** Here now are some of the results:

(1) An absolutely convergent series is convergent.
(2) More remarkably, *every rearrangement* of an absolutely convergent series is convergent, and they all converge to the same limit.
(3) Suppose a series (eventually) has alternating terms and the magnitudes of the terms (eventually) *monotonically* decrease to 0. Then, the series is convergent. *Recall that for any series, whether it has positive, zero, or negative terms, a necessary but not sufficient condition for convergence is that the terms must approach zero.*
(4) A series that is convergent but not absolutely convergent can be rearranged (i.e., its terms permuted) to give *any specified real sum*, and can also be rearranged to go to $+\infty$ or to go to $-\infty$. It can also be rearranged to oscillate between *any two specified finite limits*. These results are called the *Riemann series rearrangement theorem*.

In the next section, we explain each of these results.

2. Justifications

2.1. **Justification for (1) and (2).** Absolutely convergent series are convergent and the sum doesn't depend on how we arrange the terms. Why does this hold?

Let's take an absolutely convergent series and separate it into two parts: the positive terms and the negative terms. The zero terms, if they existed, can be discarded without affecting the sum. Since the original series is absolutely convergent, the positive terms sum up to a finite number and the negative terms sum up to a finite number. Moreover, if we add the absolute values of these, we get the sum of the absolute values of all terms. The whole series adds up to the total of the sum of the positive sum and the negative sum. Moreover, this is independent of the order in which we arrange the terms.

For instance, if the positive terms all add up to 13 and the negative terms all add up to -9, the absolute values add up to $|13| + |-9| = 22$. The terms of the series add up to $13 + (-9) = 4$.

A real world analogy might help. Suppose you have a total revenue of 13 (across many items) and a total expenditure of 9 (across many items). You can choose the order in which you earn the various revenue items and undertake the various expenditure items. The partial sum at any time reflects your current balance. If you eventually exhaust all the revenue and expenditure items, your final balance will be 4, regardless of the order in which you earn and spend.

2.2. **Justification for (3).** This is seen using the hopping on the number line. The partial sums are oscillating with smaller and smaller magnitudes. The lower ends of the oscillation form an increasing sequence and the upper ends of the oscillation form a decreasing sequence. The distance between these sequences keeps getting smaller, and goes to zero, because the terms are going to zero. Thus, the least upper bound of the increasing sequence of lower ends of oscillation equals the greatest lower bound of the decreasing sequence of upper ends of oscillation, and this also turns out to be the series sum.

2.3. **Justification for (4).** (4) is the most interesting, since it says that if a series is convergent but not absolutely convergent, its sum is highly sensitive to the ordering of the terms. In fact, a suitable rearrangement can generate just about any sum conceivable. What's going on here? How does the situation differ fundamentally from (1) and (2)?

Let's once again break up the series into positive and negative parts. In the absolutely convergent case, both parts had finite sums. If it is not absolutely convergent, at least one of the parts must have an infinite sum. But if just one part has an infinite sum, then the whole sum is also infinite. Since we are ending up with a finite sum at the end, *both* sums must be infinite. Specifically, the positive series sums up to $+\infty$ and the negative series sums up to $-\infty$. Also, the magnitudes of the terms in both series must still go to 0.

Since both these parts go to infinities of opposite signs, the order in which we pick the terms matters. Let us look once again at a credit/debit analogy. Suppose the potential revenue you can earn from all your revenue sources is infinite and so is the potential expenditure you can incur from all your potential expenditure sinks. You can adopt an *earn enough to binge strategy* – you see the biggest expenditure you can make, then earn enough of your biggest revenue items to be able to make that expenditure, then make the expenditure, then again earn enough to make the next expenditure on the list, and so on. You can convince yourself that this way, your balance will eventually get closer and closer to zero, because both your biggest possible expenditures and your biggest possible revenue items will get smaller and smaller as you proceed.

Or you can be a *prudent saver* – earn a lot, and spend a little. You may choose to earn 100 units (using the biggest revenue items) and then spend the biggest revenue items that fit within 10 units (so you build savings of about 90 units). Then, again, you earn 100 units, and spend 10, and so on. This way, you build to an eventual saving of ∞. Since the *total* amount that can be earned and the *total* amount that can be spent is infinite, the time line choices about when to earn and when to spend affect your overall savings (or debt, as the case may be), even if you *eventually* earn everything and spend everything.

Incidentally, Ponzi schemes are based on this sort of idea – the idea that with infinite revenue and infinite expenditures, you can make an infinite profit, by suitably ordering the points in time when you acquire revenue and expenditure. And they crash in the real world because, of course, things are *not* infinite.