TRIGONOMETRY REVIEW (PART 1)

MATH 152, SECTION 55 (VIPUL NAIK)

Difficulty level: Easy to moderate, given that you are already familiar with trigonometry.

Covered in class?: Probably not (for the most part). Some small segments may be covered in class or in problem session if it helps with some problems. Please go through this if you experience difficulties while doing trigonometry problems.

Corresponding material in the book: Section 1.6 (part).

Corresponding material in homework problems: Homework 1, advanced homework problem 6.

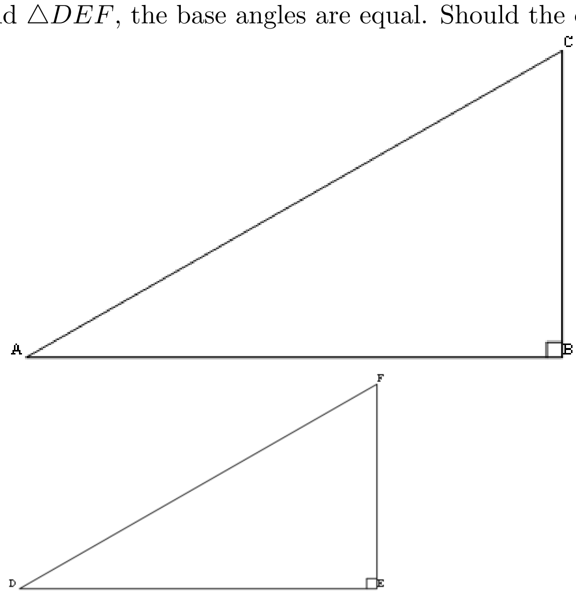
1. Trigonometric functions for acute angles

Earlier we talked about the fact that sometimes you know that something is a function (because it sends every element in the domain to a unique output – and satisfies the condition that equal inputs give equal outputs) but you don't have an expression for it. It's like you know somebody is a person but you don't know that person's name. So what do you do? You just make up a name. Well, that's what we're going to do.

So, let's consider an angle, that I'll call θ, and assume that θ is *strictly* between 0 and $\pi/2$ (90°, a right angle). By the way, the word *strictly* when used in mathematics means that the equality case (the *trivial* or *degenerate* case) is excluded. So, in this case, it means $0 < \theta < \pi/2$. The high school term for an angle strictly between 0 and $\pi/2$ is *acute angle*.

So now I define the following function whose domain is the set of acute angles. $f(\theta)$ is the ratio of the height of a right-angled triangle with base angle θ to the hypotenuse of that triangle. In other words, it is the ratio of the opposite side to the angle θ to the hypotenuse.

So, you may say, why is this a function at all? Why does it make sense? There are infinitely many triangles of different sizes with base angle θ. Could different choices of triangle give different values of $f(\theta)$? And if so, doesn't that undermine the claim that f is a function? For instance, in the two triangles $\triangle ABC$ and $\triangle DEF$, the base angles are equal. Should the corresponding side ratios also be equal?

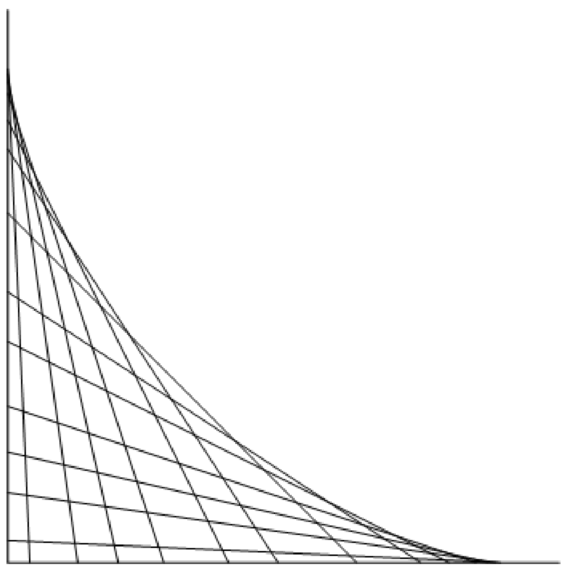


By the way, this first question that I asked is the kind of question you should ask whenever a *function* is defined in a roundabout manner, with some arbitrary choices in between. People often phrase this as *is the function well-defined?* though a more precise formulation is: *is the so-called function a function at all?*

In this case, the answer is *yes*, and the reason is the notion of *similarity* of triangles. For those of you who've taken some high school geometry, you've probably seen this notion. For those who haven't, the idea is that if two triangles have the same angles, they essentially have the same *shape*, even if they have different *sizes*, so the *ratios* of side lengths are the same.

So f is a function. But so what? Do we have an expression for it? Well, yes and no. There's no expression for f as a polynomial or rational function, because it isn't that kind of function. But we can give f its own name, and then we'll be happy. So what's a good name? f? No, f, is too plain and all too common. We need a special name. The name we use is sine, written *sine* in English and sin in mathematics. So $f(\theta)$ is written as $\sin(\theta)$. By the way, when the sin is being taken of a single letter variable or constant, we don't usually put parentheses. So we just say $\sin\theta$.

Okay, so what is the domain of the sin function as we've defined it? It is $(0, \pi/2)$. What is the range? In other words, what are the possible values that $\sin\theta$ can take? Well, think about it this way. Think of a ladder that you have placed with one end touching a vertical wall and the other end on the floor. Now, imagine this ladder sliding down. When the ladder becomes horizontal, the triangle has base angle of zero. When it's almost touched down, the base angle is really small and the opposite side is, too. So $\sin\theta \to 0$ as $\theta \to 0$. That $\to$ here means *tends to* or *approaches* – it's a concept we'll be looking at in more detail when we do limits.



At the other extreme, when the ladder is almost upright, the opposite side is almost equal to 1, so $\sin\theta \to 1$ as $\theta \to \pi/2$. So what we're seeing is that sin is an increasing function starting off from just about the right of zero and ending at just about the left of 1. So the range of this function is $(0, 1)$.

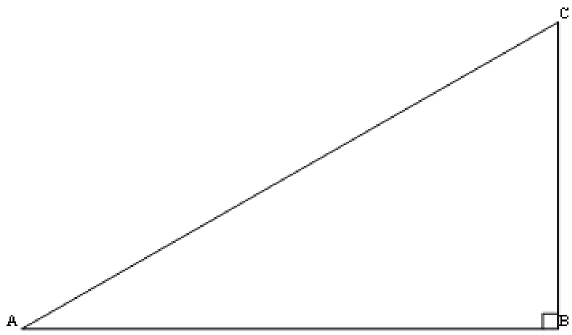
So here's one more point that is worth thinking about. In high school, if you started looking at trigonometric functions before the radian measure was introduced, then you might have seen that angles are denoted differently, e.g., by Greek letters. Why's that? Well, one reason to think of that is that angles aren't ordinary numbers. They are measurements, and denoting an angle by a common letter like x is debasing, because angles come in *degrees.* But after you switch to the radian measure, an angle (in radians) is just any old real number. So we feel free to use x and y to denote angles. We've demystified angles.

Now, let's recall the definitions of cosine. The cosine, denoted cos, is the ratio of the adjacent side to the hypotenuse. So if θ is the base angle of a right triangle, $\cos\theta$ is the ratio of the base to the hypotenuse.

And then there is the tangent function. This is denoted tan, and it is the ratio of the opposite side to the adjacent side. Now, mathematicians can be very creative with naming sometimes but sometimes they just copy names without much deep meaning. So this tangent function doesn't have any deep relation with the concept of *tangent* to a circle or a curve. Yes, they are loosely related, but the relation isn't strong enough to merit the same name. Call that an accident of history.

The other three trigonometric functions are the reciprocals of these. The reciprocal of the sine function is the cosecant function, denoted in mathematical shorthand as csc. The reciprocal of the cosine function is the secant function, denoted in mathematical shorthand as sec. The reciprocal of the tangent function is the cotangent function, denoted in mathematical shorthand as cot.

We summarize the important definitions here.



Next, we define the six trigonometric functions of θ as ratios of side lengths in this triangle:

$$\sin\theta = \frac{\text{Opposite leg}}{\text{Hypotenuse}} = \frac{BC}{AC}$$
$$\cos\theta = \frac{\text{Adjacent leg}}{\text{Hypotenuse}} = \frac{AB}{AC}$$
$$\tan\theta = \frac{\text{Opposite leg}}{\text{Adjacent leg}} = \frac{BC}{AB}$$
$$\cot\theta = \frac{\text{Adjacent leg}}{\text{Opposite leg}} = \frac{AB}{BC}$$
$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent leg}} = \frac{AC}{AB}$$
$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite leg}} = \frac{AC}{BC}$$

2. Relation between trigonometric functions

The six trigonometric functions are related via three broad classes of relationships. Each of these relationships pairs up the six trigonometric functions into three pairs. We discuss each of these pairings.

2.1. **Complementary angle relationships.** The right triangle $\triangle ABC$ has two acute angles. The ratios of side lengths of this triangle give the trigonometric function values for both acute angles. However, a leg that's opposite to one angle becomes adjacent to the other. Thus, the trigonometric functions for $(\pi/2) - \theta$ are related to the trigonometric functions for θ as follows:

$$\begin{aligned}
\sin((\pi/2) - \theta) &= \cos\theta \\
\cos((\pi/2) - \theta) &= \sin\theta \\
\tan((\pi/2) - \theta) &= \cot\theta \\
\cot((\pi/2) - \theta) &= \tan\theta \\
\sec((\pi/2) - \theta) &= \csc\theta \\
\csc((\pi/2) - \theta) &= \sec\theta
\end{aligned}$$

The prefix *co-* indicates a complementary angle relationship. Thus, the functions sine and *co*sine have a complementary angle relationship. The functions tangent and *co*tangent have a complementary angle relationship. The functions secant and *co*secant have a complementary angle relationship.

It is an easy but useful exercise to verify the complementary angle relationships from the definitions of the trigonometric functions.

2.2. **Reciprocal relationships.** Reciprocal relationships between the trigonometric functions are as follows:

(1) $\sin\theta$ and $\csc\theta$ are reciprocals. In other words, $(\sin\theta)(\csc\theta) = 1$ for all acute angles θ.
(2) $\cos\theta$ and $\sec\theta$ are reciprocals. In other words, $(\cos\theta)(\sec\theta) = 1$ for all acute angles θ.
(3) $\tan\theta$ and $\cot\theta$ are reciprocals. In other words, $(\tan\theta)(\cot\theta) = 1$ for all acute angles θ.

It is an easy but useful exercise to verify the complementary angle relationships from the definitions of the trigonometric functions.

2.3. **Square sum and difference relationships.** These are the trickiest and the most important of the relationships. We consider the most important of these first: the square sum relationship between sin and cos.

By the Pythagorean theorem for the right triangle $\triangle ABC$ with the angle at B being the right angle, we have:

$$AB^2 + BC^2 = AC^2$$

Or, in terms of the angle θ:

$$(\text{Adjacent leg})^2 + (\text{Opposite leg})^2 = (\text{Hypotenuse})^2$$

Dividing both sides by AC^2, we obtain:

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

Simplifying, we obtain:

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

Recall that $\cos\theta = AB/AC$ and $\sin\theta = BC/AC$, so we get:

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

With trigonometric functions, it is a typical convention to write the exponent before the angle, so we write $(\cos\theta)^n$ as $\cos^n\theta$. Using this convention, we can rewrite the above relationship as:

$$\cos^2\theta + \sin^2\theta = 1$$

Two other square sum and difference relationships of importance are:

$$\begin{aligned}\tan^2\theta + 1 &= \sec^2\theta \\ \cot^2\theta + 1 &= \csc^2\theta\end{aligned}$$

It is a good exercise to prove both of these using the Pythagorean theorem.

2.4. **Everything in terms of** sin **and** cos**.** It is often useful to deal with sin and cos only, so it is helpful to know how to write the other trigonometric functions in terms of sin and cos. The expressions are given below:

It is a good exercise to verify that these expressions are correct using the definitions of the trigonometric functions.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \cot\theta &= \frac{\cos\theta}{\sin\theta} \\ \sec\theta &= \frac{1}{\cos\theta} \\ \csc\theta &= \frac{1}{\sin\theta}\end{aligned}$$

2.5. **Everything in terms of** sin **or** cos**.** Finally, when it comes to acute angles, we can write all the trigonometric functions in terms of sin alone or cos alone. The key is to use the fact that $\sin^2\theta + \cos^2\theta = 1$. Since both $\sin\theta$ and $\cos\theta$ are positive for an acute angle θ, we can use this to get the expressions:

$$\begin{aligned}\cos\theta &= \sqrt{1-\sin^2\theta} \\ \sin\theta &= \sqrt{1-\cos^2\theta}\end{aligned}$$

Once we have this, we can get expressions for all the other trigonometric functions in terms of sin. We can also get expressions for all the other trigonometric functions in terms of cos.

We give here all the expressions in terms of sin. In all these, we just take the previos expressions and replace every occurrence of $\cos\theta$ by $\sqrt{1-\sin^2\theta}$.

$$\begin{aligned}\cos\theta &= \sqrt{1-\sin^2\theta} \\ \tan\theta &= \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \\ \cot\theta &= \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \\ \sec\theta &= \frac{1}{\sqrt{1-\sin^2\theta}} \\ \csc\theta &= \frac{1}{\sin\theta}\end{aligned}$$

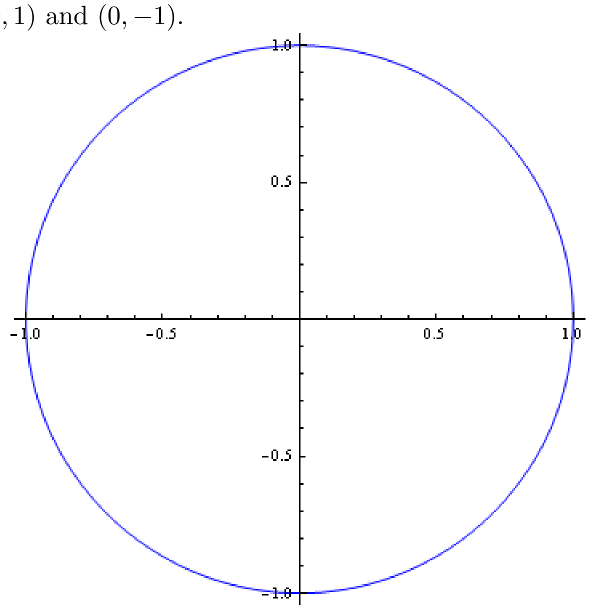
You can work out how the other five trigonometric functions look in terms of $\cos\theta$ by yourself.

Note that these expressions are valid only for acute angles. The problem is that for bigger angles, as we shall soon see, the value of sin or cos can be negative, so even thought $\sin^2\theta + \cos^2\theta = 1$, we cannot write $\sin\theta = \sqrt{1-\cos^2\theta}$ because the $\sqrt{}$ symbol always gives a nonnegative output and $\sin\theta$ may well be negative.

3. Unit circle trigonometry

3.1. **The unit circle.** The unit circle centered at the origin is defined as the set of points (x, y) in the coordinate plane that satisfy $x^2 + y^2 = 1$. This is a circle of radius 1 centered at the origin.

The unit circle intersects the x-axis at the points $(1, 0)$ and $(-1, 0)$. It intersects the y-axis at the points $(0, 1)$ and $(0, -1)$.



3.2. **Sine and cosine using the unit circle.** Suppose θ is an angle. We define $\sin\theta$ and $\cos\theta$ using the unit circle as follows. Start on the unit circle at the point $(1, 0)$. Move an angle of θ in the counter clockwise direction on the unit circle. Call the point you finally reach (x_0, y_0). Then, $\cos\theta$ is defined as x_0 and $\sin\theta$ is defined as y_0.

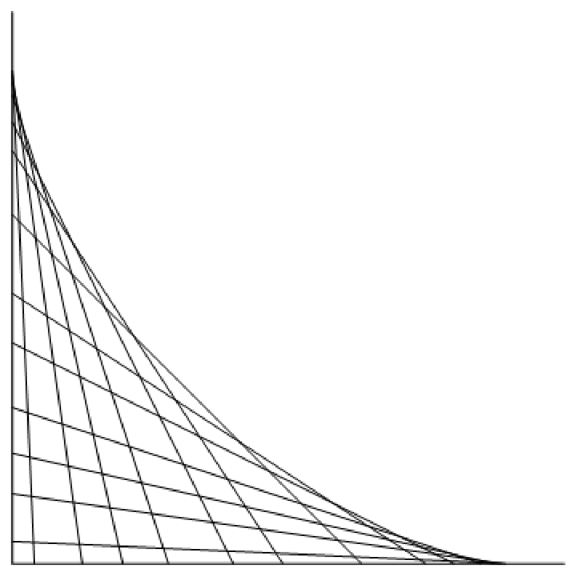
When θ is an acute angle, then the point (x_0, y_0) is in the first quadrant. We see that the definitions of $\cos\theta$ and $\sin\theta$ match the definitions we gave earlier in terms of triangles.

4. Qualitative behavior of sine and cosine

4.1. **For acute angles.** Note that prior to the introduction of unit circle trigonometry, we defined sin and cos only for acute angles. We first discuss the qualitative behavior of these functions for acute angles, but also include the limiting cases of 0 and $\pi/2$. After that, we discuss the behavior for obtuse angles.

For acute angles, we have the following:

(1) sin is a strictly increasing function for acute angles, starting off with $\sin 0 = 0$ and $\sin(\pi/2) = 1$. This can be seen graphically in many ways. For instance, imagine a ladder that is initially vertical along a wall, and gradually slides down. The angle that the foot of the ladder makes with the floor decreases from $\pi/2$ to 0, and the vertical height of the top of the ladder also decreases. [More class discussion on this]



(2) cos is a strictly decreasing function for acute angles, starting off with $\cos 0 = 1$ and $\cos(\pi/2) = 0$. The fact that the behavior of cos is the mirror opposite of that of sin is not surprising – this essentially follows from the complementary angle relationship.

(3) tan is a strictly increasing function for acute angles, starting off with $\tan 0 = 0$. $\tan(\pi/2)$ is not defined, and as an acute angle θ comes closer and closer to being $\pi/2$, $\tan\theta$ approaches ∞. (The precise meaning and explanation of this statement involve familiarity with ideas of limits, which are beyond the current scope of discussion).
(4) cot is a strictly decreasing function for acute angles, and its behavior mirrors that of tan because of the complementary angle relationship. $\cot 0$ is undefined and $\cot(\pi/2) = 0$.

4.2. **For angles between 0 and π.** Using the unit circle trigonometry definition, we can see that:

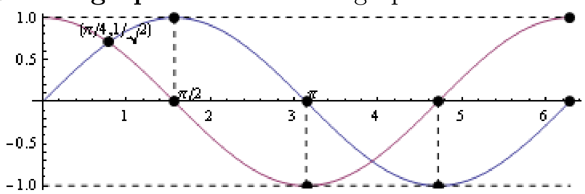
$$\begin{aligned}\sin(\pi - \theta) &= \sin\theta \\ \cos(\pi - \theta) &= -\cos\theta\end{aligned}$$

In particular, sin is positive for all angles θ that are strictly between 0 and π, including both acute and obtuse angles. In terms of unit circle trigonometry, this is because the first half of the unit circle if we go counter-clockwise from $(1, 0)$ has positive y-coordinate. On the other hand, cos is negative for obtuse angles, and this can again be seen from the unit circle trigonometry.

Note that $\tan\theta$ is defined as $\sin\theta/\cos\theta$ wherever $\cos\theta \neq 0$, so we get the formula:

$$\tan(\pi - \theta) = -\tan\theta$$

4.3. **The graphs.** Here are the graphs of the sine and cosine function from 0 to 2π.



We will study the graphs of the other trigonometric functions later.

4.4. **General facts: value taking.**

(1) sin takes its peak value of 1 at numbers of the form $2n\pi + (\pi/2)$, where n is an integer, and its trough values of -1 at numbers of the form $2n\pi - (\pi/2)$, where n is an integer. sin takes the value 0 at multiples of π, i.e., numbers of the form $n\pi$ where n is an integer.
(2) The solutions to $\sin x = \sin\alpha$ come in two families: $x = 2n\pi + \alpha$ and $x = 2n\pi + (\pi - \alpha)$, with n ranging over the integers.
(3) cos takes its maximum value of 1 at multiples of 2π, i.e., numbers of the form $2n\pi$, n ranging over integers. It takes its minimum value of -1 at odd multiples of π, i.e., numbers of the form $(2n+1)\pi$, n ranging over integers. it takes the value 0 at odd multiples of $\pi/2$, i.e., numbers of the form $n\pi + (\pi/2)$, n ranging over integers.
(4) The solutions to $\cos x = \cos\alpha$ come in two families: $x = 2n\pi + \alpha$ and $x = 2n\pi - \alpha$, n ranging over integers. These two families can be combined compactly by writing the general expression as $x = 2n\pi \pm \alpha$.

4.5. **Even, odd, mirror symmetry, half turn symmetry.** Here are some general facts about the sine and cosine functions:

(1) The sine function is an *odd function*, i.e., $\sin(-\theta) = -\sin\theta$. In particular, the graph of the sine function enjoys a half turn symmetry about the origin. In fact, the graph of the sine function enjoys a half turn symmetry about any point of the form $(n\pi, 0)$.
(2) The graph of the sine function enjoys mirror symmetry about any vertical line through a peak or trough, i.e., any line of the form $x = n\pi + (\pi/2)$.
(3) The cosine function is an *even function*, i.e., $\cos(-\theta) = \cos\theta$. In particular, the graph of the cosine function enjoys a mirror symmetry about the origin. In fact, the graph enjoys a mirror symmetry about all vertical lines through a peak or trough, i.e., any line of the form $x = n\pi$.

(4) The graph of the cosine function enjoys a half turn symmetry about any point of the form $(n\pi + (\pi/2), 0)$.
(5) The tangent, cotangent, and cosecant functions are odd functions on their domains of definition. The secant function is even on its domain of definition.

4.6. **Periodicity.** Here are some important facts:

(1) The sine and cosine are periodic functions and they both have period 2π.
(2) The tangent and cotangent are periodic functions and they both have period π.
(3) The secant and cosecant are periodic functions and they both have period 2π.

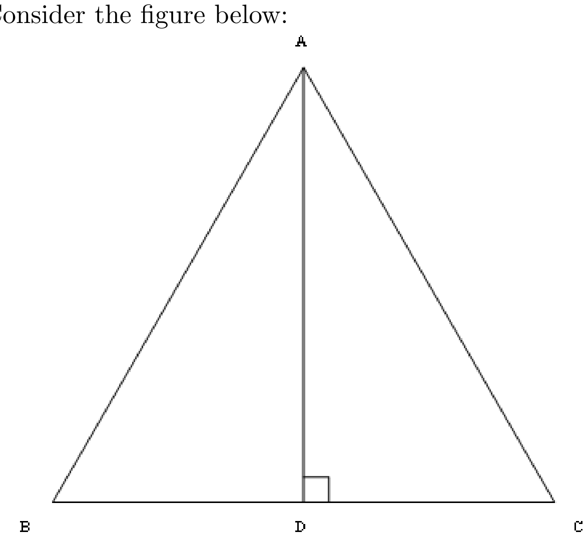
5. Values of trigonometric functions for important angles

5.1. $\pi/4$. To determine the values of trigonometric functions for $\pi/4$, we need to examine closely the right isosceles triangle. In this triangle, both legs have the same length, and by the Pythagorean theorem, the length of the hypotenuse is $\sqrt{2}$ times this length. Thus, we obtain the following:

$$\begin{aligned} \sin(\pi/4) = \cos(\pi/4) &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} && \approx 0.707 \\ \tan(\pi/4) = \cot(\pi/4) &= 1 \\ \sec(\pi/4) = \csc(\pi/4) &= \sqrt{2} && \approx 1.414 \end{aligned}$$

Notice something else too. For the angle $\pi/4$, the values of any two trigonometric functions that are related by the complementary angle relationship are equal. Thus, $\sin(\pi/4) = \cos(\pi/4)$, $\tan(\pi/4) = \cot(\pi/4)$, and $\sec(\pi/4) = \csc(\pi/4)$. Geometrically, this is because we are working with a right isosceles triangle, and there is a symmetry between the two legs. Algebraically, this is because the angle $(\pi/4)$ is its own complement, i.e., $(\pi/4) = (\pi/2) - (\pi/4)$.

5.2. $\pi/6$ **and** $\pi/3$. We now consider the triangle where one of the angles is $\pi/6$ and the other angle is $\pi/3$. Consider the figure below:



In the figure, the triangle $\triangle ABC$ is an equilateral triangle and the line AD is an altitude so $AD \perp BC$. Since $\triangle ABC$ is equilateral, all its angles are also equal and hence all the angles are $\pi/3$. Also, some elementary geometry tells us that AD bisects BC so $DC = (1/2)BC$, so $DC/BC = 1/2$.

Now consider the triangle $\triangle ADC$. The angle at C is $\pi/3$ and the angle $\angle CAD$ is $\pi/6$. Thus, we obtain that $\cos(\pi/3) = \sin(\pi/6) = DC/AC$. Since $\triangle ABC$ is equilateral, $AC = BC$, so $DC/AC = DC/BC$, which is $1/2$.

We thus have the first important fact: $\cos(\pi/3) = \sin(\pi/6) = 1/2$. Now, using the relations between trigonometric functions, we can obtain the other values. The full list is given below:

$$\begin{aligned}
\sin(\pi/6) = \cos(\pi/3) &= \frac{1}{2} && = 0.5 \\
\cos(\pi/6) = \sin(\pi/3) &= \frac{\sqrt{3}}{2} && \approx 0.866 \\
\tan(\pi/6) = \cot(\pi/3) &= \frac{1}{\sqrt{3}} && \approx 0.577 \\
\cot(\pi/6) = \tan(\pi/3) &= \sqrt{3} && \approx 1.732 \\
\sec(\pi/6) = \csc(\pi/3) &= \frac{2}{\sqrt{3}} && \approx 1.154 \\
\csc(\pi/6) = \sec(\pi/3) &= 2
\end{aligned}$$

You should be able to reconstruct all these values. You can choose either to memorize all of them, or to memorize the first two rows and reconstruct the rest from them on the spot. Alternatively, you can remember the side length configuration of the triangle $\triangle ADC$ and read off the trigonometric function values by looking at that triangle.