REVIEW SHEET FOR FINAL: ADVANCED

MATH 152, SECTION 55 (VIPUL NAIK)

1. Area computations

No error-spotting exercises.

2. Volume computations

Error-spotting exercises ...

(1) Consider the function $f(x) := \sin x$ and $g(x) := -\sin x$, for $x \in [0, \pi]$. The region between the graphs of these functions is revolved about the x-axis. The volume of the solid of revolution is:

$$\pi \int_0^\pi (f(x) - g(x))^2 \, dx = \pi \int_0^\pi 4 \sin^2 x \, dx = 2\pi^2$$

(2) If a right angled triangle is revolved about any of its sides, then we get a right circular cone.
(3) If a rectangle is revolved about one of its diagonals, then we get a union of two right circular cones.
(4) The volume of the solid of revolution obtained by revolving a region of area A is proportional to A.

3. One-one functions and inverses

3.1. Vague generalities.

Error-spotting exercises ...

(1) If f and g are one-one functions, then so is $f \circ g$ and $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

3.2. In the real world.

Error-spotting exercises ...

(1) Consider the function:

$$f(x) := x^3 + x$$

We have $f'(x) = 3x^2 + 1 > 0$ is always positive, so f is one-one. Also, note that the inverse function to $x \mapsto x^3$ is $x \mapsto x^{1/3}$ and the inverse function to $x \mapsto x$ is $x \mapsto x$. Thus, we get:

$$f^{-1}(x) := x^{1/3} + x$$

(2) If f is a one-one function on $\mathbb{R}$, then it must be either an increasing function or a decreasing function on $\mathbb{R}$.
(3) If f is a differentiable function on $\mathbb{R}$, then f is one-one if and only if $f'(x) > 0$ for all $x \in \mathbb{R}$.
(4) Suppose f and g are continuous one-one functions on $\mathbb{R}$. Then, clearly, they are either increasing or decreasing functions on $\mathbb{R}$. Thus, the sum $f + g$ is also either an increasing or decreasing function on $\mathbb{R}$, and hence it must be one-one.
(5) Suppose f, g, and h are continuous one-one functions on $\mathbb{R}$. Then, the pairwise sums $f + g$, $g + h$, and $f + h$ are all one-one functions.
(6) Suppose f is a one-one function such that the graph of f is concave up. Then, f^{-1} is also a one-one function and its graph is concave down.
(7) If c is a point in the domain of a function f such that the left hand derivative and right hand derivative of f at c do not agree, then the left hand derivative and right hand derivative of f^{-1} at c also do not agree.

(8) Consider the function:

$$f(x) := \begin{cases} x+1, & x \text{ rational} \\ x^3, & x \text{ irrational} \end{cases}$$

We know that both $x+1$ and x^3 are one-one functions on their respective domains. Thus, f is a one-one function.

4. Logarithm, exponential, derivative, and integral

4.1. Logarithm and exponential: basics.

(1) We have that $\ln(xy) = \ln x + \ln y$. Thus, $\ln((-1)^2) = \ln(-1) + \ln(-1)$. The left side is $\ln 1 = 0$, so $\ln(-1) = 0$.
(2) If f is a function on $\mathbb{R} \setminus \{0\}$ such that $f'(x) = 1/x$ for all $x \neq 0$, then $f(x) = \ln|x| + C$ for some fixed constant C.
(3) Using that $\ln 2 \sim 0.7$ and $\ln 3 \sim 1.1$, we obtain that $\ln 5 = \ln(2+3) = \ln 2 + \ln 3 \sim 0.7 + 1.1 = 1.8$.
(4) We have $\exp((\ln x)^2) = \exp(\ln(x^2)) = x^2$.

4.2. Integration involving logarithms and exponents.

(1) Consider the integration $\int_0^\pi \tan x \, dx$. This is:

$$\int_0^\pi \tan x \, dx = [\ln|\cos x|]_0^\pi = 0$$

(2) Consider the indefinite integration $\int e^{x+\ln(\sin x)} \, dx$. This becomes:

$$\int e^{x+\ln(\sin x)} \, dx = \int e^x \, dx + \int e^{\ln(\sin x)} \, dx = e^x + \int \sin x \, dx = e^x - \cos x + C = -e^x \cos x + C$$

4.3. Exponentiation with arbitrary bases, exponents.

No error-spotting exercises

5. Miscellaneous error-spotting exercises

(1) Consider the function $x \mapsto x^{1/3} + x^{2/3}$. The $x^{1/3}$ part has a vertical tangent at $x = 0$ and $x^{2/3}$ part has a vertical cusp at $x = 0$. The tangent and cusp cancel and thus overall we get neither a vertical tangent nor a vertical cusp at 0.
(2) Consider the integration:

$$\int \frac{x^2}{x+1} \, dx = \int \frac{x^2}{x} \, dx + \int \frac{x^2}{1} \, dx = \int x + 1 \, dx = x^2/2 + x + C$$

(3) Consider the function:

$$f(x) := x \sin(\pi/(x^2 + x))$$

This is undefined at $x = 0$ and $x = 1$. At both these points, the graph of f has a vertical asymptote.

6. Quickly

This "Quickly" list improves upon previous "Quickly" lists.

6.1. Our common values.

Preferably remember these (or be capable of computing quickly) to at least two digits.

(1) Square roots of 2, 3, 5, 6, 7, 10.
(2) Natural logarithms of 2, 3, 5, 7, and 10.
(3) Value of π, $1/\pi$, $\sqrt{\pi}$, and π^2.
(4) Value of e, $1/e$.
(5) Some relative logarithms, such as $\log_2 3$ or $\log_2(10)$. Although you don't need these values to a significant degree of precision, it is useful to have some idea of their magnitude.

6.2. **Adding things up: arithmetic.** You should be able to:

(1) Do quick arithmetic involving fractions.
(2) Sense when an expression will simplify to 0.
(3) Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
(4) Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

6.3. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3. *This could be an area for potential improvement for many people.*
(4) Factorize an even polynomial of degree four. *This could be an area for potential improvement for many people.*
(5) Do polynomial long division (not usually necessary, but helpful).
(6) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

6.4. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi \pm x)$ and $\cos(\pi \pm x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles or other larger angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is $1/2$.
(6) Formulas for double angles, half angles: $\sin(2x)$, $\cos(2x)$ in terms of sin and cos; also the reverse: $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$.
(7) *More advanced*: Remember the formulas for $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, and $\cos(A-B)$.
(8) *More advanced*: Convert between products of sin and cos functions and their sums: for instance, the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.

6.5. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

6.6. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a\sin x + b\cos x$ or $a\exp(-x)$.
(6) Do multiple differentiations of expressions whose derivative cycle is periodic up to constant factors, e.g. $a\exp(mx+b)$ or $a\sin(mx+\varphi)$.
(7) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).
(8) Differentiate ln, exp, and expressions of the form f^g and $\log_f(g)$.

6.7. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a sum of power functions quickly.

(4) Know that the integral of sine or cosine on any quadrant is ± 1.
(5) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a+bx)^m$.
(6) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$, any odd power of sin, any odd power of cos, any odd power of tan.
(7) Integrate on sight things such as $x\sin(x^2)$, getting the constants right without much effort.

6.8. **Being observant.** You should be able to look at a function and:
(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

6.9. **Graphing.** You should be able to:
(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: $1/3$, $2/3$, $4/3$, $5/3$, $1/2$, 1, 2, 3, -1, $-1/3$ $-2/3$.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.

6.10. **Graphing: transformations.** Given the graph of f, you should be able to quickly graph the following:
(1) $f(mx)$, $f(mx+b)$: pre-composition with a linear function; how does $m<0$ differ from $m>0$?
(2) $Af(x)+C$: post-composition with a linear function, how does $A>0$ differ from $A<0$?
(3) $f(|x|)$, $|f(x)|$, $f(x^+)$, and $(f(x))^+$: pre- and post-composition with absolute value function and positive part fnuction.
(4) More slowly: $f(1/x)$, $1/f(x)$, $\ln(|f(x)|)$, $f(\ln|x|)$, $\exp(f(x))$, and other popular composites.

6.11. **Fancy pictures.** Keep in mind approximate features of the graphs of:
(1) $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ and $x^3\sin(1/x)$, and the corresponding cos counterparts – both the behavior near 0 and the behavior near $\pm\infty$.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.