REVIEW SHEET FOR FINAL: BASIC

MATH 152, SECTION 55 (VIPUL NAIK)

With minor exceptions, this document does not re-review material already covered in the review sheet for midterm 1 and midterm 2. It is your responsibility to go through that review sheet again and make sure you have mastered all the material there.

See the advanced version for error-spotting exercises and the quickly list.

1. Area computations

Note that this section partially repeats material from the prevoius midterm review, because part of the area computations syllabus was in the previous midterm syllabus.

Words ...

(1) We can use integration to determine the area of the region between the graph of a function f and the x-axis from $x = a$ to $x = b$: this integral is $\int_a^b f(x)\,dx$. The integral measures the signed area: parts where $f \geq 0$ make positive contributions and parts where $f \leq 0$ make negative contributions. The magnitude-only area is given as $\int_a^b |f(x)|\,dx$. The best way of calculating this is to split $[a, b]$ into sub-intervals such that f has constant sign on each sub-interval, and add up the areas on each sub-interval.
(2) Given two functions f and g, we can measure the area between f and g between $x = a$ and $x = b$ as $\int_a^b |f(x) - g(x)|\,dx$. For practical purposes, we divide into sub-intervals so that on each sub-interval one function is bigger than the other. We then use integration to find the magnitude of the area on each sub-interval and add up. If f and g are both continuous, the points where the functions *cross* each other are points where $f = g$.
(3) Sometimes, we may want to compute areas against the y-axis. The typical strategy for doing this is to interchange the roles of x and y in the above discussion. In particular, we try to express x as a function of y.
(4) An alternative strategy for computing areas against the y-axis is to use formulas for computing areas against the x-axis, and then compute differences of regions.
(5) A general approach for thinking of integration is in terms of slicing and integration. Here, integration along the x-axis is based on the following idea: divide the region into vertical slices, and then integrate the lengths of these slices along the horizontal dimension. Regions for which this works best are the regions called *Type I regions*. These are the regions for which the intersection with any vertical line is either empty or a point or a line segment, hence it has a well-defined length.
(6) Correspondingly, integration along the y-axis is based on dividing the region into horizontal slices, and integrating the lengths of these slices along the vertical dimension. Regions for which this works best are the regions called *Type II regions*. These are the regions for which the intersection with any horizontal line is either empty or a point or a line segment, hence it has a well-defined length.
(7) Generalizing from both of these, we see that our general strategy is to choose two perpendicular directions in the plane, one being the direction of our slices and the other being the direction of integration.

Actions ...

(1) In some situations we are directly given functions and/or curves and are asked to find areas. In others, we are given real-world situations where we need to find areas of regions. Here, we have to find functions and set up the integration problem as an intermediate step.
(2) In all these situations, it is important to draw the graphs in a reasonably correct way. This brings us to all the ideas that are contained in graph drawing. Remember, here we may be interested in simultaneously graphing more than one function. Thus, in addition to being careful about each

function, we should also correctly estimate where one function is bigger than the other, and find (approximately or exactly) the intersection points. (Go over the notes on graph-drawing, and some additional notes on graphing that weren't completely covered in class).

(3) In some situations, we are asked to find the area(s) of region(s) bounded by the graphs of one, two, three, or more functions. Here, we first need to sketch the figure. Then, we need to find the interval of integration, and if necessary, split this interval into sub-intervals, such that on each sub-interval, we know exactly what integral we need to do. For instance, consider the region between the graphs of sin, cos, and the x-axis. Basically, the idea is to find, for all the vertical slices, the upper and lower limits of the slice.

2. Volume computations

Words ...

(1) The cross section method for computing volume is an analogue of the two-dimensional area computation method: our slices are replaced by cross sections by planes parallel to a fixed plane, and the line of integration is a line perpendicular to the planes. One-dimensional slices are replaced by two-dimensional cross sections.

(2) Suppose Ω is a region in the plane. We can construct a right cylinder with base Ω and height h. This is obtained by translating Ω in a direction perpendicular to its plane by a length of h. The cross section of this right cylinder along any plane parallel to the original plane looks like Ω if that plane is within range. The volume is the product of the area of Ω and the height h. This is also called the right cylinder with constant cross section Ω.

(3) We can also construct an oblique cylinder. Here, the direction of translation is not perpendicular to the original plane. The total volume is the product of the area of Ω and the height perpendicular to Ω. Oblique cylinders are to right cylinders what parallelograms are to rectangles.

(4) More generally, the volume of a solid can be computed using the cross section method. Here, we choose a direction. We measure areas of cross sections along planes perpendicular to that direction, and integrate these areas along that direction.

(5) This general approach has another special case that is perhaps as important as right cylinders. These are the *cones* (there are right cones and oblique cones). A cone is obtained by taking a region in a plane and connecting all points in it to a point outside the plane. It is a right cone if that point is directly above the center of the region. The volume of a cone is $1/3$ times the product of the base area and the height, i.e., the perpendicular distance from the outside point to the plane. In particular, a cone has one-third the volume of a cylinder of the same base and height.

(6) A solid of revolution is a solid obtained by revolving a region in a plane about a line (called the axis of revolution). The volume of a solid of revolution can be computed by choosing the axis as the axis of integration and using the planes of cross section as planes perpendicular to it. These cross sections are either circular disks or annuli in the nice cases. *Added*: In nastier cases, the cross sections could be unions of multiple annuli.

(7) The *disk method* is a special case of the above, where the region being revolved is supported on the axis of revolution. For instance, consider the region between the x-axis, the graph of a function f, and the lines $x = a$ and $x = b$. The volume of the corresponding solid of revolution is $\pi \int_a^b [f(x)]^2 \, dx$. This is because the radius of the cross section disk at $x = x_0$ is $|f(x_0)|$.

(8) The *washer method* is the more general case where the region need not adhere to the axis of revolution. For instance, consider two nonnegative functions f, g and suppose $0 \le g \le f$. Consider the region bounded by the graphs of these two functions and the lines $x = a$ and $x = b$. The volume of the corresponding solid of revolution is $\pi \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$. Note that in the more general case where the functions cross each other, we may need to split into sub-intervals so that we can apply the washer method on each sub-interval.

(9) The shell method works for situations where we revolve about the y-axis the region made between the graph of a function and the x-axis. The formula here is $2\pi \int_a^b x f(x) \, dx$ for f nonnegative and $0 < a < b$. If f could be positive or negative, we use $2\pi \int_a^b x|f(x)| \, dx$.. More generally, if we are looking

at the region between the graphs of f and g (vertically) with $g \leq f$, we get $2\pi \int_a^b x[f(x) - g(x)]\,dx$. If we don't know which one is bigger where, we use $2\pi \int_a^b x|f(x) - g(x)|\,dx$.

Actions ...

(1) To compute the volume using cross sections, we first need to set things up so that we know the cross section areas as a function of the position of the plane. For this, it is usually necessary to use either coordinate geometry or basic trigonometry, or a combination.
(2) A solid occurs as a solid of revolution if it has complete rotational symmetry about some axis. In that case, that axis is the axis of revolution and the original region that we need is obtained by taking a cross section in any plane containing the axis of revolution and looking at the part of that cross section that is on one side of the axis of revolution.
(3) For solids of revolution, be particularly wary if the original figure being revolved has parts on both sides of the axis of revolution. If it is symmetric about the axis of revolution, delete one side. *Added*: In general, fold the figure about the axis of revolution – folding does not affect the final solid of revolution we obtain.
(4) Be careful about the situations where you have to be sign-sensitive and the situations where you do not. In the disk method sensitivity to signs is not important. In the washer method and shell method, it is. *Added*: Also be careful about applying the disk, washer, and shell methods when the axis of revolution is *not* the x- or y-axis but is parallel to one of them.
(5) The farther the shape being revolved is from the axis, the greater the volume of the solid of revolution.
(6) The average value point of view is sometimes useful for understanding such situations.

3. One-one functions and inverses

3.1. **Vague generalities.** Words...

(1) Old hat: Given two sets A and B, a function $f : A \to B$ is something that takes inputs in A and gives outputs in B. The *domain* of a function is the set of possible inputs, while the *range* of a function is the set of possible outputs. The notation $f : A \to B$ typically means that the domain of the function is A. However, the whole of B need not be the range; rather, all we know is that the range is a *subset* of B. One way of thinking of functions is that *equal inputs give equal outputs.*
(2) A function f is one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$. In other words, *unequal inputs give unequal outputs.* Another way of thinking of this is that *equal outputs could only arise from equal inputs.* Or, *knowledge of the output allows us to determine the input uniquely.* One-to-one functions are also called one-one functions or injective functions.
(3) Suppose f is a function with domain A and range B. If f is one-to-one, there is a *unique* function g with domain B and range A such that $f(g(x)) = x$ for all $x \in B$. This function is denoted f^{-1}. We further have that g is also one-to-one, and that $f = g^{-1}$. Note that f^{-1} differs from the reciprocal function of f.
(4) Suppose $f : A \to B$ and $g : B \to C$ are one-to-one functions. Then $g \circ f$ is also one-to-one, and its inverse is the function $f^{-1} \circ g^{-1}$.

Actions ...

(1) To determine whether a function is one-to-one, solve $f(x) = f(a)$ for x in terms of a. If, for every a in the domain, the only solution is $x = a$, the function is one-to-one. If, on the other hand, there are some values of a for which there is a solution $x \neq a$, the function is not one-to-one.
(2) To compute the inverse of a one-to-one function, solve $f(x) = y$ and the expression for x in terms of y is the inverse function.

3.2. **In graph terms.** Thousand words ...

(1) A picture in a coordinatized plane is the graph of a function if every vertical line intersects the picture at most once. The vertical lines that intersect it exactly once correspond to the x-values in the domain. This is known as the *vertical line test.*
(2) A function is one-to-one if and only if its graph satisfies the *horizontal line test*: every horizontal line intersects the graph at most once. The horizontal lines that intersect the graph exactly once correspond to y-values in the range.

(3) For a one-to-one function, the graph of the inverse function is obtained by reflecting the graph of the function about the $y = x$ line. In particular, a function equals its own inverse iff its graph is symmetric about the $y = x$ line.
(4) Many of the results on inverse functions and their properties have graphical interpretations. For instance, the fact that the derivative of the inverse function is the reciprocal of the derivative corresponds to the geometrical fact that reflection about the $y = x$ line inverts slopes of tangent lines. Similarly, the results relating increase/decrease and concave up/down for a function and its inverse function can all be deduced graphically.

3.3. **In the real world.** Words... (from now on, we restrict ourselves to functions whose domain and range are both subsets of the real numbers)

(1) An increasing function is one-to-one. A decreasing function is one-to-one.
(2) A *continuous* function on an *interval* is one-to-one if and only if it is either increasing throughout the interval or decreasing throughout the interval.
(3) If the derivative of a continuous function on an interval is of constant sign everywhere, except possibly at a few isolated points where it is either zero or undefined, then the function is one-to-one on the interval. Note that we need the function to be continuous *everywhere* on the interval, even though it is tolerable for the derivative to be undefined at a few isolated points.
(4) In particular, a one-to-one function cannot have local extreme values.
(5) A continuous one-to-one function is increasing if and only if its inverse function is increasing, and is decreasing if and only if its inverse function is decreasing.
(6) *Point added, not present in original executive summary of lecture notes*: If a one-to-one function on an interval satisfies the intermediate value property, then it is continuous. This is because the function cannot *jump* suddenly since it needs to cover all intermediate values. Note that the analogous statement is *not* true if we drop either the assumption of one-to-one or the assumption of the intermediate value property. (Think of $\sin(1/x)$, for instance).
(7) If f is one-to-one *and differentiable at a point* a (emphasis added) with $f'(a) \neq 0$, with $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$. This agrees with the previous point and also shows that the rates of relative increase are inversely proportional.
(8) Two extreme cases of interest are: $f'(a) = 0$, $f(a) = b$. In this case, f has a horizontal tangent at a and f^{-1} has a vertical tangent at b. The horizontal tangent is typically also a point of inflection. It is definitely *not* a point of local extremum. Similarly, if $(f^{-1})'(b) = 0$, then f^{-1} has a horizontal tangent at b and f has a vertical tangent at a.
(9) A slight complication occurs when f has one-sided derivatives but is not differentiable. If both one-sided derivatives of f exist and are nonzero, then both one-sided derivatives of f^{-1} (at the image point) exist and are nonzero. When f is increasing, the left hand derivative of f^{-1} is the reciprocal of the left hand derivative of f, and the right hand derivative of f^{-1} is the reciprocal of the right hand derivative of f. When f is decreasing, the right hand derivative of f^{-1} is the reciprocal of the left hand derivative of f, and the left hand derivative of f^{-1} is the reciprocal of the right hand derivative of f.
(10) The second derivative of f^{-1} at $f(a)$ is $-f''(a)/(f'(a))^3$. In particular, the second derivative of the inverse function at the image point depends on the values of both the first and the second derivatives of the function at the point.
(11) If f is increasing, the sense of concavity of f^{-1} is opposite to that of f. If f is decreasing, the sense of concavity of f^{-1} is the same as that of f.

Actions ...

(1) For functions on intervals, *to check if the function is one-to-one*, we can compute the derivative and check if it has constant sign everywhere except possibly at isolated points.
(2) In order to find $(f^{-1})'$ at a particular point, given an explicit description of f, it is *not* necessary to find an explicit description of f^{-1}. Rather, it is enough to find f^{-1} at that particular point and then calculate the derivative using the above formula. The same is true for $(f^{-1})''$, except that now we need to compute the values of both f' and f''.

(3) The idea can be extended somewhat to finding $(f^{-1})'$ when f satisfies a differential equation that expresses $f'(x)$ in terms of $f(x)$ (with no direct appearance of x).

4. Logarithms, exponents, derivatives and integrals

4.1. **Logarithm and exponential: basics.**

(1) The *natural logarithm* is a one-to-one function with domain $(0,\infty)$ and range $\mathbb{R}$, and is defined as $\ln(x) := \int_1^x (dt/t)$.
(2) The natural logarithm is an increasing function that is concave down. It satisfies the identities $\ln(1) = 0$, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^r) = r \ln a$, and $\ln(1/a) = -\ln a$.
(3) The limit $\lim_{x\to 0} \ln(x)$ is $-\infty$ and the limit $\lim_{x\to\infty} \ln(x)$ is $+\infty$. Note that ln goes off to $+\infty$ at ∞ even though its derivative goes to zero as $x \to +\infty$.
(4) The derivative of $\ln(x)$ is $1/x$ and the derivative of $\ln(kx)$ is also $1/x$. The derivative of $\ln(x^r)$ is r/x.
(5) The antiderivative of $1/x$ is $\ln|x|+C$. What this really means is that the antiderivative is $\ln(-x)+C$ when x is negative and $\ln(x)+C$ when x is positive. If we consider $1/x$ on both positive and negative reals, the constant on the negative side is unrelated to the constant on the positive side.
(6) e is defined as the unique number x such that $\ln(x) = 1$. e is approximately 2.718. In particular, it is between 2 and 3.
(7) The inverse of the natural logarithm function is denoted exp, and $\exp(x)$ is also written as e^x. When x is a rational number, $e^x = e^x$ (i.e., the two definitions of exponentiation coincide). In particular, $e^1 = e$, $e^0 = 1$, etc.
(8) The function exp equals its own derivative and hence also its own antiderivative. Further, the derivative of $x \mapsto e^{mx}$ is me^{mx}. Similarly, the integral of e^{mx} is $(1/m)e^{mx} + C$.
(9) We have $\exp(x + y) = \exp(x)\exp(y)$, $\exp(rx) = (\exp(x))^r$, $\exp(0) = 1$, and $\exp(-x) = 1/\exp(x)$. All of these follow from the corresponding identities for ln.

Actions...

(1) We can calculate $\ln(x)$ for given x by using the usual methods of estimating the values of integrals, applied to the function $1/x$. We can also use the known properties of logarithms, as well as approximate ln values for some specific x values, to estimate $\ln x$ to a reasonable approximation. For this, it helps to remember $\ln 2$, $\ln 3$, and $\ln 5$ or $\ln 10$.
(2) Since both ln and exp are one-to-one, we can *cancel* ln from both sides of an equation and similarly *cancel* exp. Technically, we cancel ln by applying exp to both sides, and we cancel exp by applying ln to both sides.

4.2. **Integrations involving logarithms and exponents.** Words/actions ...

(1) If the numerator is the derivative of the denominator, the integral is the logarithm of the (absolute value of) the denominator. In symbols, $\int g'(x)/g(x)\,dx = \ln|g(x)| + C$.
(2) More generally, whenever we see an expression of the form $g'(x)/g(x)$ inside the integrand, we should consider the substitution $u = \ln|g(x)|$. Thus, $\int f(\ln|g(x)|)g'(x)/g(x)\,dx = \int f(u)\,du$ where $u = \ln|g(x)|$.
(3) $\int f(e^x)e^x\,dx = \int f(u)\,du$ where $u = e^x$.
(4) $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.
(5) $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$.
(6) Trigonometric integrals: $\int \tan x\,dx = -\ln|\cos x| + C$, and similar integration formulas for cot, sec and csc: $\int \cot x\,dx = \ln|\sin x| + C$, $\int \sec x = \ln|\sec x + \tan x| + C$, and $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

4.3. **Exponents with arbitrary bases, exponents.** Words ...

(1) For $a > 0$ and b real, we define $a^b := \exp(b \ln a)$. This coincides with the usual definition when b is rational.
(2) All the laws of exponents that we are familiar with for integer and rational exponents continue to hold. In particular, $a^0 = 1$, $a^{b+c} = a^b \cdot a^c$, $a^1 = a$, and $a^{bc} = (a^b)^c$.

(3) The exponentiation function is continuous in the exponent variable. In particular, for a fixed value of $a > 0$, the function $x \mapsto a^x$ is continuous. When $a \neq 1$, it is also one-to-one with domain $\mathbb{R}$ and range $(0, \infty)$, with inverse function $y \mapsto (\ln y)/(\ln a)$, which is also written as $\log_a(y)$. In the case $a > 1$, it is an increasing function, and in the case $a < 1$, it is a decreasing function.
(4) The exponentiation function is also continuous in the base variable. In particular, for a fixed value of b, the function $x \mapsto x^b$ is continuous. When $b \neq 0$, it is a one-to-one function with domain and range both $(0, \infty)$, and the inverse function is $y \mapsto y^{1/b}$. In case $b > 0$, the function is increasing, and in case $b < 0$, the function is decreasing.
(5) Actually, we can say something stronger about a^b – it is *jointly* continuous in both variables. This is hard to describe formally here, but what it approximately means is that if f and g are both continuous functions, and f takes positive values only, then $x \mapsto [f(x)]^{g(x)}$ is also continuous.
(6) The derivative of the function $[f(x)]^{g(x)}$ is $[f(x)]^{g(x)}$ times the derivative of its logarithm, which is $g(x)\ln(f(x))$. We can further simplify this to obtain the formula:

$$\frac{d}{dx}\left([f(x)]^{g(x)}\right) = [f(x)]^{g(x)}\left[\frac{g(x)f'(x)}{f(x)} + g'(x)\ln(f(x))\right]$$

(7) Special cases worth noting: the derivative of $(f(x))^r$ is $r(f(x))^{r-1}f'(x)$ and the derivative of $a^{g(x)}$ is $a^{g(x)}g'(x)\ln a$.
(8) Even further special cases: the derivative of x^r is rx^{r-1} and the derivative of a^x is $a^x \ln a$.
(9) The antiderivative of x^r is $x^{r+1}/(r+1)+C$ (for $r \neq -1$) and $\ln|x|+C$ for $r = -1$. The antiderivative of a^x is $a^x/(\ln a) + C$ for $a \neq 1$ and $x + C$ for $a = 1$.
(10) The logarithm $\log_a(b)$ is defined as $(\ln b)/(\ln a)$. This is called the logarithm of b to base a. Note that this is defined when a and b are both positive and $a \neq 1$. This satisfies a bunch of identities, most of which are direct consequences of identities for the natural logarithm. In particular, $\log_a(bc) = \log_a(b)+\log_a(c)$, $\log_a(b)\log_b(c) = \log_a(c)$, $\log_a(1) = 0$, $\log_a(a) = 1$, $\log_a(a^r) = r$, $\log_a(b)\cdot\log_b(a) = 1$ and so on.
(11) *Added*: The derivative of $\log_{f(x)}(g(x))$ is given by:

$$\frac{d}{dx}\left[\log_{f(x)}(g(x))\right] = \frac{\ln(f(x))g'(x)/g(x) - \ln(g(x))f'(x)/f(x)}{(\ln(f(x)))^2}$$

Actions...

(1) We can use the formulas here to differentiate expressions of the form $f(x)^{g(x)}$, and even to differentiate longer exponent towers (such as x^{x^x} and 2^{2^x}).
(2) To solve an integration problem with exponents, it may be most prudent to rewrite a^b as $\exp(b\ln a)$ and work from there onward using the rules mastered earlier. Similarly, when dealing with relative logarithms, it may be most prudent to convert all expressions in terms of natural logairthms and then use the rules mastered earlier.