REVIEW SHEET FOR MIDTERM 1

MATH 152, SECTION 55 (VIPUL NAIK)

The document is arranged as follows. The initial sections/subsections correspond to topics. Each subsection has two sets of points, "Words" which includes basic theory and definitions, and "Actions" which provides information on strategies for specific problem types. In some cases, there are additional points. The lists of points are largely the same as the executive summaries at the beginning of the lecture notes, though some additional points (that make sense now but wouldn't have made sense at the time the lecture was delivered) have been added.

For each subsection, there is also an "Error-spotting exercises" list. We will be doing these exercises in the review session, though you may benefit by trying them out in advance. For the simpler topics, we may do *only* the error-spotting exercises in the review session so as to save time and concentrate on the harder topics.

The section titled "Tricky topics" covers a bunch of topics and question types that habitually confuse students. This includes piecewise definitions by interval, piecewise definition by rational-irrational, $\sin(1/x)$ examples, and thinking about counterexamples to statements. You need to read each point carefully and then try to locate examples from class, homeworks, or quiz questions.

The section titled "High yield practice" lists (without details) the areas where I think practice is most helpful if you feel you're already fairly thorough with the basic formulas. If you feel you are on top of AP-level material, for instance, then these are the areas where most of your energies should be devoted.

The end of the document has some "Quickly" lists. These are lists of things you should be able to accomplish quickly. This includes numerical values, formulas, graphs, examples, and counterexamples, that should be ready for immediate recall in the test environment. I simply provide a list and do not include details of all the formulas and graphs.

To maximize efficiency in the review session, here is what I suggest. Go through all the lists of points. For each point, make sure you understand it by jotting down a relevant example or illustration or providing a brief justification. If you have difficulty, go back to the lecture notes and read them in detail. You might also want to look at more worked examples in the book, and check out homework and quiz problems.

If everybody is on top of the basic material, we will go very quickly over Sections 1–4 and Section 7 and concentrate most of our energies on Section 5 ("Tricky topics") and Section 6 ("High yield practice").

1. Functions

1.1. Review part 1. Words ...

(1) The *domain* of a function is the set of possible inputs. The *range* is the set of possible outputs. When we say $f : A \to B$ is a function, we mean that the domain is A, and the range is a *subset* of B (possibly equal to B, but also possibly a proper subset).
(2) The main fact about functions is that *equal inputs give equal outputs*. We deal here with functions whose domain and range are both subsets of the real numbers.
(3) We typically define a function using an algebraic expression, e.g. $f(x) := 3+\sin x$. When an algebraic expression is given without a specified domain, we take the domain to be the largest possible subset of the real numbers for which the function makes sense.
(4) Functions can be defined piecewise, i.e., one definition on one part of the domain, another definition on another part of the domain. Interesting things happen where the function changes definition.
(5) Functions involving absolute values, max of two functions, min of two functions, and other similar constructions end up having piecewise definitions.

Actions (think back to examples where you've dealt with these issues)...

(1) To find the (maximum possible) domain of a function given using an expression, exclude points where:
   (a) Any denominator is zero.
   (b) Any expression under the square root sign is negative.
   (c) Any expression under the square root sign in the denominator is zero or negative.
(2) To find whether a given number a is in the range of a function f, try solving $f(x) = a$ for x in the domain.
(3) To find the range of a given function f, try solving $f(x) = a$ with a now being an *unknown constant*. Basically, solve for x in terms of a. The set of a for which there exists one or more value of x solving the equation is the range.
(4) To write a function defined as $H(x) := \max\{f(x), g(x)\}$ or $h(x) := \min\{f(x), g(x)\}$ using a piecewise definition, find the points where $f(x) - g(x)$ is zero, find the points where it is positive, and find the points where it is inegative. Accordingly, define h and H on those regions as f or g. *Added: Note that when f and g are both continuous everywhere, then the functions can cross each other only at points where they become equal. However, if the two functions are not everywhere continuous, the functions can cross each other at points of discontinuity as well.*
(5) To write a function defined as $h(x) := |f(x)|$ piecewise, split into regions based on the sign of $f(x)$.
(6) To solve an equation for a function with a piecewise definition, solve for each definition within the piece (domain) for which that definition is satisfied.

Error-spotting exercises ...

Warning: There may be one or more errors in each item.

(1) Consider the function $f(x) := \sqrt{x-1} + \sqrt{2-x}$. The domain of $\sqrt{x-1}$ is $[1, \infty)$ and the domain of $\sqrt{2-x}$ is $(-\infty, 2]$. The domain of the sum is therefore the union of the domains, which is $(-\infty, \infty)$, i.e., the set of all real numbers.
(2) Consider the function $f(x) := \sqrt{(x-1)(x-2)}$. This is the product of the functions $x \mapsto \sqrt{x-1}$ and $x \mapsto \sqrt{x-2}$, hence its domain is the intersection of the domains of the two functions, which are $[1, \infty)$ and $[2, \infty)$ respectively. The domain of f is thus $[2, \infty)$.
(3) Consider the function $f(x) := \max\{x-1, 2x+1\}$. Then, we get $(f(x))^2 = \max\{(x-1)^2, (2x+1)^2\}$.

1.2. **Review part 2.** Note: Although the lecture notes (and the executive summary in front of them) cover the notion of miror symmetry and half-turn symmetry in greater generality than just looking at even and odd functions, I didn't get time to cover this in class. Since this is also not in the book, we will omit these more general notions of symmetry for this midterm. We'll probably cover them when we turn to graphing functions.

Words ...

(1) Given two functions f and g, we can define pointwise combinations of f and g: the sum $f + g$, the difference $f - g$, the product $f \cdot g$, and the quotient f/g. For the sum, difference, and product, the domain is the intersection of the domains of f and g. For the quotient, the domain is the intersection of the domain of f and the set of points where g takes a nonzero value.
(2) Given a function f and a real number α, we can consider the scalar multiple αf.
(3) Given two functions f and g, we can try talking of the composite function $f \circ g$. This is defined for those points in the domain of g whose image lies in the domain of f.
(4) An *even function* is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(5) An odd function is a function having half-turn symmetry about the origin. By definition, the domain of an odd function is symmetric about $\mathbb{R}$. An odd function, if defined at 0, takes the value 0 at 0.
(6) A function f defined on $\mathbb{R}$ is periodic if there exists $h > 0$ such that $f(x+h) = f(x)$ for every $x \in \mathbb{R}$. If there is a smallest $h > 0$ satisfying this, such a h is termed the *period*. Constant functions are periodic but have no period. The sine and cosine functions are periodic with period 2π.

Actions ...

(1) To prove that a function is periodic, try to find a h that *works* for every x. To prove that a function is periodic but has no period, try to show that there are arbitrarily small $h > 0$ that work.

(2) To prove that a function is even or odd, just try proving the corresponding equation for all x. Nothing but algebra.
(3) If a function is defined for the positive or nonnegative reals and you want to extend the definition to negatives to make it even or odd, extend it so that the formula is preserved. So define $f(-x) = f(x)$, for instance, to make it even.

Error-spotting exercises ...

(1) We know that odd + odd = even, so the sum of two odd functions is an even function.
(2) Consider the function $\sin^2 x = \sin(\sin x)$. Since the period of the sin function is 2π, the period of the $\sin^2$ function is also 2π.
(3) Suppose f is a periodic function with period h_f and g is a periodic function with period h_g. Then, the period of $f + g$ is $h_f + h_g$.
(4) Suppose f is a periodic function with period h_f and g is a periodic function with period h_g. Then, the period of the composite $f \circ g$ is $h_f h_g$.
(5) The period of the function $x \mapsto \sin(x^2)$ is $\sqrt{\pi}$.

2. Limits

2.1. **Informal introduction to limits.** Words ...

(1) On the real line, there are two directions from which to approach a point: the *left* direction and the *right* direction.
(2) For a function f, $\lim_{x \to c} f(x)$ is read as "the limit as x approaches c of $f(x)$. Equivalently, as x approaches c, $\lim_{x \to c} f(x)$ is the value that $f(x)$ approaches.
(3) $\lim_{x \to c} f(x)$ makes sense only if f is defined *around* c, i.e., both to the immediate left and to the immediate right of c.
(4) We have the notion of the *left hand limit* $\lim_{x \to c^-} f(x)$ and the *right hand limit* $\lim_{x \to c^+} f(x)$. The *limit* $\lim_{x \to c} f(x)$ exists if and only if (both the left hand limit and the right hand limit exist and they are both equal).
(5) f is termed *continuous* at c if c is in the domain of f, the limit of f at c exists, and $f(c)$ equals the limit. f is termed *left continuous* at c if the left hand limit exists and equals $f(c)$. f is termed *right continuous* at c if the right hand limit exists and equals $f(c)$.
(6) f is termed *continuous* on an interval I in its domain if f is continuous at all points in the interior of I, continuous from the right at any left endpoint in I (if I is closed from the left) and continuous fromthe left at any right endpoint in I (if I is closed from the right).
(7) A *removable discontinuity* for f is a discontinuity where a two-sided limit exists but is not equal to the value. A *jump discontinuity* is a discontinuity where both the left hand limit and right hand limit exist but they are not equal.

Error-spotting exercises...

(1) Consider the function $f(x) := 1/x$. At $x = 0$, both the left hand limit and the right hand limit are equal to each other (since they both do not exist), so f has a limit at $x = 0$.
(2) If f and g both have removable discontinuities at $x = c$, then $f + g$ also has a removable discontinuity at $x = c$.
(3) If f and$g$ both have jump discontinuities at $x = c$, then $f + g$ also has a jump discontinuity at $x = c$.

2.2. **Formal definition of limits.** Words ...

(1) $\lim_{x \to c} f(x) = L$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$ (in other words, $x \in (c - \delta, c) \cup (c, c + \delta)$, we have $|f(x) - L| < \epsilon$ (in other words, $f(x) \in (L - \epsilon, L + \epsilon)$.
(2) What that means is that however small a trap (namely ϵ) the skeptic demands, the person who wants to claim that the limit does exist can find a δ such that when the x-value is δ-close to c, the $f(x)$-value is ϵ-close to L.
(3) The negation of the statement $\lim_{x \to c} f(x) = L$ is: there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$.

(4) The statement $\lim_{x \to c} f(x)$ doesn't exist: for every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$.
(5) We can think of $\epsilon - \delta$ limits as a game. The skeptic, who is unconvinced that the limit is L, throws to the prover a value $\epsilon > 0$. The prover must now throw back a $\delta > 0$ that works. L being the limit means that the prover has a winning strategy, i.e., the prover has a wayof picking, for any $\epsilon > 0$, a value of $\delta > 0$ suitable to that ϵ.
(6) The function $f(x) = \sin(1/x)$ is a classy example of a limit not existing. The problem is that, however small we choose a δ around 0, the function takes all values between -1 and 1, and hence refuses to be confined within small ϵ-traps.
(7) We say that f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

Actions...

(1) If a δ works for a given ϵ, then every smaller δ works too. Also, if a δ works for a given ϵ, the same δ works for any larger ϵ.
(2) Constant functions are continuous, we can choose δ to be anything. In this $\epsilon - \delta$ game, the person trying to prove that the limit does exist wins no matter what ϵ the skeptic throws and no matter what δ is thrown back.
(3) For the function $f(x) = x$, it's continuous, and $\delta = \epsilon$ works.
(4) For a linear function $f(x) = ax + b$ with $a \neq 0$, it's continuous, and $\delta = \epsilon/|a|$ works. That's the largest δ that works.
(5) For a function $f(x) = x^2$ taking the limit at a point p, the limit is p^2 (the function is continuous) and $\delta = \min\{1, \epsilon/(1 + |2p|)\}$ works. It isn't the best, but it works.
(6) For a function $f(x) = ax^2 + bx + c$, taking the limit at a point p, the limit is $f(p)$ (the function is continuous) and $\delta = \min\{1, \epsilon/(|a| + |2ap + b|)\}$ works. It isn't the best, but it works.
(7) If there are two functions f and g and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$, and h is a function such that $h(x) = f(x)$ or $h(x) = g(x)$ for every x, then $\lim_{x \to c} h(x) = L$. The δ that works for h is the minimum of the δs that work for f and g. This applies to many situations: functions defined differently on the left and right of the point, functions defined differently for the rationals and the irrationals, functions defined as the max or min of two functions.

Error-spotting exercises ...

(1) Consider the limit game for $\lim_{x \to c} f(x) = L$. This is a game between the prover and the skeptic. In this game, if the prover wins, then the limit statement is true. If the skeptic wins, then we say that the limit statement is false.
(2) A winning strategy for the prover in the game for $\lim_{x \to c} f(x) = L$ involves the prover fooling the skeptic choosing a suitably large value of ϵ so that the prover can trap the function appropriately.
(3) Consider two continuous functions f and g on the reals with $f(c) = g(c) = L$. Consider $h(x) := \min\{f(x), g(x)\}$ and $H(x) := \max\{f(x), g(x)\}$. The winning strategy for the prover for showing that $\lim_{x \to c} h(x) = L$ is to pick, for any given ϵ, the *minimum* of the δs that work for f and for g. The winning strategy for H is to pick, for any given ϵ, the *maximum* of the δs that work for f and for g.

2.3. **Limit theorems + quick/intuitive calculation of limits.** Words...

(1) If the limits for two functions exist at a particular point, the limit of the sum exists and equals the sum of the limits. Similarly for product and difference.
(2) For quotient, we need to add the caveat that the limit of the denominator is nonzero.
(3) If $\lim_{x \to c} f(x) = L \neq 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c}(f(x)/g(x))$ is undefined.
(4) If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, then we cannot say anything offhand about $\lim_{x \to c}(f(x)/g(x))$.
(5) Everything we said (or implied) can be reformulated for one-sided limits.

Error-spotting exercises...

(1) Suppose f and g are functions both defined around a point c. If $\lim_{x \to c} f(x)$ does not exist and $\lim_{x \to c} g(x)$ does not exist, then $\lim_{x \to c}(f(x) + g(x))$ does not exist either.

2.4. **Continuity theorems.** Words ...

(1) If f and g are functions that are both continuous at a point c, then the function $f+g$ is also continuous at c. Similarly, $f-g$ and $f \cdot g$ are continuous at c. Also, if $g(c) \neq 0$, then f/g is continuous at c.
(2) If f and g are both continuous in an interval, then $f+g$, $f-g$ and $f \cdot g$ are also continuous on the interval. Similarly for f/g provided g is not zero anywhere on the interval.
(3) The composition theorem for continuous functions states that if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c. The corresponding composition theorem for limits is *not true but almost true*: if $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = M$, then $\lim_{x \to c} f(g(x)) = M$.
(4) The one-sided analogues of the theorems for sum, difference, product, quotient work, but the one-sided analogue of the theorem for composition is not in general true.
(5) Each of these theorems at points has a suitable analogue/corollary for continuity (and, with the exception of composition, for one-sided conitnuity) on intervals.

Error-spotting exercises ...

(1) Suppose f and g are both functions defined and continuous around a point $c \in \mathbb{R}$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all defined and continuous around c.
(2) Suppose f and g are both continuous functions on the domain $[0,1]$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all continuous functions on the domain $[0,1]$.
(3) Suppose f and g are both left continuous functions on the domain $[0,1]$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all left continuous functions on the domain $[0,1]$.
(4) Suppose $\lim_{x \to 0} g(x)/x = A \neq 0$. Then, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x} = \lim_{x \to 0} \frac{g(g(x))}{g(x)} \lim_{x \to 0} \frac{g(x)}{x} = g\left(\lim_{x \to 0} \frac{g(x)}{x}\right) \cdot \lim_{x \to 0} \frac{g(x)}{x} = g(A) \cdot A = Ag(A)$$

(5) Suppose $\lim_{x \to 0} g(x)/x^2 = A \neq 0$. Then, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \frac{g(x)}{x^2} = A \cdot A = A^2$$

2.5. **Three important theorems.** Words ...

(1) The pinching theorem states that if $f(x) \leq g(x) \leq h(x)$, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$. A one-sided version of the pinching theorem also holds.
(2) The intermediate-value theorem states that if f is a continuous function, and $a < b$, and p is between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = p$. Note that we need f to be defined and continuous on the entire closed interval $[a,b]$.
(3) The extreme-value theorem states that on a closed bounded interval $[a,b]$, a continuous function attains its maximum and minimum.

Actions ...

(1) When trying to calculate a limit that's tricky, you might want to bound it from both sides by things whose limits you know and are equal. For instance, the function $x \sin(1/x)$ taking the limit at 0, or the function that's x on rationals and 0 on irrationals, again taking the limit at 0.
(2) We can use the intermediate-value theorem to show that a given equation has a solution in an interval by calculating the values of the expression at endpoints of the interval and showing that they have opposite signs.

Error-spotting exercises ...

(1) Consider the function $f(x) := 1/x$ on the interval $[-1,1]$. We have $f(-1) = -1$ and $f(1) = 1$, so by the intermediate value theorem, there exists $x \in [-1,1]$, such that $f(x) = 1/2$. Thus, we get that $1/x = 1/2$, so $x = 2$ is in the interval $[-1,1]$.
(2) By the intermediate value theorem, the image of a closed interval $[a,b]$ under a continuous function f is the closed interval $[f(a), f(b)]$ if $f(a) < f(b)$ and the closed interval $[f(b), f(a)]$ if $f(b) < f(a)$.

3. Derivatives

3.1. Derivatives: basics. Words ...

1. For a function f, we define the *difference quotient* between w and x as the quotient $(f(w)-f(x))/(w-x)$. It is also the slope of the line joining $(x, f(x))$ and $(w, f(w))$. This line is called a *secant line*. The segment of the line between the points x and w is sometimes termed a *chord*.
2. The limit of the difference quotient is defined as the *derivative*. This is the slope of the *tangent line* through that point. In other words, we define $f'(x) := \lim_{w \to x} \frac{f(w)-f(x)}{w-x}$. This can also be defined as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
3. If the derivative of f at a point x exists, the function is termed *differentiable* at x.
4. If the derivative at a point exists, then the tangent line to the graph of the function exists and its slope equals the derivative. The tangent line is horizontal if the derivative is zero. Note that if the derivative exists, then the tangent line cannot be vertical.
5. Here are some misconceptions about tangent lines: (i) that the tangent line is the line perpendicular to the radius (this makes sense only for circles) (ii) that the tangent line does not intersect the curve at any other point (this is true for some curves but not for others) (iii) that any line other than the tangent line intersects the curve at at least one more point (this is always false – the vertical line through the point does not intersect the curve elsewhere, but is not the tangent line if the function is differentiable).
6. In the Leibniz notation, if y is functionally dependent on x, then $\Delta y/\Delta x$ is the difference quotient – it is the quotient of the difference between the y-values corresponding to x-values. The limit of this, which is the derivative, is dy/dx.
7. The left-hand derivative of f is defined as the left-hand limit for the derivative expression. It is $\lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$. The right-hand derivative is $\lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$.
8. Higher derivatives are obtained by differentiating again and again. The *second* derivative is the derivative of the derivative. The n^{th} derivative is the function obtained by differentiating n times. In prime notation, the second derivative is denoted f'', the third derivative f''', and the n^{th} derivative for large n as $f^{(n)}$. In the Leibniz notation, the n^{th} derivative of y with respect to x is denoted $d^n y/dx^n$.
9. Derivative of sum equals sum of derivatives. Derivative of difference is difference of derivatives. Scalar multiples can be pulled out.
10. We have the *product rule* for differentiating products: $(f \cdot g)' = f' \cdot g + f \cdot g'$.
11. We have the *quotient rule* for differentiating quotients: $(f/g)' = (g \cdot f' - f \cdot g')/g^2$.
12. The derivative of x^n with respect to x is nx^{n-1}.
13. The derivative of sin is cos and the derivative of cos is $-\sin$.
14. The chain rule says that $(f \circ g)' = (f' \circ g) \cdot g'$

Actions ...

1. We can differentiate any polynomial function of x, or a sum of powers (possibly negative powers or fractional powers), by differentiating each power with respect to x.
2. We can differentiate any rational function using the quotient rule and our knowledge of how to differentiate polynomials.
3. We can find the equation of the tangent line at a point by first finding the derivative, which is the slope, and then finding the point's coordinates (which requires evaluating the function) and then using the point-slope form.
4. Suppose g and h are everywhere differentiable functions. Suppose f is a function that is g to the left of a point a and h to the right of the point a, and suppose $f(a) = g(a) = h(a)$. Then, the left-hand derivative of f at a is $g'(a)$ and the right-hand derivative of f at a is $h'(a)$.
5. The k^{th} derivative of a polynomial of degree n is a polynomial of degree $n-k$, if $k \leq n$, and is zero if $k > n$.
6. We can often use the sum rule, product rule, etc. to find the values of derivatives of functions constructed from other functions simply using the values of the functions and their derivatives at

specific points. For instance, $(f \cdot g)'$ at a specific point c can be determined by knowing $f(c)$, $g(c)$, $f'(c)$, and $g'(c)$.

(7) Given a function f with some unknown constants in it (so a function that is not completely known) we can use information about the value of the function and its derivatives at specific points to determine those constant parameters.

Error-spotting exercises ...

(1) The derivative of the function $x \mapsto \sin^2 x$ is:

$$\frac{d}{dx}(\sin(x^2)) = -\cos(x^2)$$

because the derivative of sin is $-\cos$.

(2) The derivative of the function $x \mapsto \sin(x^2)$ is:

$$\frac{d}{dx}(\sin(x^2)) = (\cos x)(2x) = 2x \cos x$$

(3) The derivative of the function $x \mapsto x \cos x$ is:

$$\frac{d}{dx}(x \cos x) = \frac{d}{dx}(x) \cdot \frac{d}{dx}(\cos x) = (1)(-\sin x) = -\sin x$$

3.2. **Tangents and normals: geometry.** Words...

(1) The normal line to a curve at a point is the line perpendicular to the tangent line. Since the tangent line is the best linear approximation to the curve at the point, the normal line can be thought of as the line best approximating the perpendicular line to the curve.

(2) The angle of intersection between two curves at a point of intersection is defined as the angle between the tangent lines to the curves at that point. If the slopes of the tangent lines are m_1 and m_2, the angle is $\pi/2$ if $m_1 m_2 = -1$. Otherwise, it is the angle α such that $\tan \alpha = |m_1 - m_2|/(|1 + m_1 m_2|)$.

(3) If the angle between two curves at a point of intersection is $\pi/2$, they are termed *orthogonal* at that point. If the curves are orthogonal at all points of intersection, they are termed *orthogonal curves*.

(4) If the angle between two curves at a point of intersection is 0, that means they have the same tangent line. In this case, we say that the curves *touch* each other or are *tangent* to each other.

Actions...

(1) The equation of the normal line to the graph of a function f at the point $(x_0, f(x_0))$ is $f'(x_0)(y - f(x_0)) + (x - x_0) = 0$. The slope is $-1/f'(x_0)$.

(2) To find the angle(s) of intersection between two curves, we first find the point(s) of intersection, then compute the value of derivative (or slope of tangent line) to both curves, and then finally plug that in the formula for the angle of intersection.

(3) It is also possible to find all tangents to a given curve, or all normals to a given curve, that pass through a given point *not* on the curve. To do this, we set up the generic expression for a tangent line or normal line to the curve, and then plug into that generic expression the specific coordinates of the point, and solve. For instance, the generic equation for the tangent line to the graph of a function f is $y - f(x_1) = f'(x_1)(x - x_1)$ where $(x_1, f(x_1))$ is the point of tangency. Plugging in the point (x, y) that we know the curve passes through, we can solve for x_1.

(4) In many cases, it is possible to determine geometrically the number of tangents/normals passing through a point outside the curve. Also, in some cases, the algebraic equations may not be directly solvable, but we may be able to determine the number and approximate location of the solutions.

3.3. **Deeper perspectives on derivatives.** Words...

(1) A continuous function that is everywhere differentiable need not be everywhere continuously differentiable.

(2) If f and g are functions that are both continuously differentiable (i.e., they are differentiable and their derivatives are continuous functions), then $f + g$, $f - g$, $f \cdot g$, and $f \circ g$ are all continuously differentiable.

(3) If f and g are functions that are both k times differentiable (i.e., the k^{th} derivatives of the functions f and g exist), then $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are also k times differentiable.
(4) If f and g are functions that are both k times continuously differentiable (i.e., the k^{th} derivatives of both functions exist and are continuous) then $f+g$, $f-g$, and $f \cdot g$, and $f \circ g$ are also k times continuously differentiable.
(5) If f is k times differentiable, for $k \geq 2$, then it is $k-1$ times continuously differentiable, i.e., the $(k-1)^{th}$ derivative of f is a continuous function.
(6) If a function is *infinitely differentiable*, i.e., it has k^{th} derivatives for all k, then its k^{th} derivatives are continuous functions for all k.

Error-spotting exercises...

(1) Suppose f and g are everywhere defined functions such that f is twice differentiable and g is three times differentiable. Then the function $f+g$ is $2+3=5$ times differentiability and the function $f \cdot g$ is $2 \cdot 3 = 6$ times differentiable.
(2) A polynomial function of degree n is n times differentiable but not $n+1$ times differentiable, because the $(n+1)^{th}$ and higher derivatives all vanish.
(3) The function $x \mapsto x^{11/3}$ is infinitely differentiable everywhere because we can keep applying the differentiation formula as many times as we want.

4. Trigonometry: review, limits, and derivatives

(1) The following three important limits form the foundation of trigonometric limits: $\lim_{x \to 0} (\sin x)/x = 1$, $\lim_{x \to 0} (\tan x)/x = 1$, and $\lim_{x \to 0} (1 - \cos x)/x^2 = 1/2$.
(2) The derivative of sin is cos, the derivative of cos is $-\sin$. The derivative of tan is $\sec^2$, the derivative of cot is $-\csc^2$, the derivative of sec is $\sec \cdot \tan$, and the derivative of csc is $-\csc \cdot \cot$.
(3) The second derivative of any function of the form $x \mapsto a \sin x + b \cos x$ is the negative of that function, and the fourth derivative is the original function.

Actions ...

(1) Substitution is one trick that we use for trigonometric limits: we translate $\lim_{x \to c}$ to $\lim_{h \to 0}$ where $x = c + h$.
(2) Multiplicative splitting, chaining, and stripping are some further tricks that we often use.
(3) For derivatives of functions that involve composites of trigonometric and polynomial functions, we *have* to use the chain rule as well as rules for sums, differences, products, and quotients when simplifying expressions.

5. Tricky topics

These are some tricky question types and some stumbling blocks across multiple question types. The selection here is based on class feedback, your quiz scores, and concerns raised in problem session.

Some of this is a repeat of points given earlier. However, it may be helpful to have the information presented in this alternative format.

5.1. Piecewise definition by interval: left and right.

Think of examples for each point. I've deliberately not included examples, because I want you to puzzle out each point here in terms of things like this that you've seen. The material straddles limits, continuity, and differentiability.

(1) It is often the case that we define a function piecewise by splitting the domain into intervals. Here, a function has different expressions defining it on different intervals. For the remaining observations, we will assume that each of the piece functions itself is very nice (continuous, differentiable, etc.) so that most of the trouble arises from the changes in definition between intervals.
(2) At a point that is at the common boundary of two intervals, the function changes definition. The function at the boundary point may be defined using either of the two intervals, or separately, as an isolated definition just at that point. (Think of examples).
(3) If the point is included in the definition on one side, it is automatically continuous from that side. (Remember, we're assuming that the piece functions are continuous). For the other side, we need to

calculate the appropriate one-sided limit. If that piece function extends continuously to the point, we substitute the value. Otherwise, we use the limits techniques. (Think of examples).

(4) If the function is continuous from a particular side, that one-sided derivative can be calculated by differentiating the expression formally at the point and evaluating at the point. (Think of examples).

(5) The function is differentiable at a point of definition change if: (i) it is continuous (from both sides) and (ii) the left hand derivative and the right hand derivative agree.

(6) To calculate second or higher derivatives of functions with piecewise definitions, first get a piecewise definition for the function and then differentiate it.

(7) To do an $\epsilon - \delta$ proof of continuity for a function at a point where it may be changing definition, we need to find a δ that works for each piece, and then pick the minimum of those δs.

(8) In order to add, subtract, or multiply two functions with piecewise definitions, we need to break the domains into further pieces so that the pieces for both functions match up. (In mathematical jargon, this is a common refinment). Then we can add, subtract, and multiply in each piece.

(9) Also note that the limit, continuity, and differentiation formulas hold for one-sided approach.

(10) Composition involving piecewise definitions is tricky. The limit, continuity and differentiation theorems for composition do not hold for one-sided approach. If one of the functions is decreasing, then things can get flipped. For piecewise definitions, when composing, we need to think clearly about how the intervals transform.

Error-spotting exercises ...

(1) Consider the function:

$$f(x) := \begin{cases} 1, & x = 0 \\ x + 1, & x > 0 \\ x + \cos x, & x < 0 \end{cases}$$

The derivative is given as follows:

$$f'(x) := \begin{cases} 0, & x = 0 \\ 1, & x > 0 \\ 1 - \sin x, & x < 0 \end{cases}$$

(2) Consider the function:

$$f(x) := \begin{cases} x^2, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

Then the function $f(x) + f(1 - x)$ is:

$$f(x) + f(1 - x) = \begin{cases} x^2 + (1 - x)^2, & x < 0 \\ x^3 + (1 - x)^3, & x \geq 0 \end{cases}$$

5.2. Piecewise definitions: rational and irrational.

(1) Sometimes, we may define a function f as one thing for rational inputs and another thing for irrational inputs. The important thing to remember is that *every open interval contains both rational and irrational numbers*. Hence, however small an interval we choose, both definitions are operational in that interval. This is in sharp contrast to the piecewise definition by interval, where different definitions operate in different regions. We'll assume that both piece definitions are obtained by restriction from continuous functions on $\mathbb{R}$.

(2) A function f defined this way is continuous if both the rational and irrational definitions "agree" at the point. (This is assuming that both piece definitions are drawn from continuous functions of $\mathbb{R}$).

(3) An $\epsilon - \delta$ proof of continuity would find the δ that works for the rational and irrational pieces and use the minimum of these. The proof would involve splitting into cases for x based on whether x is rational or irrational.

(4) If both piece definitions are drawn from differentiable functions on $\mathbb{R}$, then f is differentiable at a point if the rational and irrational definitions for both f and f' "agree" at the point.

(5) In order for a second or higher derivative to exist, the first derivative must be defined in a neighborhood of the point. Note that in this sense, the rational-irrational version differs radically from the left-right version. For instance, consider:

$$f(x) := \begin{cases} x^3, & x \leq 0 \\ x^5, & x > 0 \end{cases}$$

and

$$g(x) := \begin{cases} x^3, & x \text{ rational} \\ x^5, & x \text{ irrational} \end{cases}$$

For both f and g, the function is continuous, and both "piece" derivatives at 0 are 0, so f and g are both differentiable at 0 and $f'(0) = g'(0) = 0$.

However, the situation becomes different with the second derivative. It turns out that $f''(0)$ exists and equals 0. But we cannot talk of $g''(0)$, because, *although $g'(0)$ exists, g' is not defined anywhere around 0, so it does not make sense to differentiate a second time.*

Thus, although the rational-irrational situation is somewhat similar to the left-right situation.

5.3. The $\sin(1/x)$ examples.

(1) The $\sin(1/x)$ and related examples are somewhat tricky because the function definition differs at an *isolated point*, namely 0.

(2) To calculate any limit or derivative at a point other than 0, we can do formal computations. However, to calculate the derivative at 0, we *must* use the definition of derivative as a limit of a difference quotient.

(3) For all the facts below, the qualitative conclusions hold if we replace sin by cos. The expressions for the derivatives change, but we haven't included those expressions below anyway.

(4) The function $f_0(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ satisfies the intermediate value property but is not continuous at 0. At all other points, it is infinitely differentiable and we can calculate the derivative formally.

(5) The function $f_1(x) := \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous but not differentiable at 0. We can see this from the pinching theorem – it is pinched between $|x|$ and $-|x|$. f_1 is infinitely differentable at all points other than 0.

(6) The function $f_2(x) := \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at 0, and infinitely differentiable everywhere other than 0, but the derivative is not continuous at 0. The limit $\lim_{x\to 0} f_2'(x)$ does not exist. Note that f_2' is defined everywhere and satisfies the intermediate value property but is not continuous.

(7) The function $f_3(x) := \begin{cases} x^3\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuously differentiable but not twice differentiable at 0, and infinitely differentiable everywhere other than 0.

5.4. Testing hypotheses about functions.

Many of you have faced tricky quiz problems where you're asked to determine whether certain general facts about functions are true. Ideally, to show that a general fact is true, you try to give a *generic* proof. However, to show that a general fact is not true, you usually need to come up with a counterexample.

Since the quiz problems are meant to be tricky, I typically choose at least a few options where the most obvious examples you choose don't give counterexamples. However, you can be cleverer still and try to understand how to find good counterexamples. Here are some general tips in that direction:

(1) Start trying to prove (in a rough sense) the statement to be true. Locate the *precise step* where your proof encounters an obstacle. What additional assumption do you need to make here? Try to think of an example of a function that violates this additional assumption.

(2) When the question involves one-sidedness, try both increasing and decreasing functions. Try functions that increase on part of the domain and decrease on another part. Also, try piecewise behavior.

(3) Wherever it makes sense, think of functions defined differently on the rationals and irrationals. This is particularly helpful to get functions that are well behaved at only a handful of points (e.g., continuous at only 14 points, or differentiable at only 11 points).
(4) Wherever it makes sense, think of the $\sin(1/x)$ examples. This is particularly helpful to get functions that are: (i) badly behaved at only one point, and (ii) give examples to show that the implications continuously differentiable $\implies$ differentiable $\implies$ continuous $\implies$ intermediate value property are strict.
(5) If the property that you are interested in (e.g., being periodic, or polynomial) remains preserved on adding a constant function, add a whacko constant function and see if things still hold up.
(6) For functions on intervals extending to infinity in one or both directions, think of examples where the function approaches but does not reach a value. For instance, $1/x^2$ approaches 0 as x approaches infinity, but does not reach it. This is useful for showing that the analogue of the extreme value theorem does not hold for intervals stretching out to infinity in one or both directions.

6. High yield practice

These are areas where practice shortly before the test should offer high yield. These are things that you're probably not yet very good at, but where being good gives you that extra edge:

(1) $\epsilon - \delta$ proofs for quadratic and piecewise linear.
(2) Limit computations for trigonometric functions, particularly those involving chaining. (Refer to the notes on "trigonometric limits and derivatives" for a number of computational techniques).
(3) Converting a definition involving max and min into a piecewise definition.
(4) Piecewise functions (see all the items listed under piecewise functions in "Tricky topics").

7. Quickly

7.1. **Arithmetic.** You should be able to:

- Do quick arithmetic involving fractions.
- Remember $\sqrt{2}$, $\sqrt{3}$, and π to at least two digits.
- Sense when an expression will simplify to 0.
- Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.

7.2. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if you know one of its linear factors (necessary for limit computations).
(4) Do polynomial long division (not usually necessary, but helpful).
(5) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

7.3. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin and cos at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$.

7.4. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

7.5. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a \sin x + b \cos x$.

(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).

7.6. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

7.7. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Graph a piecewise linear function with some thought.
(3) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(4) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.

7.8. **Fancy pictures.** Keep in mind approximate features of the graphs of:

(1) $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ and $x^3\sin(1/x)$, particularly the behavior near 0.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.

REVIEW SHEET FOR MIDTERM 2: BASIC

MATH 152, SECTION 55 (VIPUL NAIK)

Unlike the previous midterm, I've split the review sheet into two parts: basic and advanced. The basic review sheet contains (with some changes) the executive summaries from the lecture notes.

This is the *basic* part of the review sheet for midterm 2. Hopefully, you will not need to go through this more than once, except the points that give you trouble the first time around.

The advanced part contains the error-spotting exercises, a section on "Graphing and Miscellanea on Functions" and a section on "Tricky Topics" and this is what we will focus on in the live review session. It also contains a "Quickly" list of stuff you should be able to recall and do quickly.

Section numbers from the basic and advanced are matched up so that you can look at both sheets together.

This document does not re-review material already covered in the review sheet for midterm 1. It is your responsibility to go through that review sheet again and make sure you have mastered all the material there. We could cover in the review session some topics there that you are having difficulty with, but that will not be a priority.

1. Left-overs from differentiation basics

1.1. **Derivative as rate of change.** Words...

(1) The derivative of A with respect to B is the rate of change of A with respect to B. Thus, to determine rates of change of various quantities, we can use the techniques of differentiation.
(2) If there are three linked quantities that are changing together (e.g., different measures for a circle such as radius, diameter, circumference, area) then we can use the chain rule.

Most of the actions in this case are not more than a direct application of the words.

1.2. **Implicit differentiation.** Words...

(1) Suppose there is a curve in the plane, whose equation cannot be manipulated easily to write one variable in terms of the other. We can use the technique of implicit differentiation to determine the derivative, and hence the slope of the tangent line, at different points to the curve.
(2) For a curve where neither variable is expressible as a function of the other, the notion of derivative still makes sense as long as *locally*, we can get y as a function of x. For instance, for the circle $x^2 + y^2 = 1$, y is not a function of x, but if we restrict attention to the part of the circle above the x-axis, then on this restricted region, y is a function of x.
(3) In some cases, even when one variable is expressible as a function of the other, implicit differentiation is easier to handle as it may involve fewer messy squareroot symbols.

Actions ...

(1) To determine the derivative using implicit differentiation, write down the equations of both curves, differentiate both sides with respect to x, and simplify using all the differentiation rules, to get everything in terms of x, y, and dy/dx. Isolate the dy/dx term in terms of x and y, and compute it at whatever point is needed.
(2) This procedure can be iterated to compute higher order derivatives at specific points on the curve where the curve locally looks like a function.

2. Increase/decrease, maxima/minima, concavity, inflection, tangents, cusps, asymptotes

2.1. **Rolle's, mean value, increase/decrease, maxima/minima.** Words...

(1) If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$. This is called *Rolle's theorem* and is a consequence of the extreme-value theorem.

(2) If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists $c \in (a, b)$ such that $f'(c)$ is the difference quotient $(f(b) - f(a))/(b - a)$. This result is called the *mean-value theorem*. Geometrically, it says that for any chord, there is a parallel tangent. Another way of thinking about it is that every difference quotient is equal to a derivative at some intermediate point.
(3) If f is a function and c is a point such that $f(c) \geq f(x)$ for x to the immediate left of c, we say that c is a local maximum from the left. In this case, the left-hand derivative of f at c, if it exists, is greater than or equal to zero. This is because the difference quotient is greater than or equal to zero. Local maximum from the right implies that the right-hand derivative (if it exists) is ≤ 0, local minimum from the left implies that the left-hand derivative (if it exists) is ≤ 0, and local minimum from the right implies that the right-hand derivative (if it exists) is ≥ 0. Even in the case of *strict* local maxima and minima, we still need to retain the equality sign on the derivative because it occurs as a *limit* and a limit of positive numbers can still be zero.
(4) If c is a point where f attains a local maximum (i.e., $f(c) \geq f(x)$ for all x close enough to c on both sides), then $f'(c)$, if it exists, is equal to zero. Similarly for local minimum.
(5) A *critical point* for a function is a point where either the function is not differentiable or the derivative is zero. All local maxima and local minima must occur at critical points.
(6) If $f'(x) > 0$ for all x in the open interval (a, b), f is increasing on (a, b). Further, if f is one-sided continuous at the endpoint a and/or the endpoint b, then f is increasing on the interval including that endpoint. Similarly, $f'(x) < 0$ implies f decreasing.
(7) If $f'(x) > 0$ everywhere except possibly at some isolated points (so that they don't cluster around any point) where f is still continuous, then f is increasing everywhere.
(8) If $f'(x) = 0$ on an open interval, f is constant on that interval, and it takes the same constant value at an endpoint where it's continuous from the appropriate side.
(9) If f and g are two functions that are both continuous on an interval I and have the same derivative on the interior of I, then $f - g$ is a constant function.

    Note: Due to an oversight, the remaining items in this list were not included in the executive summary to the lecture notes.
(10) There is a *first derivative test* which provides a sufficient (though not necessary) condition for a local extreme value: it says that if the first derivative is nonnegative (respectively positive) on the immediate left of a critical point, that gives a strict local maximum (respectively local maximum) from the left. If the first derivative is negative on the immediate left, we get a strict local minimum from the left. If the first derivative is positive on the immediate right, we get a strict local minimum from the right, and if it is negative on the immediate right, we get a strict local maximum from the right.

    The first derivative test is similar to the corresponding "one-sided derivative" test, but is somewhat stronger for a variety of situations because in many cases, one-sided derivatives are zero, which is inconclusive, whereas the first derivative test fails us more rarely.
(11) The second derivative test states that if f has a critical point c where it is twice differentiable, then $f''(c) > 0$ implies that f has a local minimum at c, and $f''(c) < 0$ implies that f has a local maximum at c.
(12) There are also higher derivative tests that work for critical points c where $f'(c) = 0$. These work as follows: we look for the smallest k such that $f^{(k)}(c) \neq 0$. If this k is even, then f has a local extreme value at c, and the nature (max versus min) depends on the sign of $f^{(k)}(c)$ (max if negative, min if positive). If k is odd, then we have what we'll see soon is a point of inflection.
(13) To determine absolute maxima/minima, the candidates are: points of discontinuity, boundary points of domain (whether included in domain or outside the domain; if the latter, then limiting), critical points (derivative zero or undefined), and limiting cases at $\pm\infty$.

Actions... (think of examples that you've done)

(1) Rolle's theorem, along with the more sophisticated formulations involving increasing/decreasing, tell us that there is an intimate relationship between the zeros of a function and the zeros of its derivative. Specifically, between any two zeros of the function, there is a zero of its derivative. Thus,

if a function has r zeros, the derivative has at least $r-1$ zeros, with at least one zero between any two consecutive zeros of f.

(2) The more sophisticated version tells us that between any two zeros of a differentiable function, the function must attain a local maximum or local minimum. So, if the function is increasing everywhere or decreasing everywhere, there is at most one zero.

(3) The mean-value theorem allows us to use bounds on the derivative of a function to bound the overall variation, or change, in the function. This is because if the derivative cannot exceed some value, then the difference quotient also cannot exceed that value, which means that the function cannot change too quickly on average.

(4) To determine regions where a function is increasing and decreasing, we find the derivative and determine regions where the derivative is positive, zero, and negative.

(5) To determine all the local maxima and local minima of a function, find all the critical points. To find the critical points, solve $f' = 0$ and also consider, as possible candidates, all the points where the function changes definition. *Although a point where the function changes definition need not be a critical point, it is a very likely candidate.*

(6) *This item was missed in the original executive summary*: To determine absolute maxima and absolute minima, find all candidates (discontinuity, endpoints, limiting cases, boundary points), evaluate at each, and compare. Note that any absolute maximum must arise as a local or endpoint maximum. However, instead of first determining which critical points give local maxima by the derivative tests, we can straightaway compute values everywhere and compare, if our interest is solely in finding the absolute maximum and minimum.

2.2. **Concave up/down and points of inflection.** Words ...

(1) A function is called *concave up* on an interval if it is continuous and its first derivative is continuous and increasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be positive except possibly at isolated points, where it can be zero. (Think x^4, whose first derivative, $4x^3$, is increasing, and the second derivative is positive everywhere except at 0, where it is zero).

(2) A function is called *concave down* on an interval if it is continuous and its first derivative is continuous and decreasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be negative except possibly at isolated points, where it can be zero.

(3) A *point of inflection* is a point where the sense of concavity of the function changes. A point of inflection for a function is a point of local extremum for the first derivative.

(4) Geometrically, at a point of inflection, the tangent line to the graph of the function *cuts through* the graph.

Actions ...

(1) To determine points of inflection, we first find critical points for the first derivative (which are points where this derivative is zero or undefined) and then use the first or second derivative test at these points. Note that these derivative tests are applied to the first derivative, so the first derivative here is the second derivative and the second derivative here is the third derivative.

(2) In particular, if the second derivative is zero and the third derivative exists and is nonzero, we have a point of inflection.

(3) A point where the first two derivatives are zero could be a point of local extremum or a point of inflection. To find out which one it is, we either use sign changes of the derivatives, or we use higher derivatives.

(4) Most importantly, the second derivative being zero does *not* automatically imply that we have a point of inflection.

(5) If the third derivative is zero, we can use a higher derivative test again. The upshot is that if the first $k \geq 2$ for which $f^{(k)}(c) \neq 0$ is even, then we do not have a point of inflection, but if the first k is odd, then we have a point of inflection.

2.3. **Tangents, cusps, and asymptotes.** Words...

(1) We say that f has a horizontal asymptote with value L if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. Sometimes, both might occur. (In fact, in almost all the examples you have seen, the limits at $\pm\infty$, if finite, are both equal).
(2) We say that f has a vertical asymptote at c if $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$. Note that in this case, it usually also happens that $f'(x) \to \pm\infty$ on the relevant side, with the sign the same as that of $f(x)$'s approach if the approach is from the left and opposite to that of $f(x)$'s approach if the approach is from the right. However, this is not a foregone conclusion.
(3) We say that f has a vertical tangent at the point c if f is continuous (and finite) at c and $\lim_{x\to c} f'(x) = \pm\infty$, with the *same sign of infinity* from both sides. If f is increasing, this sign is $+\infty$, and if f is decreasing, this sign is $-\infty$. Geometrically, points of vertical tangent behave a lot likepoints of inflection and usually *do* give points of inflection (in the sense that the tangent line cuts through the graph). Think $x^{1/3}$.
(4) We say that f has a vertical cusp at the point c if f is continuous (and finite) at c and $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are infinities of opposite sign. In other words, f takes a sharp about-turn at the x-value of c. Think $x^{2/3}$.
(5) We say that f is asymptotic to g if $\lim_{x\to\infty} f(x) - g(x) = \lim_{x\to-\infty} f(x) - g(x) = 0$. In other words, the graphs of f and g come progressively closer as $|x|$ becomes larger. (We can also talk of one-sided asymptoticity, i.e., asymptotic only in the positive direction or only in the negative direction). When g is a *nonconstant linear function*, we say that f has an *oblique asymptote*. Horizontal asymptotes are a special case, where one of the functions is a constant function.

Actions...

(1) Finding the horizontal asymptotes involves computing limits as the domain value goes to infinity. Finding the vertical asymptotes involves locating points in the domain, or the boundary of the domain, where the function limits off to infinity. For both of these, it is useful to remember the various rules for limits related to infinities.
(2) Remember that for a vertical tangent or vertical cusp at a point, it is necessary that the function be continuous (and take a finite value). So, we not only need to find the points where the derivative goes off to infinity, we also need to make sure those are points where the function is continuous. Thus, for the function $f(x) = 1/x$, $f'(x) \to -\infty$ on both sides as $x \to 0$, but we do *not* obtain a vertical tangent – rather, we obtain a vertical asymptote.

3. Max-min problems

Words...

(1) In real-world situations, maximization and minimization problems typically involve multiple variables, multiple constraints on those variables, and some objective function that needs to be maximized or minimized.
(2) The only thing we know to solve such problems is to reduce everything in terms of one variable. This is typically done by *using up* some of the constraints to express the other variables in terms of that variable.
(3) The problem then typically boils down to a maximization/minimization problem of a function in a single variable over an interval. We use the usual techniques for understanding this function, determining the local extreme values, determining the endpoint extreme values, and determining the absolute extreme values.

Special note: For integer optimization, please refer back to the lecture notes!
Actions... (think of examples; also review the notes on max-min problems)

(1) Extremes sometimes occur at endpoints but these endpoints could correspond to degenerate cases. For instance, of all the rectangles with given perimeter, the square has the maximum area, and the minimum occurs in the degenerate case of a rectangle where one side has length zero.
(2) Some constraints on the variables we have are explicitly stated, while others are implicit. Implicit constraints include such things as nonnegativity constraints. *Some of these implicit constraints may be on variables other than the single variable in terms of which we eventually write everything.*

(3) After we have obtained the objective function in terms of one variable, we are in a position to throw out the other variables. However, before doing so, it is *necessary to translate all the constraints into constraints on the one variable that we now have.*
(4) When our intent is to maximize a function, it is sometimes useful to maximize an equivalent function that is easier to visualize or differentiate. For instance, to maximize $\sqrt{f(x)}$ is equivalent to maximizing $f(x)$ if $f(x)$ is nonnegative. With this way of thinking about equivalent functions, we can make sure that the actual function that we differentiate is easy to differentiate. The main criterion is that the two functions should rise and fall together. (Analogous observations apply for minimizing) Remember, however, that to calculate the *value* of the maximum/minimum, you should go back to the original function.
(5) Sometimes, there are other parameters in the maximization/minimization problem that are *unknown constants*, and the final solution is expected to be in terms of those constants. In rare cases, the nature of the function, and hence the nature of maxima and minima, depends on whether those constants fall in particular intervals. *If you find this to be the case, go back to the original problem and see whether the real-world situation it came from constrains the constants to one of the intervals.*
(6) For some geometrical problems, the maximization/minimization can be done trigonometrically. Here, we make a clever choice of an angle that controls the *shape* of the figure and then use the trigonometric functions of that angle. This could provide alternate insight into maximization.

4. Definite and indefinite integration

4.1. **Definition and basics.** Words ...

(1) The definite integral of a continuous (though somewhat weaker conditions also work) function f on an interval $[a, b]$ is a measure of the signed area between the graph of f and the x-axis. It measures the *total value* of the function.
(2) For a partition P of $[a, b]$, the lower sum $L_f(P)$ adds up, for each subinterval of the partition, the length of that interval times the minimum value of f over that interval. The upper sum adds up, for each subinterval of the partition, the length of that interval times the maximum value of f on that subinterval.
(3) Every lower sum of f is less than or equal to every upper sum of f.
(4) The *norm* or *size* of a partition P, denoted $\|P\|$, is defined as the maximum of the lengths of its subintervals.
(5) If P_1 is a finer partition than P_2, i.e., every interval of P_1 is contained in an interval of P_2, then the following three things are true: (a) $L_f(P_2) \le L_f(P_1)$, (b) $U_f(P_1) \le U_f(P_2)$, and (c) $\|P_1\| \le \|P_2\|$.
(6) If $\lim_{\|P\| \to 0} L_f(P) = \lim_{\|P\| \to 0} U_f(P)$, then this common limit is termed the *integral* of f on the interval $[a, b]$.
(7) We can define $\int_a^b f(x)\,dx$ as above if $a < b$. If $a = b$ the integral is defined to be 0. If $a > b$, the integral is defined as $-\int_b^a f(x)\,dx$.
(8) A continuous function on $[a, b]$ has an integral on $[a, b]$. A piecewise continuous function where one-sided limits exist and are finite at every point is also integrable.

Actions ...

(1) For constant functions, the integral is just the product of the value of the function and the length of the interval.
(2) Points don't matter. So, if we change the value of a function at one point while leaving the other values unaffected, the integral does not change.
(3) A first-cut lower and upper bound on the integral can be obtained using the *trivial* partition, where we do not subdivide the interval at all. The upper bound is thus the maximum value times the length of the interval, and the lower bound is the minimum value times the length of the interval.
(4) The finer the partition, the closer the lower and upper bounds, and the better the approximation we obtain for the integral.
(5) A very useful kind of partition is a *regular partition*, which is a partition where all the parts have the same length. If the integral exists, we can calculate the actual integral as $\lim_{n \to \infty}$ of the upper sums or the lower sums for a regular partition into n parts.

(6) When a function is increasing on some parts of the interval and decreasing on other parts, it is useful to choose the partition in such a way that on each piece of the partition, the function is either increasing throughout or decreasing throughout. This way, the maximum and minimum occur at the endpoints in each piece. In particular, try to choose all points of local extrema as points of partition.

4.2. **Definite integral, antiderivative, and indefinite integral.** Words ..

(1) We have $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
(2) We say that F is an antiderivative for f if $F' = f$.
(3) For a continuous function f defined on a closed interval $[a, b]$, and for a point $c \in [a, b]$, the function F given by $F(x) = \int_c^x f(t)\,dt$ is an antiderivative for f.
(4) If f is continuous on $[a, b]$ and F is a function continuous on $[a, b]$ such that $F' = f$ on (a, b), then $\int_a^b f(x)\,dx = F(b) - F(a)$.
(5) The two results above essentially state that differentiation and integration are opposite operations.
(6) For a function f on an interval $[a, b]$, if F and G are antiderivatives, then $F - G$ is constant on $[a, b]$. Conversely, if F is an antiderivative of f, so is F plus any constant.
(7) The *indefinite integral* of a function f is the collection of all antiderivatives for the function. This is typically written by writing one antiderivative plus C, where C is an arbitrary constant. We write $\int f(x)\,dx$ for the indefinite integral. Note that there are no upper and lower limits.
(8) Both the definite and the indefinite integral are additive. In other words, $\int f(x)\,dx + \int g(x)\,dx = \int f(x) + g(x)\,dx$. The analogue holds for definite integrals, with limits.
(9) We can also pull constants multiplicatively out of integrals.
(10) Note the important formula:

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x)$$

Actions ...

(1) Consider the integration problem:

$$\int \frac{2x}{x^3 + 1}\,dx = \int \frac{2x}{x^3}\,dx + \int \frac{2x}{1}\,dx = \int \frac{2}{x^2}\,dx + \int 2x\,dx = \frac{-2}{x} + x^2 + C$$

(2) To do a definite integral, find any one antiderivative and evaluate it between limits.
(3) An important caveat: when using antiderivatives to do a definite integral, it is important to make sure that the antiderivative is defined and continuous everywhere on the interval of integration. (Think of the $1/x^3$ example).
(4) To do an indefinite integral, find any antiderivative and put a $+C$.
(5) To find an antiderivative, use the additive splitting and pulling constants out, and the fact that $\int x^r\,dx = x^{r+1}/(r + 1)$.

4.3. **Higher derivatives, multiple integrals, and initial/boundary conditions.** Actions ...

(1) The simplest kind of *initial value problem* (a notion we will encounter again when we study differential equations) is as follows. The k^{th} derivative of a function is given on the entire domain. Next, the *values* of the function and the first $k - 1$ derivatives are given at a single point of the domain. We can use this data to find the function. Step by step, we find derivatives of lower orders. First, we integrate the k^{th} derivative to get that the $(k - 1)^{th}$ derivative is of the form $F(x) + C$, where C is unknown. We now use the value of the $(k - 1)^{th}$ derivative at the given point to find C. Now, we have the $(k - 1)^{th}$ derivative. We proceed now to find the $(k - 2)^{th}$ derivative, and so on.
(2) Sometimes, we may be interested in finding *all* functions with a given second derivative f. For this, we have to perform an indefinite integration twice. The net result will be a general expression of the form $F(x) + C_1 x + C_2$, where F is a function with $F'' = f$, and C_1 and C_2 are arbitrary constants. In other words, we now have *up to constants or linear functions* instead of *up to constants* as our degree of ambiguity.

(3) More generally, if the k^{th} derivative of a function is given, the function is uniquely determined up to additive differences of polynomials of degree strictly less than k (so, degree at most $k-1$). The number of free constants that can take arbitrary real values is k (namely, the coefficients of the polynomial).
(4) This general expression is useful if, instead of an initial value problem, we have a boundary value problem. Suppose we are given G'' as a function, and we are given the value of G at two points. We can then first find the general expression for G as $F + C_1 x + C_2$. Next, we plug in the values to get a system of two linear equations, that we solve in order to determine C_1 and C_2, and hence G.

4.4. **Reversing the chain rule.** Words ...

(1) The chain rule states that $(f \circ g)' = (f' \circ g) \cdot g'$.
(2) Some integrations require us to reverse the chain rule. For this, we need to realize the integrand that we have in the form of the right-hand side of the chain rule.
(3) The first step usually is to find the correct function g, which is the *inner function* of the composition, then to adjust constants suitably so that the remaining term is g', and then figure out what f' is. Finally, we find an antiderivative for f', which we can call f, and then compute $f \circ g$.
(4) A slight variant of this method (which is essentially the same) is the substitution method, where we identify g just as before, try to spot g' in the integrand as before, and then put $u = g(x)$ and rewrite the integral in terms of u.

Actions ...

(1) Good targets for u in the u-substitution are things that are shielded.
(2) In general, things in denominators, with non-integer exponents, or with complicated composites appended to them are good targets for u.

4.5. **u-substitutions for definite integrals.** Words ... (try to recall the numerical formulations)

(1) When doing the u-substitution for definite integrals, we transform the upper and lower limits of integration by the u-function.
(2) Note that the u-substitution is valid only when the u-function is well-defined on the entire interval of integration.
(3) The integral of a translate of a function is the integral of a function with the interval of integration suitably translated.
(4) The integral of a multiplicative transform of a function is the integral of the function with the interval of integration transformed by the same multiplicative factor, scaled by that multiplicative factor.

4.6. **Symmetry and integration.** Words ...

(1) If a function is continuous and even, its integral on $[-a, 0]$ equals its integral on $[0, a]$. More generally, its integrals on any two segments that are reflections of each other about the origin are equal. As a corollary, the integral on $[-a, a]$ is twice the integral on $[0, a]$.
(2) If a function is continuous and odd, its integral on $[-a, 0]$ is the negative of its integral on $[0, a]$. More generally, its integrals on any two segments that are reflections of each other about the origin are negatives of each other. As a corollary, the integral on $[-a, a]$ is zero.
(3) If a function is continuous and has mirror symmetry about the line $x = c$, its integral on $[c - h, c]$ equals its integral on $[c, c + h]$.
(4) If a function is continuous and has half-turn symmetry about $(c, f(c))$, its integral on any interval of the form $[c - h, c + h]$ is $2hf(c)$. Basically, all the variation about $f(c)$ *cancels out* and the *average value* is $f(c)$.
(5) Suppose f is continuous and periodic with period h and F is an antiderivative of f. The integral of f over any interval of length h is constant. Thus, $F(x + h) - F(x)$ is the same constant for all x. (We saw this fact long ago, without proof).
(6) The constant mentioned above is zero iff F is periodic, i.e., f has a periodic antiderivative.
(7) There is thus a well-defined *average value* of a continuous periodic function on a period. This is also the average value of the same periodic function on any interval whose length is a nonzero integer multiple of the period. This is also the limit of the average value over very large intervals.

Actions...

(1) All this even-odd-periodic stuff is useful for trivializing some integral calculations without computing antiderivatives. This is more than an idle observation, since in a lot of real-world situations, we get functions that have some obvious symmetry, even though we know very little about the concrete form of the functions. We use this obvious symmetry to compute the integral.
(2) Even if the whole integrand does not succumb to the lure of symmetry, it may be that the integrand can be written as (something nice and symmetric) + (something computable). The (nice and symmetric) part is then tackled using the ideas of symmetry, and the computable part is computed.

4.7. **Mean-value theorem.** Words ...

(1) The *average value*, or *mean value*, of a continuous function on an interval is the quotient of the integral of the function on the interval by the length of the interval.
(2) The mean value theorem for integrals says that a continuous function must attain its mean value somewhere on the interior of the interval.
(3) For periodic functions, the mean value over any interval whose length is a multiple of the period is the same. Also, the mean value over a very large interval approaches this value.
(4) The mean value of a periodic continuous function f being 0 means that f has a periodic antiderivative; in fact, every antiderivative of f is periodic. If the mean value is nonzero, every antiderivative of f is periodic with shift, and the linear part of any antiderivative has slope equal to the mean value of f.

Actions ...

(1) The mean values (over a period) of sin and cos are 0. For sin, we can see this since it is an *odd* periodic function. cos is a translate of sin, hence has the same mean value.
(2) The mean values (over a period) of $\sin^2$ and $\cos^2$ are $1/2$. The functions add up to 1, and are translates of each other, so this makes sense.
(3) The mean value (over a period) of $x \mapsto f(mx)$, $m \neq 0$, is the same as the mean value of f, where f is a continuous periodic function.
(4) The mean value (over a period) of $|\sin|$ is $2/\pi$, and that of $\sin^+$ is $1/\pi$.

4.8. **Application to area computations.** Words ...

(1) We can use integration to determine the area of the region between the graph of a function f and the x-axis from $x = a$ to $x = b$: this integral is $\int_a^b f(x)\,dx$. The integral measures the signed area: parts where $f \geq 0$ make positive contributions and parts where $f \leq 0$ make negative contributions. The magnitude-only area is given as $\int_a^b |f(x)|\,dx$. The best way of calculating this is to split $[a, b]$ into sub-intervals such that f has constant sign on each sub-interval, and add up the areas on each sub-interval.
(2) Given two functions f and g, we can measure the area between f and g between $x = a$ and $x = b$ as $\int_a^b |f(x) - g(x)|\,dx$. For practical purposes, we divide into sub-intervals so that on each sub-interval one function is bigger than the other. We then use integration to find the magnitude of the area on each sub-interval and add up. If f and g are both continuous, the points where the functions *cross* each other are points where $f = g$.

Actions ...

(1) In some situations we are directly given functions and/or curves and are asked to find areas. In others, we are given real-world situations where we need to find areas of regions. Here, we have to find functions and set up the integration problem as an intermediate step.
(2) In all these situations, it is important to draw the graphs in a reasonably correct way. This brings us to all the ideas that are contained in graph drawing. Remember, here we may be interested in simultaneously graphing more than one function. Thus, in addition to being careful about each function, we should also correctly estimate where one function is bigger than the other, and find (approximately or exactly) the intersection points. (Go over the notes on graph-drawing, and some additional notes on graphing that weren't completely covered in class).

(3) In some situations, we are asked to find the area(s) of region(s) bounded by the graphs of one, two, three, or more functions. Here, we first need to sketch the figure. Then, we need to find the interval of integration, and if necessary, split this interval into sub-intervals, such that on each sub-interval, we know exactly what integral we need to do. For instance, consider the region between the graphs of sin, cos, and the x-axis. Basically, the idea is to find, for all the vertical slices, the upper and lower limits of the slice.

REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 152, SECTION 55 (VIPUL NAIK)

This is the part of the review sheet that we will concentrate on during the review session.

1. Left-overs from differentiation basics

1.1. Derivative as rate of change.
No error-spotting exercises

1.2. Implicit differentiation.
No error-spotting exercises

2. Increase/decrease, maxima/minima, concavity, inflection, tangents, cusps, asymptotes

2.1. Rolle's, mean value, increase/decrease, maxima/minima.
Error-spotting exercises

(1) If a function f has a local maximum at a point c in its domain, then f is increasing on the immediate left of c and decreasing on the immediate right of c.

(2) Consider the function:

$$f(x) := \{ \begin{array}{rl} x^3 - 12x + 14, & x \le 1 \\ x^2 - 6x + 8, & x > 1 \end{array}$$

The derivative is:

$$f'(x) = \{ \begin{array}{rl} 3x^2 - 12, & x \le 1 \\ 2x - 6, & x > 1 \end{array}$$

The solutions for $f'(x) = 0$ are $x = -2$ and $x = 2$ (for the $x \le 1$ case) and $x = 3$ (the $x > 1$ case). Thus, the critical points are at $x = -2$, $x = 2$, and $x = 3$.

(3) Consider the function:

$$f(x) := x^4 - x + 1$$

The derivative is:

$$f'(x) = 4x^3 - 1$$

Solve $f'(x) = 0$ and we get $x = (1/4)^{1/3}$. Thus, f has a local maximum at $x = (1/4)^{1/3}$. The local maximum value is:

$$4((1/4)^{1/3})^3 - 1$$

which is 0.

(4) Consider the function

$$f(x) := \frac{1}{x^3 - 1}$$

The derivative is:

$$f'(x) = \frac{3x^2}{(x^3 - 1)^2}$$

The derivative is zero at $x = 0$, so that gives a critical point. Also, the derivative is undefined at $x = 1$, so that gives another critical point for f.

(5) An everywhere differentiable function f on $\mathbb{R}$ has critical points at 2, 5, and 9 with corresponding function values 11, 16, and 3 respectively. Thus, the absolute maximum value of f is 16 and the absolute minimum value is 3.

2.2. **Concave up/down and points of inflection.** Error-spotting exercises ...

(1) Consider the function

$$f(x) := 3x^5 - 5x^4 + 12x + 17$$

The derivative is:

$$f'(x) = 15x^4 - 20x^3 + 12$$

The second derivative is:

$$f''(x) = 60x^3 - 60x^2$$

The zeros of this are $x = 0$ and $x = 1$. The function thus has points of inflection at the points on the graph corresponding to $x = 0$ and $x = 1$.

(2) To check whether a critical point for the first derivative gives a point of inflection for the graph of the function, we need to check the sign of the third derivative. If the third derivative is nonzero, we get a point of inflection. If the third derivative is zero, then we *do not* get a point of inflection.

2.3. **Tangents, cusps, and asymptotes.** Cute fact: Rational functions are asymptotically polynomial, and the polynomial to which a given rational function is asymptotic (both directions) is obtained by doing long division and looking at the quotient. If the degree of the numerator is one more than that of the denominator, we get an oblique (linear) asymptote. If the numerator and denominator have equal degree, we get a horizontal asymptote (both directions) with nonzero value. If the numerator has smaller degree, the x-axis is the horizontal asymptote (both directions).

Error-spotting exercises

(1) If $\lim_{x \to \infty} f(x) = L$ with L a finite number, then $\lim_{x \to \infty} f'(x) = 0$.
(2) If $\lim_{x \to \infty} f'(x) = 0$, then $\lim_{x \to \infty} f(x) = L$, with L a finite number.
(3) If f' has a vertical tangent at a point a in its domain, then f has a point of inflection at $(a, f(a))$.
(4) If f' has a vertical cusp at a point a in its domain, then f has a local extreme value at a.
(5) Suppose f and g are functions defined on all of $\mathbb{R}$. Suppose f has a vertical tangent at a point a in its domain and g has a vertical tangent at a point b in its domain. Then $f + g$ has a vertical tangent at $a + b$ and $f - g$ has a vertical tangent at $a - b$.
(6) Suppose f and g are functions, both defined on $\mathbb{R}$. Suppose f and g both have vertical tangents at a point a in their domain (i.e., at the same point in the domain). Then, the sum $f + g$ also has a vertical tangent at a.
(7) Suppose f and g are functions, both defined on $\mathbb{R}$. Suppose f and g both have vertical tangents at a point a in their domain (i.e., at the same point in the domain). Then, the pointwise product $f \cdot g$ also has a vertical tangent at a. *This is trickier than it looks!*

3. Max-min problems

Smart thoughts for smart people ...

(1) Before getting started on the messy differentiation to find critical points, think about the constraints and the endpoints. Is it obvious that the function will attain a minimum/maximum at one of the endpoints? What are the values of the function at the endpoints? (If no endpoints, take limiting values as you go in one direction of the domain). Is there an intuitive reason to believe that the function attains its optimal value somewhere *in between* rather than at an endpoint? Is there some kind of trade-off to be made? Are there some things that can be said qualitatively about where the trade-off is likely to occur?
(2) Feel free to convert your function to an equivalent function such that the two functions rise and fall together. This reduces the burden of messy expressions.
(3) *Cobb-Douglas production*: For $p, q > 0$, the function $x \mapsto x^p(1 - x)^q$ attains a local maximum at $p/(p + q)$. In fact, this is the absolute maximum on $[0, 1]$, and the function value is $p^p q^q/(p + q)^{p+q}$. This is important because this function appears in disguise all the time (e.g., maximizing area of rectangle with given perimeter, etc.)

(4) A useful idea is that when dividing a resource into two competing uses, and one use is hands-down better than the other, the *best* use happens when the entire resource is devoted to the better use. However, the *worst* may well happen somewhere in between, because divided resources often perform even worse than resources devoted wholeheartedly to a bad use. This is seen in perimeter allocation to boundaries with the objective function being the total area, and area allocation to surfaces with the objective function being the total volume.

(5) When we want to *maximize* something subject to a collection of many constraints, the most relevant constraint is the *minimum* one. Think of the ladder-through-the-hallway problem, or the truck-going-under-bridges problem.

Error-spotting exercises

(1) The absolute maximum among the values of a (?) function (of reals) at integers is attained at the integer closest to the point at which it attains its absolute maximum among all reals.

(2) The absolute maximum among the values of a (?) function (of reals) at integers is attained at one of the integers closest to the point at which it attains a local maximum.

(3) To maximize the sum of two functions is equivalent to maximizing each one separately and then finding the common point of maximum.

(4) If f is a function that is continuous and concave up on an interval $[a, b]$, then the absolute minimum of f always occurs at an interior point and the absolute maximum of f always occurs at an endpoint. *This is a little subtle, because it's almost but not completely correct. Think through it clearly!*

(5) Consider the function:

$$f(x) := \begin{cases} x^3, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$$

Then, f' is increasing on $[0, 2]$, so f is concave up on $[0, 2]$.

4. Definite and indefinite integration

4.1. Definition and basics.
Error-spotting exercises ...

(1) If P_1 and P_2 are partitions of $[a, b]$ and $\|P_2\| \le \|P_1\|$, then P_2 is finer than P_1.

(2) If P_1 and P_2 are partitions of $[a, b]$ such that P_2 is finer than P_1, and f is a bounded function on $[a, b]$, then $L_f(P_2) \le L_f(P_1)$ and $U_f(P_2) \le U_f(P_1)$.

(3) For any continuous function f on $[a, b]$, the number of parts n we need in a regular partition of $[a, b]$ so that the integral is bounded in an interval of length L is proportional to $1/n$.

4.2. Definite integral, antiderivative, and indefinite integral.
Error-spotting exercises ...

(1) Consider the function $f(x) := \int_x^{x^2} \sin x \, dx$. Then $f'(x) = \sin(x^2) - \sin(x)$.

(2) Suppose f is a function on the nonzero reals such that $f'(x) = 1/x^2$ for all $x \in \mathbb{R}$. Then, we must have $f(x) = 1/x + C$ for some constant C.

4.3. Higher derivatives, multiple integrals, and initial/boundary conditions.
Error-spotting exercises ...

(1) Suppose F and G are everywhere k times differentiable functions for k a positive integer. If the k^{th} derivatives of the functions F and G are equal, then $F - G$ is a polynomial of degree k.

(2) Suppose F is a function defined on nonzero reals and $F''(x) = 1/x^3$ for all x. Then, F is of the form $F(x) = 1/x + C$ where C is a real constant.

4.4. Reversing the chain rule.
No error-spotting exercises

4.5. u-substitutions for definite integrals.
No error-spotting exercises

4.6. Symmetry and integration.
No error-spotting exercises

4.7. Mean-value theorem.
No error-spotting exercises

4.8. Application to area computations.
No error-spotting exercises

5. Graphing and miscellanea on functions

5.1. **Symmetry yet again.** Words...

1. All mathematics is the study of symmetry (well, not all).
2. One interesting kind of symmetry that we often see in the graph of a function is *mirror symmetry* about a vertical line. This means that the graph of the function equals its reflection about the vertical line. If the vertical line is $x = c$ and the function is f, this is equivalent to asserting that $f(x) = f(2c - x)$ for all x in the domain, or equivalently, $f(c + h) = f(c - h)$ whenever $c + h$ is in the domain. In particular, the domain itself must be symmetric about c.
3. A special case of mirror symmetry is the case of an *even function*. An even function is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
4. Another interesting kind of symmetry that we often see in the graph of a function is *half-turn symmetry* about a point on the graph. This means that the graph equals the figure obtained by rotating it by an angle of π about that point. A point (c, d) is a point of half-turn symmetry if $f(x) + f(2c - x) = 2d$ for all x in the domain. In particular, the domain itself must be symmetric about c. If f is defined at c, then $d = f(c)$.
5. A special case of half-turn symmetry is an odd function, which is a function having half-turn symmetry about the origin.
6. Another symmetry is *translation symmetry*. A function is *periodic* if there exists $h > 0$ such that $f(x + h) = f(x)$ for all x in the domain of the function (in particular, the domain itself should be invariant under translation by h). If a smallest such h exists, then such an h is termed the period of f.
7. A related notion is that of a function that is *periodic with shift*. A function is periodic with shift if there exists $h > 0$ and $k \in \mathbb{R}$ such that $f(x + h) - f(x) = k$ for all $x \in \mathbb{R}$. Note that if k is nonzero, the function isn't periodic.

   If f is differentiable for all real numbers, then f' is periodic if and only if f is periodic with shift. In particular, if f' is periodic with period h, then $f(x + h) - f(x)$ is constant. If this constant value is k, then the graph of f has a two-dimensional translational symmetry by (h, k) and its multiples.

   A function that is periodic with shift can be expressed as the sum of a linear function (slope k/h) and a periodic function. The linear part represents the secular trend and the periodic part represents the seasonal variation.

Derivative facts...

1. The derivative of an even function, if defined everywhere, is odd. Any antiderivative of an odd function is even.
2. The derivative of an odd function is even. Any antiderivative of an even function is an odd function plus a constant.
3. The derivative of a function with mirror symmetry has half turn symmetry about the corresponding x-value and has value 0 at that x-value. (For a more detailed description of these, see the solutions to the November 12 whoppers).
4. Assuming that f' is defined and does not change sign infinitely often on a neighborhood of c, we have that if $x = c$ is an axis of mirror symmetry for the graph of f, then c is a point of local extremum. The reason is that if f is increasing on the immediate left, it must be decreasing on the immediate right, and similarly ...
5. Assuming that f'' is defined and does not change sign infinitely often on a neighborhood of c, we have that if $(c, f(c))$ is a point of half-turn symmetry for the graph of f, then it is also a point of inflection for the graph. The reason is that if f is concave up on the immediate left, it must be concave down on the immediate right, and similarly ...
6. The conver statements to the above two do not hold: most points of inflection do not give points of half-turn symmetry, and most local extrema do not give axes of mirror symmetry.
7. If f has more than one axis of mirror symmetry, then it is periodic. Conversely, if f is periodic with period h, and has an axis of mirror symmetry $x = c$, then all $x = c + (nh/2)$, n an integer, are axes of mirror symmetry.

(8) If f has more than one point of half-turn symmetry, then it is periodic with shift. Conversely, if f is periodic with shift and has a point of half-turn symmetry, it has infinitely many points of half-turn symmetry.

Cute facts...

(1) Constant functions enjoy mirror symmetry about every vertical line and half-turn symmetry about every point on the graph.
(2) Nonconstant linear functions enjoy half-turn symmetry about every point on their graph. They do not enjoy any mirror symmetry (in the sense of mirror symmetry about vertical lines) because they are everywhere increasing or everywhere decreasing. (They do have mirror symmetry about *oblique* lines, but this is not a kind of symmetry that we are considering).
(3) Quadratic (nonlinear) functions enjoy mirror symmetry about the line passing through the vertex (which is the unique absolute maximum/minimum, depending on the sign of the leading coefficient). They do not enjoy any half-turn symmetry.
(4) Cubic functions enjoy half-turn symmetry about the point of inflection, and no mirror symmetry. Either the first derivative does not change sign anywhere, or it becomes zero at exactly one point, or there is exactly one local maximum and one local minimum, symmetric about the point of inflection.
(5) Functions of higher degree do not necessarily have either half-turn symmetry or mirror symmetry.
(6) More generally, we can say the following for sure: a nonconstant polynomial of even degree greater than zero can have at most one line of mirror symmetry and no point of half-turn symmetry. A nonconstant polynomial of odd degree greater than one can have at most one point of half-turn symmetry and no line of mirror symmetry.
(7) The sine function is an example of a function where the points of inflection and the points of half-turn symmetry are the same: the multiples of π. Similarly, the points with vertical axis of symmetry are the same as the points of local extrema: odd multiples of $\pi/2$.
(8) A polynomial is an even function iff all its terms have even degree. Such a polynomial is termed an *even polynomial*. A polynomial is an odd function iff all its terms have odd degree. Such a polynomial is termed an *odd polynomial*.

Actions ...

(1) Worried about periodicity? Don't be worried if you only see polynomials and rational functions. Trigonometric functions should make you alert. Try to fit in the nicest choices of period. Check if smaller periods can work (e.g., for $\sin^2$, the period is π). Even if the function in and of itself is not periodic, it might have a periodic derivative or a periodic second derivative. The sum of a linear function and a periodic function has periodic derivative, and the sum of a quadratic function and a periodic function has a periodic second derivative.
(2) Want to milk periodicity? Use the fact that for a periodic function, the behavior everywhere is just the behavior over one period translates over and over again. If the first derivative is periodic, the increase/decrease behavior is periodic. If the second derivative is periodic, the concave up/down behavior is periodic.
(3) Worried about even and odd, and half-turn symmetry and mirror symmetry? If you are dealing with a quadratic polynomial, or a function constructed largely from a quadratic polynomial, you are probably seeing some kind of mirror symmetry. For cubic polynomials and related constructions, think half-turn symmetry.
(4) Use also the cues about even and odd polynomials.

5.2. **Graphing a function.** Actions ...

(1) To graph a function, a useful first step is finding the domain of the function.
(2) It is useful to find the intercepts and plot a few additional points.
(3) Try to look for symmetry: even, odd, periodic, mirror symmetry, half-turn symmetry, and periodic derivative.
(4) Compute the derivative. Use that to find the critical points, the local extreme values, and the intervals where the function increases and decreases.

(5) Compute the second derivative. Use that to find the points of inflection and the intervals where the function is concave up and concave down.
(6) Look for vertical tangents and vertical cusps. Look for vertical asymptotes and horizontal asymptotes. For this, you may need to compute some limits.
(7) Connect the dots formed by the points of interest. Use the information on increase/decrease and concave up/down to join these points. To make your graph a little better, compute the first derivative (possibly one-sided) at each of these points and start off your graph appropriately at that point.

Subtler points...

(1) When graphing a function, there may be many steps where you need to do some calculations and solve equations and you are unable to carry them out effectively. You can skip some of the steps and come back to them later.
(2) If you cannot solve an equation exactly, try to approximate the locations of roots using the intermediate value theorem or other results such as Rolle's theorem.
(3) In some cases, it is helpful to graph multiple functions together, on the same graph. For instance, we may be interested in graphing a function and its second and higher derivatives. There are other examples, such as graphing a function and its translates, or a function and its multiplicative shifts.
(4) A graph can be used to suggest things about a function that are not obvious otherwise. However, the graph should not be used as conclusive evidence. Rather, the steps used in drawing the graph should be retraced and used to give an algebraic proof.
(5) We are sometimes interested in sketching curves that are not graphs of functions. This can be done by locally expressing the curve piecewise as the graph of a function. Or, we could use many techniques similar to those for graphing functions.
(6) For a function with a piecewise description, we plot each piece within its domain. At the points where the definition changes, determine the one-sided limits of the function and its first and second derivatives. Use this to make the appropriate open circles, asymptotes, etc.

6. Tricky topics

6.1. **Piecewise definition by interval: new issues.** Before looking at these, please review the corresponding material on piecewise definition by interval in the previous midterm review sheet.

(1) Composition involving piecewise definitions is tricky. The limit, continuity and differentiation theorems for composition do not hold for one-sided approach. If one of the functions is decreasing, then things can get flipped. For piecewise definitions, when composing, we need to think clearly about how the intervals transform.

    Please review the midterm question on composition (midterm 1, question 7) of piecewise definitions. The key idea is as follows: for the composition $f \circ g$, we make cases to determine the values of g for which the image under g would land in a particular piece for the definition of f. Considering all cases is extremely painful and we are usually able to take shortcuts based on the nature of the problem.
(2) For a function with piecewise definition, the points where the definition changes are endpoints for each definition, and hence, these points are possible candidates for critical points, points of inflection, and local extreme. They're just *candidates* (so they may not be any of these) but they're worth checking out.
(3) A related helpful concept is that of *how smoothly* a function transitions at a point where its definition changes.
(4) At the one extreme are the discontinuous transitions, where the function has a non-removable discontinuity at the point. Such a transition may be a jump discontinuity (if both one-sided limits are defined but unequal) or something even worse, such as an infinite or oscillatory discontinuity.

    For functions with a discontinuity at a point, it makes sense to talk of one-sided derivatives only from the side where the function is continuous; of course, this one-sided derivative may still not exist.
(5) A somewhat smoother transition occurs where the function is continuous but not differentiable at the point where it changes definition. This is a particular kind of *critical point* for the function definition. Critical points could arise in the form of vertical tangents, vertical cusps, or just plain

points of turning such as for $|x|$ or x^+ at $x = 0$. At such points, it makes sense to try to compute the one-sided derivatives, and these can be computed just by differentiating the piece functions and plugging in at the point. The second derivative does not exist at such points. Also, there is an abrupt change in the nature of concavity at these points.
(6) An even smoother transition occurs if the first derivative is defined at the point. If the first derivative is also defined around the point, then we can start thinking about the second derivative.
(7) More generally, we could think of situations where we want the first k derivatives to be defined at or around the point.
(8) To integrate a function with a piecewise definition, partition the interval of integration in a manner that each part lies within one definition piece. Please review the following two routine problems from Homework 6: Exercise 5.4.55 and 5.4.60. You might want to do a few more suggested problems of the same type.

6.2. **The $\sin(1/x)$ examples.**

(1) The $\sin(1/x)$ and related examples are somewhat tricky because the function definition differs at an *isolated point*, namely 0.
(2) To calculate any limit or derivative at a point other than 0, we can do formal computations. However, to calculate the derivative at 0, we *must* use the definition of derivative as a limit of a difference quotient.
(3) For all the facts below, the qualitative conclusions at finite places hold if we replace sin by cos. Those at ∞ change qualitatively.
(4) The function $f_0(x) := \{ \begin{matrix} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is odd and satisfies the intermediate value property but is not continuous at 0. Its limit at $\pm\infty$ is 0, i.e., it has horizontal asymptote the x-axis in both directions.
(5) The function $f_1(x) := \{ \begin{matrix} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is even and continuous but not differentiable at 0. We can see this from the pinching theorem – it is pinched between $|x|$ and $-|x|$. f_1 is infinitely differentable at all points other than 0. Its limit at $\pm\infty$ is 1, and it approaches this from below in both directions.
(6) The function $f_2(x) := \{ \begin{matrix} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is differentiable at 0, and infinitely differentiable everywhere other than 0, but the derivative is not continuous at 0. The limit $\lim_{x\to 0} f_2'(x)$ does not exist. Note that f_2' is defined everywhere and satisfies the intermediate value property but is not continuous at 0.
   f_2 is asymptotic to the line $y = x$ both additively and multiplicatively, as $x \to \pm\infty$.
(7) The function $f_3(x) := \{ \begin{matrix} x^3\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is continuously differentiable but not twice differentiable at 0, and infinitely differentiable everywhere other than 0.
   f_3 is asymptotic to the line $y = x^2 + C$ as $x \to \pm\infty$, where C is an actual constant (whose value you were supposed to compute in a homework problem).
(8) More generally, consider something such as $p(x)\sin(1/(q(x)))$. This function is not defined at the zeros of q. However, it does not have vertical asymptotes at these points. If a is a root of q and also of p, then the limiting value as $x \to a$ is 0. Otherwise, the limit is undefined but the function oscillates between finite bounds.
   In the limit as $x \to \pm\infty$, if the degree of p is less than that of q, the function has horizontal asymptote the y-axis. If their degrees are equal, it has asymptote a finite nonzero value, namely $\lim_{x\to\infty} p(x)/q(x)$. If the degree of p is bigger, it is asymptotic to a polynomial.
   For $p(x)\cos(1/(q(x)))$, the behavior at points where the function isn't defined is the same as for sin, but the behavior at $\pm\infty$ is different – the cos part goes to 1, so the function is asymptotically polynomial, albeit not necessarily to p itself.
(9) Fun exercise: Consider $x\tan(1/x)$. What can you say about this?

6.3. **Power functions.** We here consider exponents $r = p/q$, q odd. When q is even, or when r is irrational, the conclusions drawn here continue to hold for $x > 0$; however, the function isn't defined for $x < 0$.

For each of these, you should be able to provide ready justifications/reasoning based on derivatives.

(1) Case $r < 0$: x^r is undefined at 0. It is decreasing and concave up on $(0, \infty)$, with vertical asymptote at $x = 0$ going to $+\infty$ and horizontal asymptote as $x \to \infty$ going to $y = 0$. If p is even, it is increasing and concave up on $(-\infty, 0)$ with horizontal asymptote as $x \to -\infty$ going to $y = 0$ and vertical asymptote $+\infty$ at 0. If p is odd, it is decreasing and concave down on $(-\infty, 0)$ with horizontal asymptote as $x \to -\infty$ going to $y = 0$ and vertical asymptote $-\infty$ at 0.
(2) Case $r = 0$: We get a constant function with value 1.
(3) Case $0 < r < 1$: x^r is increasing and concave down on $(0, \infty)$. If p is even, it is decreasing and concave down on $(-\infty, 0)$ and has a downward-facing vertical cusp at $(0, 0)$. If p is odd, it is increasing and concave up on $(-\infty, 0)$ and has an upward vertical tangent at $(0, 0)$.
(4) Case $r = 1$: A straight line $y = x$.
(5) Case $1 < r$: x^r is increasing and concave up on $(0, \infty)$. If p is even, it is decreasing and concave up on $(-\infty, 0)$ and has a local and absolute minimum and critical point at $(0, 0)$. If p is odd, it is increasing and concave down on $(-\infty, 0)$ and has a point of inflection-type critical point (no local extreme value) at $(0, 0)$.

6.4. **Local behavior heuristics: multiplicative.** You have a complicated looking function such that $(x - \alpha_1)^{r_1}(x - \alpha_2)^{r_2} \dots (x - \alpha_k)^{r_k}$. What is the local behavior of the function near $x = \alpha_1$?

The answer: For determining the qualitative nature of this local behavior, you can just concentrate on $(x - \alpha_i)^{r_i}$ and ignore the rest. In particular, *just* looking at r_1, you can figure out whether you have a critical point, local extremum, point of inflection, vertical tangent, or vertical cusp. The other things *do* matter if you are further interested in, say, whether we have a local maximum or minimum, or in whether the vertical tangent is an increasing tangent or a decreasing tangent, or which direction the vertical cusp points in.

Overall, if $r_i > 0$ and $r_i = p_i/q_i$, q_i odd, and both p_i, q_i positive, then:

- Critical point iff $r_i \neq 1$.
- Local extremum iff p_i even.
- Point of inflection/vertical tangent iff p_i odd.
- Vertical cusp iff p_i even and $p_i < q_i$, both positive, i.e., $r_i < 1$. Note that vertical cusp is a special kind of local extremum.
- Vertical tangent iff p_i odd, $p_i < q_i$, both positive, i.e., $r_i < 1$.

So, if we look at, say $(x - \pi)^2(x - \sqrt{6})^3(x - 2)^{1/3}(x - 3)^{2/3}$, it has a local extremum at π, a point of inflection at $\sqrt{6}$, a vertical tangent at 2, and a vertical cusp at 3.

6.5. **Local behavior heuristics: additive.** If you have something of the form $f + g$, and the vertical tangent/cusp points for f are disjoint from those of g, then the vertical tangent/cusp points for $f + g$ include both lists. Further, the nature (tangent versus cusp) is inherited from the corresponding piece.

For instance, for $x^{1/3} + (x - 131.4)^{2/3}$, there is a vertical tangent at $x = 0$ and a vertical cusp at $x = 131.4$.

In particular, if g is everywhere differentiable, then the vertical tangent/cusp behavior of $f + g$ is the same as that of f.

7. High yield practice

Here are the areas that you should focus on if you have a thorough grasp of the basics:

(1) Everything to do with piecewise definitions (differentiation, integration, reasoning).
(2) Vertical tangents and cusps in sophisticated cases.
(3) Horizontal, oblique, and weird asymptotes.
(4) Trigonometric integrations, particularly $\sin^2$, $\cos^2$, and $\tan^2$ and their variants.
(5) Tricky integration problems that involve the use of symmetry and/or the chain rule.

8. Quickly

This "Quickly" list is a bit of a repeat and augmentation of the "Quicky" list given out for the previous midterm.

8.1. **Arithmetic.** You should be able to:

- Do quick arithmetic involving fractions.
- Remember $\sqrt{2}$, $\sqrt{3}$, and π to at least two digits.
- Sense when an expression will simplify to 0.
- Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
- Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

8.2. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3.
(4) Do polynomial long division (not usually necessary, but helpful).
(5) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

8.3. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi - x)$ and $\cos(\pi - x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is $1/2$.

8.4. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

8.5. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a \sin x + b \cos x$.
(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).

8.6. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a sum of power functions quickly.
(4) Know that the integral of sine or cosine on any quadrant is ± 1.
(5) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a + bx)^m$.
(6) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$,.

8.7. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

8.8. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: 1/3, 2/3, 4/3, 5/3, 1/2, 1, 2, 3, −1, −1/3 −2/3.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.

8.9. **Fancy pictures.** Keep in mind approximate features of the graphs of:

(1) $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ and $x^3\sin(1/x)$, and the corresponding cos counterparts – both the behavior near 0 and the behavior near $\pm\infty$.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.

REVIEW SHEET FOR FINAL: BASIC

MATH 152, SECTION 55 (VIPUL NAIK)

With minor exceptions, this document does not re-review material already covered in the review sheet for midterm 1 and midterm 2. It is your responsibility to go through that review sheet again and make sure you have mastered all the material there.

See the advanced version for error-spotting exercises and the quickly list.

1. Area computations

Note that this section partially repeats material from the prevoius midterm review, because part of the area computations syllabus was in the previous midterm syllabus.

Words ...

(1) We can use integration to determine the area of the region between the graph of a function f and the x-axis from $x = a$ to $x = b$: this integral is $\int_a^b f(x)\,dx$. The integral measures the signed area: parts where $f \geq 0$ make positive contributions and parts where $f \leq 0$ make negative contributions. The magnitude-only area is given as $\int_a^b |f(x)|\,dx$. The best way of calculating this is to split $[a, b]$ into sub-intervals such that f has constant sign on each sub-interval, and add up the areas on each sub-interval.

(2) Given two functions f and g, we can measure the area between f and g between $x = a$ and $x = b$ as $\int_a^b |f(x) - g(x)|\,dx$. For practical purposes, we divide into sub-intervals so that on each sub-interval one function is bigger than the other. We then use integration to find the magnitude of the area on each sub-interval and add up. If f and g are both continuous, the points where the functions *cross* each other are points where $f = g$.

(3) Sometimes, we may want to compute areas against the y-axis. The typical strategy for doing this is to interchange the roles of x and y in the above discussion. In particular, we try to express x as a function of y.

(4) An alternative strategy for computing areas against the y-axis is to use formulas for computing areas against the x-axis, and then compute differences of regions.

(5) A general approach for thinking of integration is in terms of slicing and integration. Here, integration along the x-axis is based on the following idea: divide the region into vertical slices, and then integrate the lengths of these slices along the horizontal dimension. Regions for which this works best are the regions called *Type I regions*. These are the regions for which the intersection with any vertical line is either empty or a point or a line segment, hence it has a well-defined length.

(6) Correspondingly, integration along the y-axis is based on dividing the region into horizontal slices, and integrating the lengths of these slices along the vertical dimension. Regions for which this works best are the regions called *Type II regions*. These are the regions for which the intersection with any horizontal line is either empty or a point or a line segment, hence it has a well-defined length.

(7) Generalizing from both of these, we see that our general strategy is to choose two perpendicular directions in the plane, one being the direction of our slices and the other being the direction of integration.

Actions ...

(1) In some situations we are directly given functions and/or curves and are asked to find areas. In others, we are given real-world situations where we need to find areas of regions. Here, we have to find functions and set up the integration problem as an intermediate step.

(2) In all these situations, it is important to draw the graphs in a reasonably correct way. This brings us to all the ideas that are contained in graph drawing. Remember, here we may be interested in simultaneously graphing more than one function. Thus, in addition to being careful about each

function, we should also correctly estimate where one function is bigger than the other, and find (approximately or exactly) the intersection points. (Go over the notes on graph-drawing, and some additional notes on graphing that weren't completely covered in class).

(3) In some situations, we are asked to find the area(s) of region(s) bounded by the graphs of one, two, three, or more functions. Here, we first need to sketch the figure. Then, we need to find the interval of integration, and if necessary, split this interval into sub-intervals, such that on each sub-interval, we know exactly what integral we need to do. For instance, consider the region between the graphs of sin, cos, and the x-axis. Basically, the idea is to find, for all the vertical slices, the upper and lower limits of the slice.

2. Volume computations

Words ...

(1) The cross section method for computing volume is an analogue of the two-dimensional area computation method: our slices are replaced by cross sections by planes parallel to a fixed plane, and the line of integration is a line perpendicular to the planes. One-dimensional slices are replaced by two-dimensional cross sections.
(2) Suppose Ω is a region in the plane. We can construct a right cylinder with base Ω and height h. This is obtained by translating Ω in a direction perpendicular to its plane by a length of h. The cross section of this right cylinder along any plane parallel to the original plane looks like Ω if that plane is within range. The volume is the product of the area of Ω and the height h. This is also called the right cylinder with constant cross section Ω.
(3) We can also construct an oblique cylinder. Here, the direction of translation is not perpendicular to the original plane. The total volume is the product of the area of Ω and the height perpendicular to Ω. Oblique cylinders are to right cylinders what parallelograms are to rectangles.
(4) More generally, the volume of a solid can be computed using the cross section method. Here, we choose a direction. We measure areas of cross sections along planes perpendicular to that direction, and integrate these areas along that direction.
(5) This general approach has another special case that is perhaps as important as right cylinders. These are the *cones* (there are right cones and oblique cones). A cone is obtained by taking a region in a plane and connecting all points in it to a point outside the plane. It is a right cone if that point is directly above the center of the region. The volume of a cone is $1/3$ times the product of the base area and the height, i.e., the perpendicular distance from the outside point to the plane. In particular, a cone has one-third the volume of a cylinder of the same base and height.
(6) A solid of revolution is a solid obtained by revolving a region in a plane about a line (called the axis of revolution). The volume of a solid of revolution can be computed by choosing the axis as the axis of integration and using the planes of cross section as planes perpendicular to it. These cross sections are either circular disks or annuli in the nice cases. *Added*: In nastier cases, the cross sections could be unions of multiple annuli.
(7) The *disk method* is a special case of the above, where the region being revolved is supported on the axis of revolution. For instance, consider the region between the x-axis, the graph of a function f, and the lines $x = a$ and $x = b$. The volume of the corresponding solid of revolution is $\pi \int_a^b [f(x)]^2 \, dx$. This is because the radius of the cross section disk at $x = x_0$ is $|f(x_0)|$.
(8) The *washer method* is the more general case where the region need not adhere to the axis of revolution. For instance, consider two nonnegative functions f, g and suppose $0 \le g \le f$. Consider the region bounded by the graphs of these two functions and the lines $x = a$ and $x = b$. The volume of the corresponding solid of revolution is $\pi \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$. Note that in the more general case where the functions cross each other, we may need to split into sub-intervals so that we can apply the washer method on each sub-interval.
(9) The shell method works for situations where we revolve about the y-axis the region made between the graph of a function and the x-axis. The formula here is $2\pi \int_a^b x f(x) \, dx$ for f nonnegative and $0 < a < b$. If f could be positive or negative, we use $2\pi \int_a^b x|f(x)| \, dx$.. More generally, if we are looking

at the region between the graphs of f and g (vertically) with $g \leq f$, we get $2\pi \int_a^b x[f(x) - g(x)]\,dx$. If we don't know which one is bigger where, we use $2\pi \int_a^b x|f(x) - g(x)|\,dx$.

Actions ...

(1) To compute the volume using cross sections, we first need to set things up so that we know the cross section areas as a function of the position of the plane. For this, it is usually necessary to use either coordinate geometry or basic trigonometry, or a combination.
(2) A solid occurs as a solid of revolution if it has complete rotational symmetry about some axis. In that case, that axis is the axis of revolution and the original region that we need is obtained by taking a cross section in any plane containing the axis of revolution and looking at the part of that cross section that is on one side of the axis of revolution.
(3) For solids of revolution, be particularly wary if the original figure being revolved has parts on both sides of the axis of revolution. If it is symmetric about the axis of revolution, delete one side. *Added*: In general, fold the figure about the axis of revolution – folding does not affect the final solid of revolution we obtain.
(4) Be careful about the situations where you have to be sign-sensitive and the situations where you do not. In the disk method sensitivity to signs is not important. In the washer method and shell method, it is. *Added*: Also be careful about applying the disk, washer, and shell methods when the axis of revolution is *not* the x- or y-axis but is parallel to one of them.
(5) The farther the shape being revolved is from the axis, the greater the volume of the solid of revolution.
(6) The average value point of view is sometimes useful for understanding such situations.

3. One-one functions and inverses

3.1. Vague generalities.

Words...

(1) Old hat: Given two sets A and B, a function $f : A \to B$ is something that takes inputs in A and gives outputs in B. The *domain* of a function is the set of possible inputs, while the *range* of a function is the set of possible outputs. The notation $f : A \to B$ typically means that the domain of the function is A. However, the whole of B need not be the range; rather, all we know is that the range is a *subset* of B. One way of thinking of functions is that *equal inputs give equal outputs*.
(2) A function f is one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$. In other words, *unequal inputs give unequal outputs*. Another way of thinking of this is that *equal outputs could only arise from equal inputs*. Or, *knowledge of the output allows us to determine the input uniquely*. One-to-one functions are also called one-one functions or injective functions.
(3) Suppose f is a function with domain A and range B. If f is one-to-one, there is a *unique* function g with domain B and range A such that $f(g(x)) = x$ for all $x \in B$. This function is denoted f^{-1}. We further have that g is also one-to-one, and that $f = g^{-1}$. Note that f^{-1} differs from the reciprocal function of f.
(4) Suppose $f : A \to B$ and $g : B \to C$ are one-to-one functions. Then $g \circ f$ is also one-to-one, and its inverse is the function $f^{-1} \circ g^{-1}$.

Actions ...

(1) To determine whether a function is one-to-one, solve $f(x) = f(a)$ for x in terms of a. If, for every a in the domain, the only solution is $x = a$, the function is one-to-one. If, on the other hand, there are some values of a for which there is a solution $x \neq a$, the function is not one-to-one.
(2) To compute the inverse of a one-to-one function, solve $f(x) = y$ and the expression for x in terms of y is the inverse function.

3.2. In graph terms.

Thousand words ...

(1) A picture in a coordinatized plane is the graph of a function if every vertical line intersects the picture at most once. The vertical lines that intersect it exactly once correspond to the x-values in the domain. This is known as the *vertical line test*.
(2) A function is one-to-one if and only if its graph satisfies the *horizontal line test*: every horizontal line intersects the graph at most once. The horizontal lines that intersect the graph exactly once correspond to y-values in the range.

(3) For a one-to-one function, the graph of the inverse function is obtained by reflecting the graph of the function about the $y = x$ line. In particular, a function equals its own inverse iff its graph is symmetric about the $y = x$ line.
(4) Many of the results on inverse functions and their properties have graphical interpretations. For instance, the fact that the derivative of the inverse function is the reciprocal of the derivative corresponds to the geometrical fact that reflection about the $y = x$ line inverts slopes of tangent lines. Similarly, the results relating increase/decrease and concave up/down for a function and its inverse function can all be deduced graphically.

3.3. **In the real world.** Words... (from now on, we restrict ourselves to functions whose domain and range are both subsets of the real numbers)

(1) An increasing function is one-to-one. A decreasing function is one-to-one.
(2) A *continuous* function on an *interval* is one-to-one if and only if it is either increasing throughout the interval or decreasing throughout the interval.
(3) If the derivative of a continuous function on an interval is of constant sign everywhere, except possibly at a few isolated points where it is either zero or undefined, then the function is one-to-one on the interval. Note that we need the function to be continuous *everywhere* on the interval, even though it is tolerable for the derivative to be undefined at a few isolated points.
(4) In particular, a one-to-one function cannot have local extreme values.
(5) A continuous one-to-one function is increasing if and only if its inverse function is increasing, and is decreasing if and only if its inverse function is decreasing.
(6) *Point added, not present in original executive summary of lecture notes*: If a one-to-one function on an interval satisfies the intermediate value property, then it is continuous. This is because the function cannot *jump* suddenly since it needs to cover all intermediate values. Note that the analogous statement is *not* true if we drop either the assumption of one-to-one or the assumption of the intermediate value property. (Think of $\sin(1/x)$, for instance).
(7) If f is one-to-one *and differentiable at a point* a (emphasis added) with $f'(a) \neq 0$, with $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$. This agrees with the previous point and also shows that the rates of relative increase are inversely proportional.
(8) Two extreme cases of interest are: $f'(a) = 0$, $f(a) = b$. In this case, f has a horizontal tangent at a and f^{-1} has a vertical tangent at b. The horizontal tangent is typically also a point of inflection. It is definitely *not* a point of local extremum. Similarly, if $(f^{-1})'(b) = 0$, then f^{-1} has a horizontal tangent at b and f has a vertical tangent at a.
(9) A slight complication occurs when f has one-sided derivatives but is not differentiable. If both one-sided derivatives of f exist and are nonzero, then both one-sided derivatives of f^{-1} (at the image point) exist and are nonzero. When f is increasing, the left hand derivative of f^{-1} is the reciprocal of the left hand derivative of f, and the right hand derivative of f^{-1} is the reciprocal of the right hand derivative of f. When f is decreasing, the right hand derivative of f^{-1} is the reciprocal of the left hand derivative of f, and the left hand derivative of f^{-1} is the reciprocal of the right hand derivative of f.
(10) The second derivative of f^{-1} at $f(a)$ is $-f''(a)/(f'(a))^3$. In particular, the second derivative of the inverse function at the image point depends on the values of both the first and the second derivatives of the function at the point.
(11) If f is increasing, the sense of concavity of f^{-1} is opposite to that of f. If f is decreasing, the sense of concavity of f^{-1} is the same as that of f.

Actions ...

(1) For functions on intervals, *to check if the function is one-to-one*, we can compute the derivative and check if it has constant sign everywhere except possibly at isolated points.
(2) In order to find $(f^{-1})'$ at a particular point, given an explicit description of f, it is *not* necessary to find an explicit description of f^{-1}. Rather, it is enough to find f^{-1} at that particular point and then calculate the derivative using the above formula. The same is true for $(f^{-1})''$, except that now we need to compute the values of both f' and f''.

(3) The idea can be extended somewhat to finding $(f^{-1})'$ when f satisfies a differential equation that expresses $f'(x)$ in terms of $f(x)$ (with no direct appearance of x).

4. Logarithms, exponents, derivatives and integrals

4.1. **Logarithm and exponential: basics.**

(1) The *natural logarithm* is a one-to-one function with domain $(0, \infty)$ and range $\mathbb{R}$, and is defined as $\ln(x) := \int_1^x (dt/t)$.
(2) The natural logarithm is an increasing function that is concave down. It satisfies the identities $\ln(1) = 0$, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^r) = r \ln a$, and $\ln(1/a) = -\ln a$.
(3) The limit $\lim_{x\to 0} \ln(x)$ is $-\infty$ and the limit $\lim_{x\to\infty} \ln(x)$ is $+\infty$. Note that ln goes off to $+\infty$ at ∞ even though its derivative goes to zero as $x \to +\infty$.
(4) The derivative of $\ln(x)$ is $1/x$ and the derivative of $\ln(kx)$ is also $1/x$. The derivative of $\ln(x^r)$ is r/x.
(5) The antiderivative of $1/x$ is $\ln|x| + C$. What this really means is that the antiderivative is $\ln(-x)+C$ when x is negative and $\ln(x)+C$ when x is positive. If we consider $1/x$ on both positive and negative reals, the constant on the negative side is unrelated to the constant on the positive side.
(6) e is defined as the unique number x such that $\ln(x) = 1$. e is approximately 2.718. In particular, it is between 2 and 3.
(7) The inverse of the natural logarithm function is denoted exp, and $\exp(x)$ is also written as e^x. When x is a rational number, $e^x = e^x$ (i.e., the two definitions of exponentiation coincide). In particular, $e^1 = e$, $e^0 = 1$, etc.
(8) The function exp equals its own derivative and hence also its own antiderivative. Further, the derivative of $x \mapsto e^{mx}$ is me^{mx}. Similarly, the integral of e^{mx} is $(1/m)e^{mx} + C$.
(9) We have $\exp(x + y) = \exp(x)\exp(y)$, $\exp(rx) = (\exp(x))^r$, $\exp(0) = 1$, and $\exp(-x) = 1/\exp(x)$. All of these follow from the corresponding identities for ln.

Actions...

(1) We can calculate $\ln(x)$ for given x by using the usual methods of estimating the values of integrals, applied to the function $1/x$. We can also use the known properties of logarithms, as well as approximate ln values for some specific x values, to estimate $\ln x$ to a reasonable approximation. For this, it helps to remember $\ln 2$, $\ln 3$, and $\ln 5$ or $\ln 10$.
(2) Since both ln and exp are one-to-one, we can *cancel* ln from both sides of an equation and similarly *cancel* exp. Technically, we cancel ln by applying exp to both sides, and we cancel exp by applying ln to both sides.

4.2. **Integrations involving logarithms and exponents.** Words/actions ...

(1) If the numerator is the derivative of the denominator, the integral is the logarithm of the (absolute value of) the denominator. In symbols, $\int g'(x)/g(x)\,dx = \ln|g(x)| + C$.
(2) More generally, whenever we see an expression of the form $g'(x)/g(x)$ inside the integrand, we should consider the substitution $u = \ln|g(x)|$. Thus, $\int f(\ln|g(x)|)g'(x)/g(x)\,dx = \int f(u)\,du$ where $u = \ln|g(x)|$.
(3) $\int f(e^x)e^x\,dx = \int f(u)\,du$ where $u = e^x$.
(4) $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.
(5) $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$.
(6) Trigonometric integrals: $\int \tan x\,dx = -\ln|\cos x| + C$, and similar integration formulas for cot, sec and csc: $\int \cot x\,dx = \ln|\sin x| + C$, $\int \sec x = \ln|\sec x + \tan x| + C$, and $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

4.3. **Exponents with arbitrary bases, exponents.** Words ...

(1) For $a > 0$ and b real, we define $a^b := \exp(b \ln a)$. This coincides with the usual definition when b is rational.
(2) All the laws of exponents that we are familiar with for integer and rational exponents continue to hold. In particular, $a^0 = 1$, $a^{b+c} = a^b \cdot a^c$, $a^1 = a$, and $a^{bc} = (a^b)^c$.

(3) The exponentiation function is continuous in the exponent variable. In particular, for a fixed value of $a > 0$, the function $x \mapsto a^x$ is continuous. When $a \neq 1$, it is also one-to-one with domain $\mathbb{R}$ and range $(0, \infty)$, with inverse function $y \mapsto (\ln y)/(\ln a)$, which is also written as $\log_a(y)$. In the case $a > 1$, it is an increasing function, and in the case $a < 1$, it is a decreasing function.
(4) The exponentiation function is also continuous in the base variable. In particular, for a fixed value of b, the function $x \mapsto x^b$ is continuous. When $b \neq 0$, it is a one-to-one function with domain and range both $(0, \infty)$, and the inverse function is $y \mapsto y^{1/b}$. In case $b > 0$, the function is increasing, and in case $b < 0$, the function is decreasing.
(5) Actually, we can say something stronger about a^b – it is *jointly* continuous in both variables. This is hard to describe formally here, but what it approximately means is that if f and g are both continuous functions, and f takes positive values only, then $x \mapsto [f(x)]^{g(x)}$ is also continuous.
(6) The derivative of the function $[f(x)]^{g(x)}$ is $[f(x)]^{g(x)}$ times the derivative of its logarithm, which is $g(x)\ln(f(x))$. We can further simplify this to obtain the formula:

$$\frac{d}{dx}\left([f(x)]^{g(x)}\right) = [f(x)]^{g(x)}\left[\frac{g(x)f'(x)}{f(x)} + g'(x)\ln(f(x))\right]$$

(7) Special cases worth noting: the derivative of $(f(x))^r$ is $r(f(x))^{r-1}f'(x)$ and the derivative of $a^{g(x)}$ is $a^{g(x)}g'(x)\ln a$.
(8) Even further special cases: the derivative of x^r is rx^{r-1} and the derivative of a^x is $a^x \ln a$.
(9) The antiderivative of x^r is $x^{r+1}/(r+1)+C$ (for $r \neq -1$) and $\ln|x|+C$ for $r = -1$. The antiderivative of a^x is $a^x/(\ln a) + C$ for $a \neq 1$ and $x + C$ for $a = 1$.
(10) The logarithm $\log_a(b)$ is defined as $(\ln b)/(\ln a)$. This is called the logarithm of b to base a. Note that this is defined when a and b are both positive and $a \neq 1$. This satisfies a bunch of identities, most of which are direct consequences of identities for the natural logarithm. In particular, $\log_a(bc) = \log_a(b)+\log_a(c)$, $\log_a(b)\log_b(c) = \log_a(c)$, $\log_a(1) = 0$, $\log_a(a) = 1$, $\log_a(a^r) = r$, $\log_a(b)\cdot\log_b(a) = 1$ and so on.
(11) *Added*: The derivative of $\log_{f(x)}(g(x))$ is given by:

$$\frac{d}{dx}\left[\log_{f(x)}(g(x))\right] = \frac{\ln(f(x))g'(x)/g(x) - \ln(g(x))f'(x)/f(x)}{(\ln(f(x)))^2}$$

Actions...

(1) We can use the formulas here to differentiate expressions of the form $f(x)^{g(x)}$, and even to differentiate longer exponent towers (such as x^{x^x} and 2^{2^x}).
(2) To solve an integration problem with exponents, it may be most prudent to rewrite a^b as $\exp(b \ln a)$ and work from there onward using the rules mastered earlier. Similarly, when dealing with relative logarithms, it may be most prudent to convert all expressions in terms of natural logairthms and then use the rules mastered earlier.

REVIEW SHEET FOR FINAL: ADVANCED

MATH 152, SECTION 55 (VIPUL NAIK)

1. Area computations

No error-spotting exercises.

2. Volume computations

Error-spotting exercises ...

(1) Consider the function $f(x) := \sin x$ and $g(x) := -\sin x$, for $x \in [0, \pi]$. The region between the graphs of these functions is revolved about the x-axis. The volume of the solid of revolution is:

$$\pi \int_0^{\pi} (f(x) - g(x))^2 \, dx = \pi \int_0^{\pi} 4 \sin^2 x \, dx = 2\pi^2$$

(2) If a right angled triangle is revolved about any of its sides, then we get a right circular cone.
(3) If a rectangle is revolved about one of its diagonals, then we get a union of two right circular cones.
(4) The volume of the solid of revolution obtained by revolving a region of area A is proportional to A.

3. One-one functions and inverses

3.1. Vague generalities.

Error-spotting exercises ...

(1) If f and g are one-one functions, then so is $f \circ g$ and $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

3.2. In the real world.

Error-spotting exercises ...

(1) Consider the function:

$$f(x) := x^3 + x$$

We have $f'(x) = 3x^2 + 1 > 0$ is always positive, so f is one-one. Also, note that the inverse function to $x \mapsto x^3$ is $x \mapsto x^{1/3}$ and the inverse function to $x \mapsto x$ is $x \mapsto x$. Thus, we get:

$$f^{-1}(x) := x^{1/3} + x$$

(2) If f is a one-one function on $\mathbb{R}$, then it must be either an increasing function or a decreasing function on $\mathbb{R}$.
(3) If f is a differentiable function on $\mathbb{R}$, then f is one-one if and only if $f'(x) > 0$ for all $x \in \mathbb{R}$.
(4) Suppose f and g are continuous one-one functions on $\mathbb{R}$. Then, clearly, they are either increasing or decreasing functions on $\mathbb{R}$. Thus, the sum $f + g$ is also either an increasing or decreasing function on $\mathbb{R}$, and hence it must be one-one.
(5) Suppose f, g, and h are continuous one-one functions on $\mathbb{R}$. Then, the pairwise sums $f + g$, $g + h$, and $f + h$ are all one-one functions.
(6) Suppose f is a one-one function such that the graph of f is concave up. Then, f^{-1} is also a one-one function and its graph is concave down.
(7) If c is a point in the domain of a function f such that the left hand derivative and right hand derivative of f at c do not agree, then the left hand derivative and right hand derivative of f^{-1} at c also do not agree.

(8) Consider the function:

$$f(x) := \begin{cases} x+1, & x \text{ rational} \\ x^3, & x \text{ irrational} \end{cases}$$

We know that both $x+1$ and x^3 are one-one functions on their respective domains. Thus, f is a one-one function.

4. Logarithm, exponential, derivative, and integral

4.1. Logarithm and exponential: basics.

(1) We have that $\ln(xy) = \ln x + \ln y$. Thus, $\ln((-1)^2) = \ln(-1) + \ln(-1)$. The left side is $\ln 1 = 0$, so $\ln(-1) = 0$.
(2) If f is a function on $\mathbb{R} \setminus \{0\}$ such that $f'(x) = 1/x$ for all $x \neq 0$, then $f(x) = \ln|x| + C$ for some fixed constant C.
(3) Using that $\ln 2 \sim 0.7$ and $\ln 3 \sim 1.1$, we obtain that $\ln 5 = \ln(2+3) = \ln 2 + \ln 3 \sim 0.7 + 1.1 = 1.8$.
(4) We have $\exp((\ln x)^2) = \exp(\ln(x^2)) = x^2$.

4.2. Integration involving logarithms and exponents.

(1) Consider the integration $\int_0^\pi \tan x \, dx$. This is:

$$\int_0^\pi \tan x \, dx = [\ln|\cos x|]_0^\pi = 0$$

(2) Consider the indefinite integration $\int e^{x+\ln(\sin x)} \, dx$. This becomes:

$$\int e^{x+\ln(\sin x)} \, dx = \int e^x \, dx + \int e^{\ln(\sin x)} \, dx = e^x + \int \sin x \, dx = e^x - \cos x + C = -e^x \cos x + C$$

4.3. Exponentiation with arbitrary bases, exponents.

No error-spotting exercises

5. Miscellaneous error-spotting exercises

(1) Consider the function $x \mapsto x^{1/3} + x^{2/3}$. The $x^{1/3}$ part has a vertical tangent at $x = 0$ and $x^{2/3}$ part has a vertical cusp at $x = 0$. The tangent and cusp cancel and thus overall we get neither a vertical tangent nor a vertical cusp at 0.
(2) Consider the integration:

$$\int \frac{x^2}{x+1} \, dx = \int \frac{x^2}{x} \, dx + \int \frac{x^2}{1} \, dx = \int x + 1 \, dx = x^2/2 + x + C$$

(3) Consider the function:

$$f(x) := x \sin(\pi/(x^2 + x))$$

This is undefined at $x = 0$ and $x = 1$. At both these points, the graph of f has a vertical asymptote.

6. Quickly

This "Quickly" list improves upon previous "Quickly" lists.

6.1. Our common values.

Preferably remember these (or be capable of computing quickly) to at least two digits.

(1) Square roots of 2, 3, 5, 6, 7, 10.
(2) Natural logarithms of 2, 3, 5, 7, and 10.
(3) Value of π, $1/\pi$, $\sqrt{\pi}$, and π^2.
(4) Value of e, $1/e$.
(5) Some relative logarithms, such as $\log_2 3$ or $\log_2(10)$. Although you don't need these values to a significant degree of precision, it is useful to have some idea of their magnitude.

6.2. **Adding things up: arithmetic.** You should be able to:

(1) Do quick arithmetic involving fractions.
(2) Sense when an expression will simplify to 0.
(3) Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
(4) Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

6.3. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3. *This could be an area for potential improvement for many people.*
(4) Factorize an even polynomial of degree four. *This could be an area for potential improvement for many people.*
(5) Do polynomial long division (not usually necessary, but helpful).
(6) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

6.4. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi \pm x)$ and $\cos(\pi \pm x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles or other larger angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is 1/2.
(6) Formulas for double angles, half angles: $\sin(2x)$, $\cos(2x)$ in terms of sin and cos; also the reverse: $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$.
(7) *More advanced*: Remember the formulas for $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, and $\cos(A-B)$.
(8) *More advanced*: Convert between products of sin and cos functions and their sums: for instance, the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.

6.5. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

6.6. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a\sin x + b\cos x$ or $a\exp(-x)$.
(6) Do multiple differentiations of expressions whose derivative cycle is periodic up to constant factors, e.g. $a\exp(mx+b)$ or $a\sin(mx+\varphi)$.
(7) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).
(8) Differentiate ln, exp, and expressions of the form f^g and $\log_f(g)$.

6.7. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a sum of power functions quickly.

(4) Know that the integral of sine or cosine on any quadrant is ± 1.
(5) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a + bx)^m$.
(6) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$, any odd power of sin, any odd power of cos, any odd power of tan.
(7) Integrate on sight things such as $x \sin(x^2)$, getting the constants right without much effort.

6.8. **Being observant.** You should be able to look at a function and:
(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

6.9. **Graphing.** You should be able to:
(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: $1/3$, $2/3$, $4/3$, $5/3$, $1/2$, 1, 2, 3, -1, $-1/3$ $-2/3$.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A \sin(mx)$ and $A \cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.

6.10. **Graphing: transformations.** Given the graph of f, you should be able to quickly graph the following:
(1) $f(mx)$, $f(mx + b)$: pre-composition with a linear function; how does $m < 0$ differ from $m > 0$?
(2) $Af(x) + C$: post-composition with a linear function, how does $A > 0$ differ from $A < 0$?
(3) $f(|x|)$, $|f(x)|$, $f(x^+)$, and $(f(x))^+$: pre- and post-composition with absolute value function and positive part fnuction.
(4) More slowly: $f(1/x)$, $1/f(x)$, $\ln(|f(x)|)$, $f(\ln |x|)$, $\exp(f(x))$, and other popular composites.

6.11. **Fancy pictures.** Keep in mind approximate features of the graphs of:
(1) $\sin(1/x)$, $x \sin(1/x)$, $x^2 \sin(1/x)$ and $x^3 \sin(1/x)$, and the corresponding cos counterparts – both the behavior near 0 and the behavior near $\pm\infty$.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.