REVIEW SHEET FOR MIDTERM 2: BASIC

MATH 152, SECTION 55 (VIPUL NAIK)

Unlike the previous midterm, I've split the review sheet into two parts: basic and advanced. The basic review sheet contains (with some changes) the executive summaries from the lecture notes.

This is the *basic* part of the review sheet for midterm 2. Hopefully, you will not need to go through this more than once, except the points that give you trouble the first time around.

The advanced part contains the error-spotting exercises, a section on "Graphing and Miscellanea on Functions" and a section on "Tricky Topics" and this is what we will focus on in the live review session. It also contains a "Quickly" list of stuff you should be able to recall and do quickly.

Section numbers from the basic and advanced are matched up so that you can look at both sheets together.

This document does not re-review material already covered in the review sheet for midterm 1. It is your responsibility to go through that review sheet again and make sure you have mastered all the material there. We could cover in the review session some topics there that you are having difficulty with, but that will not be a priority.

1. Left-overs from differentiation basics

1.1. **Derivative as rate of change.** Words...

(1) The derivative of A with respect to B is the rate of change of A with respect to B. Thus, to determine rates of change of various quantities, we can use the techniques of differentiation.
(2) If there are three linked quantities that are changing together (e.g., different measures for a circle such as radius, diameter, circumference, area) then we can use the chain rule.

Most of the actions in this case are not more than a direct application of the words.

1.2. **Implicit differentiation.** Words...

(1) Suppose there is a curve in the plane, whose equation cannot be manipulated easily to write one variable in terms of the other. We can use the technique of implicit differentiation to determine the derivative, and hence the slope of the tangent line, at different points to the curve.
(2) For a curve where neither variable is expressible as a function of the other, the notion of derivative still makes sense as long as *locally*, we can get y as a function of x. For instance, for the circle $x^2 + y^2 = 1$, y is not a function of x, but if we restrict attention to the part of the circle above the x-axis, then on this restricted region, y is a function of x.
(3) In some cases, even when one variable is expressible as a function of the other, implicit differentiation is easier to handle as it may involve fewer messy squareroot symbols.

Actions ...

(1) To determine the derivative using implicit differentiation, write down the equations of both curves, differentiate both sides with respect to x, and simplify using all the differentiation rules, to get everything in terms of x, y, and dy/dx. Isolate the dy/dx term in terms of x and y, and compute it at whatever point is needed.
(2) This procedure can be iterated to compute higher order derivatives at specific points on the curve where the curve locally looks like a function.

2. Increase/decrease, maxima/minima, concavity, inflection, tangents, cusps, asymptotes

2.1. **Rolle's, mean value, increase/decrease, maxima/minima.** Words...

(1) If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$. This is called *Rolle's theorem* and is a consequence of the extreme-value theorem.

(2) If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists $c \in (a, b)$ such that $f'(c)$ is the difference quotient $(f(b) - f(a))/(b - a)$. This result is called the *mean-value theorem*. Geometrically, it says that for any chord, there is a parallel tangent. Another way of thinking about it is that every difference quotient is equal to a derivative at some intermediate point.
(3) If f is a function and c is a point such that $f(c) \geq f(x)$ for x to the immediate left of c, we say that c is a local maximum from the left. In this case, the left-hand derivative of f at c, if it exists, is greater than or equal to zero. This is because the difference quotient is greater than or equal to zero. Local maximum from the right implies that the right-hand derivative (if it exists) is ≤ 0, local minimum from the left implies that the left-hand derivative (if it exists) is ≤ 0, and local minimum from the right implies that the right-hand derivative (if it exists) is ≥ 0. Even in the case of *strict* local maxima and minima, we still need to retain the equality sign on the derivative because it occurs as a *limit* and a limit of positive numbers can still be zero.
(4) If c is a point where f attains a local maximum (i.e., $f(c) \geq f(x)$ for all x close enough to c on both sides), then $f'(c)$, if it exists, is equal to zero. Similarly for local minimum.
(5) A *critical point* for a function is a point where either the function is not differentiable or the derivative is zero. All local maxima and local minima must occur at critical points.
(6) If $f'(x) > 0$ for all x in the open interval (a, b), f is increasing on (a, b). Further, if f is one-sided continuous at the endpoint a and/or the endpoint b, then f is increasing on the interval including that endpoint. Similarly, $f'(x) < 0$ implies f decreasing.
(7) If $f'(x) > 0$ everywhere except possibly at some isolated points (so that they don't cluster around any point) where f is still continuous, then f is increasing everywhere.
(8) If $f'(x) = 0$ on an open interval, f is constant on that interval, and it takes the same constant value at an endpoint where it's continuous from the appropriate side.
(9) If f and g are two functions that are both continuous on an interval I and have the same derivative on the interior of I, then $f - g$ is a constant function.

    Note: Due to an oversight, the remaining items in this list were not included in the executive summary to the lecture notes.
(10) There is a *first derivative test* which provides a sufficient (though not necessary) condition for a local extreme value: it says that if the first derivative is nonnegative (respectively positive) on the immediate left of a critical point, that gives a strict local maximum (respectively local maximum) from the left. If the first derivative is negative on the immediate left, we get a strict local minimum from the left. If the first derivative is positive on the immediate right, we get a strict local minimum from the right, and if it is negative on the immediate right, we get a strict local maximum from the right.

    The first derivative test is similar to the corresponding "one-sided derivative" test, but is somewhat stronger for a variety of situations because in many cases, one-sided derivatives are zero, which is inconclusive, whereas the first derivative test fails us more rarely.
(11) The second derivative test states that if f has a critical point c where it is twice differentiable, then $f''(c) > 0$ implies that f has a local minimum at c, and $f''(c) < 0$ implies that f has a local maximum at c.
(12) There are also higher derivative tests that work for critical points c where $f'(c) = 0$. These work as follows: we look for the smallest k such that $f^{(k)}(c) \neq 0$. If this k is even, then f has a local extreme value at c, and the nature (max versus min) depends on the sign of $f^{(k)}(c)$ (max if negative, min if positive). If k is odd, then we have what we'll see soon is a point of inflection.
(13) To determine absolute maxima/minima, the candidates are: points of discontinuity, boundary points of domain (whether included in domain or outside the domain; if the latter, then limiting), critical points (derivative zero or undefined), and limiting cases at $\pm\infty$.

Actions... (think of examples that you've done)

(1) Rolle's theorem, along with the more sophisticated formulations involving increasing/decreasing, tell us that there is an intimate relationship between the zeros of a function and the zeros of its derivative. Specifically, between any two zeros of the function, there is a zero of its derivative. Thus,

if a function has r zeros, the derivative has at least $r-1$ zeros, with at least one zero between any two consecutive zeros of f.

(2) The more sophisticated version tells us that between any two zeros of a differentiable function, the function must attain a local maximum or local minimum. So, if the function is increasing everywhere or decreasing everywhere, there is at most one zero.

(3) The mean-value theorem allows us to use bounds on the derivative of a function to bound the overall variation, or change, in the function. This is because if the derivative cannot exceed some value, then the difference quotient also cannot exceed that value, which means that the function cannot change too quickly on average.

(4) To determine regions where a function is increasing and decreasing, we find the derivative and determine regions where the derivative is positive, zero, and negative.

(5) To determine all the local maxima and local minima of a function, find all the critical points. To find the critical points, solve $f' = 0$ and also consider, as possible candidates, all the points where the function changes definition. *Although a point where the function changes definition need not be a critical point, it is a very likely candidate.*

(6) *This item was missed in the original executive summary*: To determine absolute maxima and absolute minima, find all candidates (discontinuity, endpoints, limiting cases, boundary points), evaluate at each, and compare. Note that any absolute maximum must arise as a local or endpoint maximum. However, instead of first determining which critical points give local maxima by the derivative tests, we can straightaway compute values everywhere and compare, if our interest is solely in finding the absolute maximum and minimum.

2.2. **Concave up/down and points of inflection.** Words ...

(1) A function is called *concave up* on an interval if it is continuous and its first derivative is continuous and increasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be positive except possibly at isolated points, where it can be zero. (Think x^4, whose first derivative, $4x^3$, is increasing, and the second derivative is positive everywhere except at 0, where it is zero).

(2) A function is called *concave down* on an interval if it is continuous and its first derivative is continuous and decreasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be negative except possibly at isolated points, where it can be zero.

(3) A *point of inflection* is a point where the sense of concavity of the function changes. A point of inflection for a function is a point of local extremum for the first derivative.

(4) Geometrically, at a point of inflection, the tangent line to the graph of the function *cuts through* the graph.

Actions ...

(1) To determine points of inflection, we first find critical points for the first derivative (which are points where this derivative is zero or undefined) and then use the first or second derivative test at these points. Note that these derivative tests are applied to the first derivative, so the first derivative here is the second derivative and the second derivative here is the third derivative.

(2) In particular, if the second derivative is zero and the third derivative exists and is nonzero, we have a point of inflection.

(3) A point where the first two derivatives are zero could be a point of local extremum or a point of inflection. To find out which one it is, we either use sign changes of the derivatives, or we use higher derivatives.

(4) Most importantly, the second derivative being zero does *not* automatically imply that we have a point of inflection.

(5) If the third derivative is zero, we can use a higher derivative test again. The upshot is that if the first $k \geq 2$ for which $f^{(k)}(c) \neq 0$ is even, then we do not have a point of inflection, but if the first k is odd, then we have a point of inflection.

2.3. **Tangents, cusps, and asymptotes.** Words...

(1) We say that f has a horizontal asymptote with value L if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. Sometimes, both might occur. (In fact, in almost all the examples you have seen, the limits at $\pm\infty$, if finite, are both equal).
(2) We say that f has a vertical asymptote at c if $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$. Note that in this case, it usually also happens that $f'(x) \to \pm\infty$ on the relevant side, with the sign the same as that of $f(x)$'s approach if the approach is from the left and opposite to that of $f(x)$'s approach if the approach is from the right. However, this is not a foregone conclusion.
(3) We say that f has a vertical tangent at the point c if f is continuous (and finite) at c and $\lim_{x\to c} f'(x) = \pm\infty$, with the *same sign of infinity* from both sides. If f is increasing, this sign is $+\infty$, and if f is decreasing, this sign is $-\infty$. Geometrically, points of vertical tangent behave a lot likepoints of inflection and usually *do* give points of inflection (in the sense that the tangent line cuts through the graph). Think $x^{1/3}$.
(4) We say that f has a vertical cusp at the point c if f is continuous (and finite) at c and $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are infinities of opposite sign. In other words, f takes a sharp about-turn at the x-value of c. Think $x^{2/3}$.
(5) We say that f is asymptotic to g if $\lim_{x\to\infty} f(x) - g(x) = \lim_{x\to-\infty} f(x) - g(x) = 0$. In other words, the graphs of f and g come progressively closer as $|x|$ becomes larger. (We can also talk of one-sided asymptoticity, i.e., asymptotic only in the positive direction or only in the negative direction). When g is a *nonconstant linear function*, we say that f has an *oblique asymptote*. Horizontal asymptotes are a special case, where one of the functions is a constant function.

Actions...

(1) Finding the horizontal asymptotes involves computing limits as the domain value goes to infinity. Finding the vertical asymptotes involves locating points in the domain, or the boundary of the domain, where the function limits off to infinity. For both of these, it is useful to remember the various rules for limits related to infinities.
(2) Remember that for a vertical tangent or vertical cusp at a point, it is necessary that the function be continuous (and take a finite value). So, we not only need to find the points where the derivative goes off to infinity, we also need to make sure those are points where the function is continuous. Thus, for the function $f(x) = 1/x$, $f'(x) \to -\infty$ on both sides as $x \to 0$, but we do *not* obtain a vertical tangent – rather, we obtain a vertical asymptote.

3. Max-min problems

Words...

(1) In real-world situations, maximization and minimization problems typically involve multiple variables, multiple constraints on those variables, and some objective function that needs to be maximized or minimized.
(2) The only thing we know to solve such problems is to reduce everything in terms of one variable. This is typically done by *using up* some of the constraints to express the other variables in terms of that variable.
(3) The problem then typically boils down to a maximization/minimization problem of a function in a single variable over an interval. We use the usual techniques for understanding this function, determining the local extreme values, determining the endpoint extreme values, and determining the absolute extreme values.

Special note: For integer optimization, please refer back to the lecture notes!

Actions... (think of examples; also review the notes on max-min problems)

(1) Extremes sometimes occur at endpoints but these endpoints could correspond to degenerate cases. For instance, of all the rectangles with given perimeter, the square has the maximum area, and the minimum occurs in the degenerate case of a rectangle where one side has length zero.
(2) Some constraints on the variables we have are explicitly stated, while others are implicit. Implicit constraints include such things as nonnegativity constraints. *Some of these implicit constraints may be on variables other than the single variable in terms of which we eventually write everything.*

(3) After we have obtained the objective function in terms of one variable, we are in a position to throw out the other variables. However, before doing so, it is *necessary to translate all the constraints into constraints on the one variable that we now have.*
(4) When our intent is to maximize a function, it is sometimes useful to maximize an equivalent function that is easier to visualize or differentiate. For instance, to maximize $\sqrt{f(x)}$ is equivalent to maximizing $f(x)$ if $f(x)$ is nonnegative. With this way of thinking about equivalent functions, we can make sure that the actual function that we differentiate is easy to differentiate. The main criterion is that the two functions should rise and fall together. (Analogous observations apply for minimizing) Remember, however, that to calculate the *value* of the maximum/minimum, you should go back to the original function.
(5) Sometimes, there are other parameters in the maximization/minimization problem that are *unknown constants*, and the final solution is expected to be in terms of those constants. In rare cases, the nature of the function, and hence the nature of maxima and minima, depends on whether those constants fall in particular intervals. *If you find this to be the case, go back to the original problem and see whether the real-world situation it came from constrains the constants to one of the intervals.*
(6) For some geometrical problems, the maximization/minimization can be done trigonometrically. Here, we make a clever choice of an angle that controls the *shape* of the figure and then use the trigonometric functions of that angle. This could provide alternate insight into maximization.

4. Definite and indefinite integration

4.1. **Definition and basics.** Words ...

(1) The definite integral of a continuous (though somewhat weaker conditions also work) function f on an interval $[a, b]$ is a measure of the signed area between the graph of f and the x-axis. It measures the *total value* of the function.
(2) For a partition P of $[a, b]$, the lower sum $L_f(P)$ adds up, for each subinterval of the partition, the length of that interval times the minimum value of f over that interval. The upper sum adds up, for each subinterval of the partition, the length of that interval times the maximum value of f on that subinterval.
(3) Every lower sum of f is less than or equal to every upper sum of f.
(4) The *norm* or *size* of a partition P, denoted $\|P\|$, is defined as the maximum of the lengths of its subintervals.
(5) If P_1 is a finer partition than P_2, i.e., every interval of P_1 is contained in an interval of P_2, then the following three things are true: (a) $L_f(P_2) \le L_f(P_1)$, (b) $U_f(P_1) \le U_f(P_2)$, and (c) $\|P_1\| \le \|P_2\|$.
(6) If $\lim_{\|P\|\to 0} L_f(P) = \lim_{\|P\|\to 0} U_f(P)$, then this common limit is termed the *integral* of f on the interval $[a, b]$.
(7) We can define $\int_a^b f(x)\,dx$ as above if $a < b$. If $a = b$ the integral is defined to be 0. If $a > b$, the integral is defined as $-\int_b^a f(x)\,dx$.
(8) A continuous function on $[a, b]$ has an integral on $[a, b]$. A piecewise continuous function where one-sided limits exist and are finite at every point is also integrable.

Actions ...

(1) For constant functions, the integral is just the product of the value of the function and the length of the interval.
(2) Points don't matter. So, if we change the value of a function at one point while leaving the other values unaffected, the integral does not change.
(3) A first-cut lower and upper bound on the integral can be obtained using the *trivial* partition, where we do not subdivide the interval at all. The upper bound is thus the maximum value times the length of the interval, and the lower bound is the minimum value times the length of the interval.
(4) The finer the partition, the closer the lower and upper bounds, and the better the approximation we obtain for the integral.
(5) A very useful kind of partition is a *regular partition*, which is a partition where all the parts have the same length. If the integral exists, we can calculate the actual integral as $\lim_{n\to\infty}$ of the upper sums or the lower sums for a regular partition into n parts.

(6) When a function is increasing on some parts of the interval and decreasing on other parts, it is useful to choose the partition in such a way that on each piece of the partition, the function is either increasing throughout or decreasing throughout. This way, the maximum and minimum occur at the endpoints in each piece. In particular, try to choose all points of local extrema as points of partition.

4.2. **Definite integral, antiderivative, and indefinite integral.** Words ..

(1) We have $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
(2) We say that F is an antiderivative for f if $F' = f$.
(3) For a continuous function f defined on a closed interval $[a, b]$, and for a point $c \in [a, b]$, the function F given by $F(x) = \int_c^x f(t)\,dt$ is an antiderivative for f.
(4) If f is continuous on $[a, b]$ and F is a function continuous on $[a, b]$ such that $F' = f$ on (a, b), then $\int_a^b f(x)\,dx = F(b) - F(a)$.
(5) The two results above essentially state that differentiation and integration are opposite operations.
(6) For a function f on an interval $[a, b]$, if F and G are antiderivatives, then $F - G$ is constant on $[a, b]$. Conversely, if F is an antiderivative of f, so is F plus any constant.
(7) The *indefinite integral* of a function f is the collection of all antiderivatives for the function. This is typically written by writing one antiderivative plus C, where C is an arbitrary constant. We write $\int f(x)\,dx$ for the indefinite integral. Note that there are no upper and lower limits.
(8) Both the definite and the indefinite integral are additive. In other words, $\int f(x)\,dx + \int g(x)\,dx = \int f(x) + g(x)\,dx$. The analogue holds for definite integrals, with limits.
(9) We can also pull constants multiplicatively out of integrals.
(10) Note the important formula:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x)$$

Actions ...

(1) Consider the integration problem:

$$\int \frac{2x}{x^3+1}\,dx = \int \frac{2x}{x^3}\,dx + \int \frac{2x}{1}\,dx = \int \frac{2}{x^2}\,dx + \int 2x\,dx = \frac{-2}{x} + x^2 + C$$

(2) To do a definite integral, find any one antiderivative and evaluate it between limits.
(3) An important caveat: when using antiderivatives to do a definite integral, it is important to make sure that the antiderivative is defined and continuous everywhere on the interval of integration. (Think of the $1/x^3$ example).
(4) To do an indefinite integral, find any antiderivative and put a $+C$.
(5) To find an antiderivative, use the additive splitting and pulling constants out, and the fact that $\int x^r\,dx = x^{r+1}/(r+1)$.

4.3. **Higher derivatives, multiple integrals, and initial/boundary conditions.** Actions ...

(1) The simplest kind of *initial value problem* (a notion we will encounter again when we study differential equations) is as follows. The k^{th} derivative of a function is given on the entire domain. Next, the *values* of the function and the first $k-1$ derivatives are given at a single point of the domain. We can use this data to find the function. Step by step, we find derivatives of lower orders. First, we integrate the k^{th} derivative to get that the $(k-1)^{th}$ derivative is of the form $F(x) + C$, where C is unknown. We now use the value of the $(k-1)^{th}$ derivative at the given point to find C. Now, we have the $(k-1)^{th}$ derivative. We proceed now to find the $(k-2)^{th}$ derivative, and so on.
(2) Sometimes, we may be interested in finding *all* functions with a given second derivative f. For this, we have to perform an indefinite integration twice. The net result will be a general expression of the form $F(x) + C_1x + C_2$, where F is a function with $F'' = f$, and C_1 and C_2 are arbitrary constants. In other words, we now have *up to constants or linear functions* instead of *up to constants* as our degree of ambiguity.

(3) More generally, if the k^{th} derivative of a function is given, the function is uniquely determined up to additive differences of polynomials of degree strictly less than k (so, degree at most $k-1$). The number of free constants that can take arbitrary real values is k (namely, the coefficients of the polynomial).
(4) This general expression is useful if, instead of an initial value problem, we have a boundary value problem. Suppose we are given G'' as a function, and we are given the value of G at two points. We can then first find the general expression for G as $F + C_1x + C_2$. Next, we plug in the values to get a system of two linear equations, that we solve in order to determine C_1 and C_2, and hence G.

4.4. **Reversing the chain rule.** Words ...

(1) The chain rule states that $(f \circ g)' = (f' \circ g) \cdot g'$.
(2) Some integrations require us to reverse the chain rule. For this, we need to realize the integrand that we have in the form of the right-hand side of the chain rule.
(3) The first step usually is to find the correct function g, which is the *inner function* of the composition, then to adjust constants suitably so that the remaining term is g', and then figure out what f' is. Finally, we find an antiderivative for f', which we can call f, and then compute $f \circ g$.
(4) A slight variant of this method (which is essentially the same) is the substitution method, where we identify g just as before, try to spot g' in the integrand as before, and then put $u = g(x)$ and rewrite the integral in terms of u.

Actions ...

(1) Good targets for u in the u-substitution are things that are shielded.
(2) In general, things in denominators, with non-integer exponents, or with complicated composites appended to them are good targets for u.

4.5. **u-substitutions for definite integrals.** Words ... (try to recall the numerical formulations)

(1) When doing the u-substitution for definite integrals, we transform the upper and lower limits of integration by the u-function.
(2) Note that the u-substitution is valid only when the u-function is well-defined on the entire interval of integration.
(3) The integral of a translate of a function is the integral of a function with the interval of integration suitably translated.
(4) The integral of a multiplicative transform of a function is the integral of the function with the interval of integration transformed by the same multiplicative factor, scaled by that multiplicative factor.

4.6. **Symmetry and integration.** Words ...

(1) If a function is continuous and even, its integral on $[-a, 0]$ equals its integral on $[0, a]$. More generally, its integrals on any two segments that are reflections of each other about the origin are equal. As a corollary, the integral on $[-a, a]$ is twice the integral on $[0, a]$.
(2) If a function is continuous and odd, its integral on $[-a, 0]$ is the negative of its integral on $[0, a]$. More generally, its integrals on any two segments that are reflections of each other about the origin are negatives of each other. As a corollary, the integral on $[-a, a]$ is zero.
(3) If a function is continuous and has mirror symmetry about the line $x = c$, its integral on $[c - h, c]$ equals its integral on $[c, c + h]$.
(4) If a function is continuous and has half-turn symmetry about $(c, f(c))$, its integral on any interval of the form $[c - h, c + h]$ is $2hf(c)$. Basically, all the variation about $f(c)$ *cancels out* and the *average value* is $f(c)$.
(5) Suppose f is continuous and periodic with period h and F is an antiderivative of f. The integral of f over any interval of length h is constant. Thus, $F(x + h) - F(x)$ is the same constant for all x. (We saw this fact long ago, without proof).
(6) The constant mentioned above is zero iff F is periodic, i.e., f has a periodic antiderivative.
(7) There is thus a well-defined *average value* of a continuous periodic function on a period. This is also the average value of the same periodic function on any interval whose length is a nonzero integer multiple of the period. This is also the limit of the average value over very large intervals.

Actions...

(1) All this even-odd-periodic stuff is useful for trivializing some integral calculations without computing antiderivatives. This is more than an idle observation, since in a lot of real-world situations, we get functions that have some obvious symmetry, even though we know very little about the concrete form of the functions. We use this obvious symmetry to compute the integral.
(2) Even if the whole integrand does not succumb to the lure of symmetry, it may be that the integrand can be written as (something nice and symmetric) + (something computable). The (nice and symmetric) part is then tackled using the ideas of symmetry, and the computable part is computed.

4.7. **Mean-value theorem.** Words ...

(1) The *average value*, or *mean value*, of a continuous function on an interval is the quotient of the integral of the function on the interval by the length of the interval.
(2) The mean value theorem for integrals says that a continuous function must attain its mean value somewhere on the interior of the interval.
(3) For periodic functions, the mean value over any interval whose length is a multiple of the period is the same. Also, the mean value over a very large interval approaches this value.
(4) The mean value of a periodic continuous function f being 0 means that f has a periodic antiderivative; in fact, every antiderivative of f is periodic. If the mean value is nonzero, every antiderivative of f is periodic with shift, and the linear part of any antiderivative has slope equal to the mean value of f.

Actions ...

(1) The mean values (over a period) of sin and cos are 0. For sin, we can see this since it is an *odd* periodic function. cos is a translate of sin, hence has the same mean value.
(2) The mean values (over a period) of $\sin^2$ and $\cos^2$ are $1/2$. The functions add up to 1, and are translates of each other, so this makes sense.
(3) The mean value (over a period) of $x \mapsto f(mx)$, $m \neq 0$, is the same as the mean value of f, where f is a continuous periodic function.
(4) The mean value (over a period) of $|\sin|$ is $2/\pi$, and that of $\sin^+$ is $1/\pi$.

4.8. **Application to area computations.** Words ...

(1) We can use integration to determine the area of the region between the graph of a function f and the x-axis from $x = a$ to $x = b$: this integral is $\int_a^b f(x)\,dx$. The integral measures the signed area: parts where $f \geq 0$ make positive contributions and parts where $f \leq 0$ make negative contributions. The magnitude-only area is given as $\int_a^b |f(x)|\,dx$. The best way of calculating this is to split $[a, b]$ into sub-intervals such that f has constant sign on each sub-interval, and add up the areas on each sub-interval.
(2) Given two functions f and g, we can measure the area between f and g between $x = a$ and $x = b$ as $\int_a^b |f(x) - g(x)|\,dx$. For practical purposes, we divide into sub-intervals so that on each sub-interval one function is bigger than the other. We then use integration to find the magnitude of the area on each sub-interval and add up. If f and g are both continuous, the points where the functions *cross* each other are points where $f = g$.

Actions ...

(1) In some situations we are directly given functions and/or curves and are asked to find areas. In others, we are given real-world situations where we need to find areas of regions. Here, we have to find functions and set up the integration problem as an intermediate step.
(2) In all these situations, it is important to draw the graphs in a reasonably correct way. This brings us to all the ideas that are contained in graph drawing. Remember, here we may be interested in simultaneously graphing more than one function. Thus, in addition to being careful about each function, we should also correctly estimate where one function is bigger than the other, and find (approximately or exactly) the intersection points. (Go over the notes on graph-drawing, and some additional notes on graphing that weren't completely covered in class).

(3) In some situations, we are asked to find the area(s) of region(s) bounded by the graphs of one, two, three, or more functions. Here, we first need to sketch the figure. Then, we need to find the interval of integration, and if necessary, split this interval into sub-intervals, such that on each sub-interval, we know exactly what integral we need to do. For instance, consider the region between the graphs of sin, cos, and the x-axis. Basically, the idea is to find, for all the vertical slices, the upper and lower limits of the slice.