PROOFS AND OTHER IDEAS OF MATHEMATICS

MATH 152, SECTION 55 (VIPUL NAIK)

1. General ideas related to proofs

1.1. **The idea of models and proof.** The main idea behind the concept of proof is to establish something clearly, give a fool-proof, error-free explanation of *why* something is true. That means that all cases must be covered, every step of the argument should be justified, and there shouldn't be any hidden assumptions that aren't true. We'll go over some of these in detail, but the first question you might be wondering is: *why proof?*

So the one thing you have to remember about mathematics is that the world of mathematics is a world of its own creation. It's true that mathematical ideas and formalisms are applied a lot to real-world settings, but often the mathematical ideas go far beyond what we can determine or verify through real-world observation. In other words, for a lot of the things we want to do in the mathematical world, it is hard to be sure of them simply by looking around at the real world. So, in mathematics, it is important to develop a way of being sure of things that depends purely on internal reasoning. In fact, this concept of internal reasoning, or *reasoning within the framework of rules of the system*, is what defines mathematics.

If you think about mathematics as a system of rules-based internal reasoning, your most significant introduction to mathematics isn't when you learn numbers, it is when you learn how to play games, and how to deploy strategies within games.

And it so happens that most systems of rules that we deal with involve the concepts of numbers, many of them involve some geometric ideas, and some of them need the tools of algebra and trigonometry and calculus. But you shouldn't think of mathematics as being the same as algebra or trigonometry or calculus. These are just tools. The main feature of mathematics is that in mathematics, there is a strong place for *internal reasoning* – reasoning within the system to try to determine what is true and what is false.

And, if you look at major developments in a number of academic disciplines, you see that mathematics is seeping into most of them. And I don't just mean that they are getting more quantitative, though that's part of the story. Those who've seen Newton's laws and classical mechanics know that there's a specific model (which, it turns out, isn't exactly how the real world operates) and we can predict how things will behave in that model. Nowadays, many papers in the social sciences also start by creating some artificial model that has a reasonable resemblance to reality, and then try to derive formally what happens in that model. And the main difference between mathematicians and those in other sciences is that for people in the other sciences, they need to justify that their model has some kind of resemblance with, or explanatory power about, the real world. But mathematicians aren't subject to that constraint.

So think of mathematical rigor as something that allows mathematicians to explore things where intuition, or real-world checks and balances, are hard to find.

1.2. **Proof by example isn't; cover your bases, consider all cases.** So one of the things people often do in the real world is when they want to know if something is true they take some example and check it. And you see the media and politicians do that kind of thing everyday. So whenever somebody wants to prove that some thing works, they'll find one person to give a testimonial for it.

But in mathematics, we don't consider a few isolated examples to be proof. And the reason is simple: different cases behave differently, so the examples we choose are probably not representative. That's true in the real world, but it is often even more true in mathematics, where things aren't constrained to be realistic.

So mathematicians try to *cover all cases* in proofs. What does this mean? If you want to prove a statement for all real numbers, it isn't enough to prove it for all rational numbers. After all, there are real numbers that aren't rational. So you need to prove the statement for all rational numbers *and* all irrational numbers.

Now, *how* we choose to break down the problem into cases is up to us. For some problems, the natural way of breaking up the problem may be to first consider rational numbers and then consider irrational numbers. Sometimes, it may be helpful to first consider positive numbers and then consider negative numbers. If you have to prove a statement for all numbers in a finite set, the ultimate break-up would be to check it separately for every element of the finite set.

The thing you should remember is that *if you are breaking things up into cases*, you should *remember to cover all cases*. And one way to remember this is to think of mathematics as just about the smartest adversary you can find in the battlefield. If you don't cover every possible line of attack on your adversary, your adversary will hide in the one place you forgot to cover.

1.3. **Conditional implication.** In mathematics, we often consider statements of the form:

"If A, then B"

Now, these kinds of statements can sometimes be confusing, so let's try to understand what exactly this means. This roughly means that, assuming that A is given to be true, B is true. For instance, "if I don't oversleep, I will attend the calculus lecture on Friday". That is a conditional statement.

There are a lot of subtleties about conditional implications that we need to understand. The first is that "If A, then B" only means that A is *sufficient* for B. It doesn't mean that A is necessary for B. There may be other ways that B could become true, even if A were false. For instance, you may say "If I have enough money, I'll eat lunch". But you may be able to eat lunch even though you don't have enough money – by going to one of U of C's Free Food events.

So "if A, then B" means that if, somehow, one could guarantee A to be true, B would follow – but there may be other ways to guarantee B. In particular, if you prove a statement "if A then B" and then you separately prove that A is true, then you would have proved that B is true.

1.4. **Rough work and fair work.** In many situations where we need to do a proof, there are two parts to doing the proof. The first is the exploratory phase, or the discovery phase, where we need to find a strategy that works for the proof. For instance, in the case of $\epsilon - \delta$ proofs, the exploratory phase involved coming up with a winning strategy for the prover or the skeptic as the case may be. In this exploratory phase, we may do some rough calculations, make some wild guesses, check out our intuition on examples, etc. The exploratory phase may involve *working backwards*, *splitting into cases*, etc. At the end of exploratory phase, we have an overall proof strategy.

At the end of the exploratory phase, we hopefully have a clear proof strategy. The next phase is that of clearly expressing the strategy and showing that it works. When writing this final stategy and the proof, you do not need to cover everything you went through in the exploratory phase. Stick only to that which is more relevant to the final proof strategy. Also, state the strategy right upfront and proceed, to the extent possible, starting from what you know and proceeding towards what you need to show.

1.5. **Opposite statement.** Another concept that I should mention, and that you've had a bit of past experience with, is the *opposite* of a statement. This is related to the question: *how do I prove that A is not true*? Well, in order to prove that, you first need a clear formulation of what it means for A to *not* be true. This new statement is sometimes called the *negation* or *opposite* of A.

Now, some of you may have seen some Boolean algebra or logic, so you might have some idea of the formal process of negating a statement, but even if you haven't, most of the rules are intuitive provided you pause to think and don't just try to rush. Keep your cool, and it's not hard. I'll just mention some important ideas:

(1) Negation turns *and* to *or*, and *or* to *and*. For instance, the negation of the statement $x = 1$ or $x = 2$ is the statement $x \neq 1$ *and* $x \neq 2$.
(2) Negation on a $\forall$ quantifier gives a $\exists$ quantifier and negation on a $\exists$ quantifier gives a $\forall$ quantifier. For instance, the negation of the statement $\forall x \in \mathbb{R}, f(x^2) = f(x)^2$ is the statement $\exists x \in \mathbb{R}, f(x^2) \neq f(x)$. This came up when we looked at $\epsilon - \delta$ proofs.

1.6. **Proof by contradiction.** One of the useful proof techniques is proof by contradiction. This comes up sometimes, and I'll talk more about it when it does, but the way it works is like this: suppose you are trying to prove A. So the first thing you may try to do is prove A straightforward, but that may seem tricky. So what you do is this. You assume that the opposite of A is true. So you write down the opposite of A, and start with that as given. And then, from that, you derive some statement that is plainly *not* true. Since the conclusion isn't true, the statement you started by assuming, namely, the opposite of A, couldn't have been true either. And since the opposite of A is false, A itself must be true.

Some of you may have seen the proof that $\sqrt{2}$ is irrational. That proof is a classic example of proof by contradiction.

2. Specific issues

The material in the previous section is very general and I think most of you would lap it up pretty easily. Most of you seem to have a reasonable understanding of these ideas, but there are some more specific issues that you may have with expressing your proofs. Below are listed some of the specific issues that students in past years have had in the first two advanced homeworks.

2.1. **Making your strategy and specific claims clear upfront.** This issue has occurred in the past with some of the $\epsilon - \delta$ proofs. If you're the prover, then the stategy involves finding an expression for δ that works in terms of ϵ. If you're the skeptic, the strategy involves finding an ϵ for which no δ works, and then being able to choose a value of x in $(c - \delta, c + \delta) \setminus \{c\}$.

In the exploratory phase, you try to figure out a strategy that works. Then, in the actual proof phase, you show that the strategy works.

When writing up the final proof, please do not show the exploratory phase. Please write the final winning strategy upfront. Then, proceed to translate the general statement about the existence or non-existence of limit into a specific claim based on your strategy. Then, do some algebraic manipulation or case-by-case reasoning to prove that your strategy works.

Some examples:

- For the homework problem $\lim_{x \to 2} x^2 = 4$, state right at the beginning that the winning strategy is $\delta = \min\{1, \epsilon/5\}$. Then, state the specific claim: if $0 < |x - 2| < \min\{1, \epsilon/5\}$, then $|x^2 - 4| < \epsilon$. Now, prove the specific claim.

  Some people do some algebraic manipulation to discover the δ that works. Others are comfortable using the general formula that works for the quadratic. Whichever thing you choose to do, please remember that the less of this exploratory work you show, the clearer your proof is. This is because exploratory work, as a general rule, is mesy, with conditionals much more complicated, steps going forward and backward, etc. So please skip this and write your winning strategy clearly.
- Consider problems where, for instance, we need to select a δ value for a given ϵ value, and the function is defined differently on rationals and irrationals. Here, we need to find a δ_1 that works for rationals, a δ_2 that works for irrationals, and then take $\delta = \min\{\delta_1, \delta_2\}$.

  You should write down the strategy for choosing δ right on top, *make the specific claim*, and split into cases *to prove the specific claim*.

  Some of you split into cases first, proved things in each case, and gave the overall winning strategy at the end. *This is probably the way that you discover things in the exploratory phase, but it's not the prettiest way of presenting a final proof.*

Caveat: There are situations where it is advantageous to show your exploratory phase. For instance, if you were a teacher and were guiding students through a learning process, this exploratory phase might be helpful. If you were trying to break ground with a similar new problem, it might help to revisit the exploratory phase.

However, you should think of showing your exploratory phase as filming the process of the manufacture of sausage, and the fair work proof phase as the phase of enjoying the final sausage.

2.2. **Doing the general case clearly.** This problem arose with some advanced homework solutions in Homework 1, and a subsequent clarification was made. However, it's worth reiterating here.

In the exploratory phase, we may use some specific numerical examples to check if something is true. Then, we discover that the actual steps work in somewhat greater generality, and we need that greater generality in order to do the whole proof.

When writing down the final proof, jump directly to proving that the actual steps work in somewhat greater generality.

The example from the first advanced homework was: "Show that the function $f(x) := \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$ is periodic but has no period."

One possible discovery approach is as follows:

(1) We notice that the number 1 works in the sense that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. We prove this by splitting into the cases where x is rational and x is irrational.
(2) After finishing that proof, we notice that, in fact, the proof depended only on the fact that rational + rational = rational and irrational + rational = irrational. Crucially, the only thing we were using about 1 was that it is rational.
(3) We thus conclude that any rational $h > 0$ works in place of 1.
(4) Since there are arbitrarily small positive rational numbers, we concluded that there is no period.

In the final write-up of the proof, we remove steps (1) and (2) and directly proceed with the claim of step (3), with the proof of that claim basically mimicking our original proof of (1).

2.3. **Meta-strategies.** Some of the advanced problems involve constructing a strategy for one game using strategies for other games *as black boxes*. For instance, in problems 1 and 5 of advanced homework 2, you are asked to come up with winning strategies for the prover for $|f|$, $\max\{f, g\}$, and $\min\{f, g\}$, assuming that there exist winning strategies for f and g.

Here, you assume that the winning strategies for f and g are given to you on a platter, but you have to treat them as black boxes. In other words, you assume some statement of the form:

"For every $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \epsilon$."

and:

"For every $\epsilon > 0$, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - L| < \epsilon$."

Our "winning strategy" for $H := \max\{f, g\}$, is to choose, for a given $\epsilon > 0$, $\delta = \min\{\delta_1, \delta_2\}$, i.e., the minimum of the δs that work for f and g.

We then make the specific claim: "If $0 < |x - c| < \min\{\delta_1, \delta_2\}$, then $|H(x) - L| < \epsilon$."

After this, we prove the specific claim by splitting into cases for x, based on whether $H(x) = f(x)$ or $H(x) = g(x)$.

Meta-strategies are tricky to understand at first, because the strategies that we are using as black boxes are *unknown knowns* – we can use them, but have to treat them as black boxes.

2.4. **Fixed but arbitrary.** Another note about the ϵ–δ proofs. In all these proofs, ϵ is "fixed but arbitrary." What this basically means is that ϵ is fixed, but it is fixed by the skeptic, so we (as the provers) have no control over the choice so we should be prepared for the worst.