THEOREMS ON LIMITS AND CONTINUITY

MATH 152, SECTION 55 (VIPUL NAIK)

Difficulty level: Moderate to hard. There are a couple of proofs here that are hard to understand; however, you are not responsible for the proofs of these theorems this quarter. The statements of the theorems will be easy if you have seen them before, and somewhat hard if you have not.

Corresponding material in the book: Sections 2.3, 2.4, 2.5, 2.6.

Things that students should definitely get: The uniqueness theorem for limits. The statements of the theorems for limits and continuity in relation to the pointwise combinations of functions and composition. The fact that all the results for pointwise combination hold for one-sided limits and one-sided continuity. The statement and simple applications of the pinching theorem, intermediate-value theorem, and extreme-value theorem.

Things that students should hopefully get: The fact that the limit theorems have both a conditional existence component and a formula component. The way that the triangle inequality is used critically to prove the uniqueness theorem and the theorem on limits of sums. The importance of the continuity assumption for the intermediate-value theorem and of that as well as the closed interval assumption for the extreme-value theorem. How to think about counterexamples and weird functions in a way that builds intuition about the significance of the hypotheses to the theorems.

Executive summary

Limit theorems + quick/intuitive calculation of limits. Words...

(1) If the limits for two functions exist at a particular point, the limit of the sum exists and equals the sum of the limits. Similarly for product and difference.
(2) For quotient, we need to add the caveat that the limit of the denominator is nonzero.
(3) If $\lim_{x\to c} f(x) = L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}(f(x)/g(x))$ is undefined.
(4) If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, then we cannot say anything offhand about $\lim_{x\to c}(f(x)/g(x))$.
(5) Everything we said (or implied) can be reformulated for one-sided limits.

Continuity theorems. Words ...

(1) If f and g are functions that are both continuous at a point c, then the function $f + g$ is also continuous at c. Similarly, $f - g$ and $f \cdot g$ are continuous at c. Also, if $g(c) \neq 0$, then f/g is continuous at c.
(2) If f and g are both continuous in an interval, then $f + g$, $f - g$ and $f \cdot g$ are also continuous on the interval. Similarly for f/g provided g is not zero anywhere on the interval.
(3) The composition theorem for continuous functions states that if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c. The corresponding composition theorem for limits is *not true but almost true*: if $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = M$, then $\lim_{x\to c} f(g(x)) = M$.
(4) The one-sided analogues of the theorems for sum, difference, product, quotient work, but the one-sided analogue of the theorem for composition is not in general true.
(5) Each of these theorems at points has a suitable analogue/corollary for continuity (and, with the exception of composition, for one-sided conitnuity) on intervals.

Three important theorems. Words ...

(1) The pinching theorem states that if $f(x) \leq g(x) \leq h(x)$, and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$. A one-sided version of the pinching theorem also holds.
(2) The intermediate-value theorem states that if f is a continuous function, and $a < b$, and p is between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = p$. Note that we need f to be defined and continuous on the entire closed interval $[a, b]$.

(3) The extreme-value theorem states that on a closed bounded interval $[a, b]$, a continuous function attains its maximum and minimum.

Actions ...

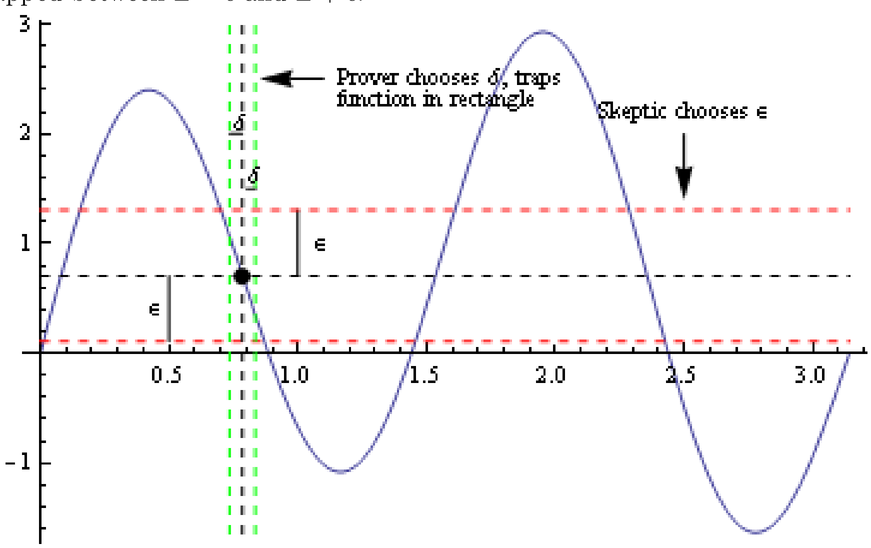
(1) When trying to calculate a limit that's tricky, you might want to bound it from both sides by things whose limits you know and are equal. For instance, the function $x \sin(1/x)$ taking the limit at 0, or the function that's x on rationals and 0 on irrationals, again taking the limit at 0.
(2) We can use the intermediate-value theorem to show that a given equation has a solution in an interval by calculating the values of the expression at endpoints of the interval and showing that they have opposite signs.

1. Limit theorems

This section discusses some important limit theorems.

Recall the definition. We say that $\lim_{x\to c} f(x) = L$ (as a two-sided limit) if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every x such that $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

Now, that's quite a mouthful. Let's interpret it graphically. What it is saying is that: "for every ϵ"so we consider this region $(L - \epsilon, L + \epsilon)$, so there are these two horizontal bars at heights $L - \epsilon$ and $L + \epsilon$. Next it says, there exists a δ, so there exist these vertical bars at $c + \delta$ and $c - \delta$. And what we're saying is that if the x-value is trapped between the vertical bars $c - \delta$ and $c + \delta$ (but is not equal to c), the $f(x)$-value is trapped between $L - \epsilon$ and $L + \epsilon$.



The important thing to note here is that the value of δ depends on the value of ϵ. As I said last time, we can think of this as a game, where I am trying to prove to you that the limit $\lim_{x\to c} f(x) = L$ and you are a skeptic who is trying to catch me out. So you throw ϵs at me, and challenge me to show that I have a δ to trap that ϵ. And if I have a winning strategy, that enables me to find a δ for every ϵ that you throw at me, then yes, the limit is equal to L.

1.1. What are limit theorems? And why do we need them?

In the absence of limit theorems, we have two alternatives:

- Use our "intuition" – this is problematic, because while intuition works great for nice functions such as polynomials, it tsarts failing us as soon as we get to weirder functions.
- Use "first principles,", i.e., the $\epsilon - \delta$ definition of limits every time – this is very tedious even for experienced mathematicians.

Limit theorems provide a sort of middle ground that avoids the pitfalls at either end. Basically, what these theorems do is, show, using the $\epsilon - \delta$ definition of limits, that certain "intuitive" facts about limits are always true. Then, we can use these theorems guilt-free without having to wade through a mess of ϵs and δs.

So think of proving a theorem as an investment. It's like putting money in a savings account. You put money once, and you keep getting the interest from it. But the first thing you need to do is put in the hard work of earning and saving the money. And proving the theorems is like doing that hardwork.

1.2. **Review of some inequalities involving the absolute value.** So, before we plunge into the proofs, I want to review some facts about the absolute value function that are very important. I'll stick to two facts. The first is what is called the *triangle inequality*, and it goes like this:

$$|a + b| \leq |a| + |b| \ \forall \ a, b \in \mathbb{R}$$

Now, equality holds if either a or b is zero, or if they both have the same sign. Equality does *not* hold if a and b are of opposite signs. So, you may wonder, why the name *triangle inequality*? And to understand this, we need to think about triangles on the real line.

But before we get into that, first, let's recall what the triangle inequality in geometry states. It says that the sum of two sides of a triangle is greater than the third side. That's basically a manifestation of the fact that *straightest is shortest* – the straight line path between two points is shorter than a path that involves two straight lines.

Now, the inequality sign is strict, because in geometry, we don't use the word *triangle* if all the three vertices are collinear. By the way, if all the three vertices are collinear, we call the triangle a *degenerate* triangle. Let's say we included degenerate triangles. Then, the equality case could occur. In fact, it'll occur in precisely the case where the single side is between the two more extreme points.

So, here's how this relates to the statement involving absolute values. Consider the degenerate triangle with vertices the points 0, a, and $a + b$ on the number line. What are the side lengths? Well, the length of the side from 0 to a is $|a|$, the length of the side from a to $a + b$ is $|b|$, and the length of the side from 0 to $a + b$ is $|a + b|$. And so the result we have is:

$$|a + b| \leq |a| + |b|$$

which is our triangle inequality.

Verbally, what this is saying is that if you travel a distance of a and then travel a distance of b along the real line, you cannot be more than $a + b$ away from where you started. The farthest you can get is if both your two pieces of travel were in the same direction.

The other result, which we've already used a few times, is that the absolute value of the product of two real numbers equals the product of their absolute values.

$$|ab| = |a||b| \ \forall \ a, b \in \mathbb{R}$$

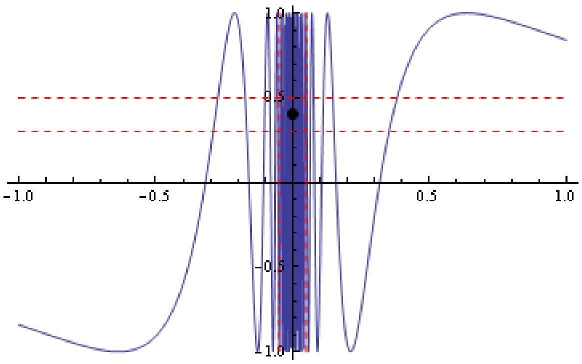
1.3. **Uniqueness theorem.** [**Note**: We will probably not go over this proof in class in full detail, and you are not expected to know this proof for any of your tests. However, I strongly suggest that you at least try to understand this proof temporarily. The ideas involved here are extremely useful for understanding some of the more advanced limits stuff that we will see in 153.]

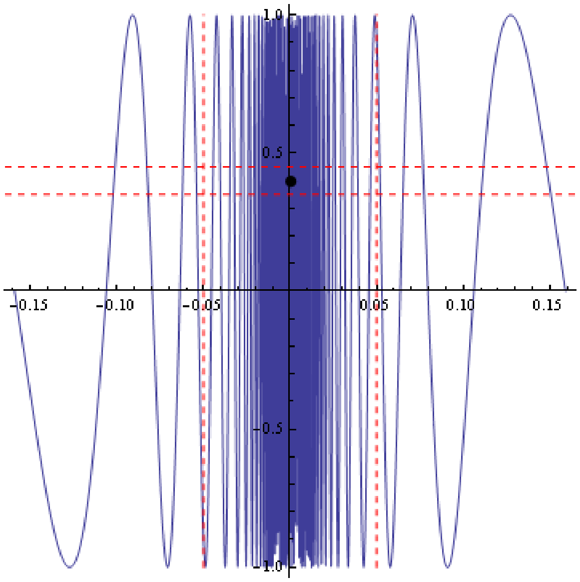
The first result that we establish is a theorem on the *uniqueness* of limits. And the intuition behind this goes back to our original discussion where I gave you a wrong definition of limit and found that the problem with that definition was that it gave rise to multiple limits. And we tweaked it and got the correct definition.

So the first thing we need to establish is that if the limit exists, then it is unique.

Okay, can you give an intuitive reason why that should be true?

Well, think back to our discussion on traps. And think back to the wrong definition of limit, and why $\sin(1/x)$ was problematic. The reason was that it was jumping a lot, and our right definition of limits, by creating traps, avoided that.





So you can think of what we are doing here as a proof by contradiction. We will show that if there are two real numbers $L \neq M$, it cannot be the case that *both* L and M satisfy the $\epsilon - \delta$ definition for $\lim_{x \to c} f(x)$.

Now, this is an example where we want to show that the limit *cannot* be something. So this is an example of a situation where you are trying to prove the opposite of a statement. We already wrote down what it means to say that as $x \to c$, $f(x)$ does not approach L. Let's recall it: there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

Okay, now let's take a step back and see what we're really trying to achieve. Simply put, think about it as two games. There's the L-game, which is the game where I'm trying to prove the limit is L. And there's the M-game, which is where I'm trying to prove the limit is M. And what is true is that no matter what, you, the skeptic, have a winning strategy for at least one of the games.

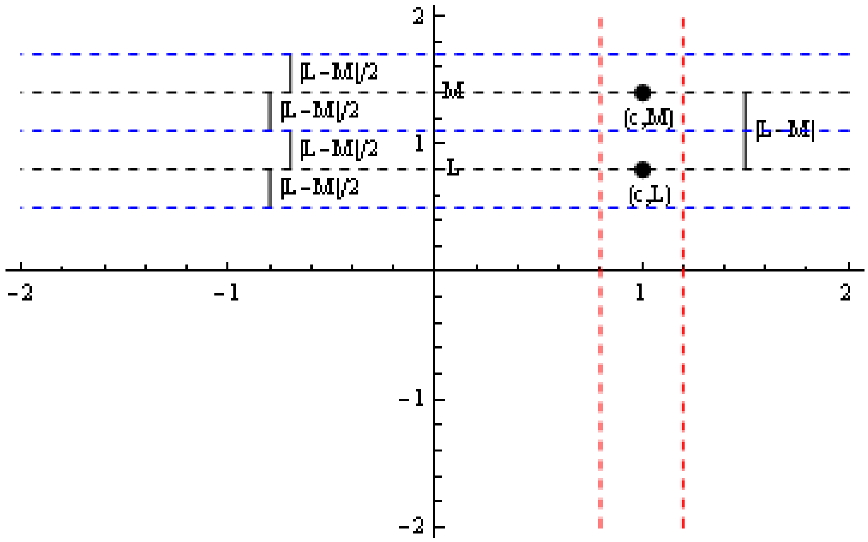
So what you do is to try to trap me with my own trap, literally speaking. Because what I'm claiming is that for x sufficiently close to c, the function is trapped really close to L, but it's also trapped really close to M. So what you need to do is call the bluff on me, by forcing me to get too close to both L and M for comfort. Basically, what you want to call me out on is the assertion that a function can be trapped in two places at the same time. And, to cut a long story short, the ϵ that you stump me with is:

$$\epsilon = \frac{|L - M|}{2}$$

So, you stump me with this ϵ in the L-game and in the M-game. And suppose I throw back δ_1 at you in the L-game and δ_2 at you in the M-game. So I'm claiming that when the x-value comes within δ_1 of c, the function value is trapped with ϵ of L, and when the x-value comes within δ_2 of c, the function value is trapped within ϵ of M.

So let $\delta = \min\{\delta_1, \delta_2\}$. Then, what we have is that:

if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$, and $|f(x) - M| < \epsilon$.



So for x in this small region around c, $f(x)$ is within ϵ of L and within ϵ of M. But the picture makes clear that this is not possible, because the ϵ-disks around L and M don't meet. The formalism behind this is the triangle inequality:

$$|L - M| = |(L - f(x)) + (f(x) - M)| \leq |L - f(x)| + |f(x) - M| < 2\epsilon = |L - M|$$

So, we get $|L - M| < |L - M|$, a contradiction.

1.4. **Limits for sum, difference, product, ratio.** Okay, now we'll state the results for the limits of sums, differences, scalar multiples, products, and ratios. None of the proofs are in the syllabus, but I've sketched the proof for sums. The book has proofs for all the limits.

Suppose f and g are two functions defined in a neighborhood of the point c. Then, if $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are well-defined, we have the following:

(1) $\lim_{x \to c}(f(x) + g(x))$ is defined, and equals the sum of the values $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$.
(2) $\lim_{x \to c}(f(x) - g(x))$ is defined, and equals $\lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.
(3) $\lim_{x \to c} f(x)g(x)$ is defined, and equals the product $\lim_{x \to c} f(x) \lim_{x \to c} g(x)$.

The scalar multiples result basically states that if $\lim_{x \to c} f(x)$ exists, and $\alpha \in \mathbb{R}$, $\lim_{x \to c} \alpha f(x) = \alpha \lim_{x \to c} f(x)$.

By the way, a week ago, we defined the notions of sum, difference, and product, of functions. So with that notation, we can rewrite the results as:

$$\begin{aligned} \lim_{x \to c}(f + g)(x) &= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \\ \lim_{x \to c}(f - g)(x) &= \lim_{x \to c} f(x) - \lim_{x \to c} g(x) \\ \lim_{x \to c}(f \cdot g)(x) &= \lim_{x \to c} f(x) \lim_{x \to c} g(x) \end{aligned}$$

A couple of important additional points. The first is that the results I mentioned state a little more than what is captured in the formulas. The subtlety arises because every limit that we write need not exist.

What the sum result says is that *if* the two limits $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ *both exist*, the the limit for $f + g$ exists *and* is given by the formula. So, the result is a *conditional existence result plus a formula.* Note that it may very well be the case that the limit for $f + g$ exists but the individual limits – those for f and g, do not exist. For instance, if $f(x) = 1/x$ and $g(x) = -1/x$, then f and g do not have limits at 0, but $f + g$ does have a limit at 0.

Similarly for the results about difference, product, and scalar multiples.

1.5. **Result for the ratio.** For the ratio (also called the quotient), we have the result that if $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, *and* if $\lim_{x \to c} g(x) \neq 0$, then we have:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

1.6. **Indeterminate form $\to 0/\to 0$ limits.** In addition to the previous limit theorems, there are some important facts that should feel familiar if you have done limit computations.

Suppose $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$. Then the quotient limit $\lim_{x \to c} f(x)/g(x)$ has what is called a $0/0$ form – more precisely, it is of the form $(\to 0)/(\to 0)$. This is an example of an *indeterminate form*. An indeterminate form is a form of a limit that does not allow us to conclude anything specific about the value of the limit. A limit of the form $(\to 0)/(\to 0)$ may or may not exist. Further, it could "not exist" in practically all the possible values that limits happen to "not exist" (infinite limits, oscillatory limits, etc.). $(\to 0)/(\to 0)$ may be finite and nonzero, it may be zero, it may be going to $+\infty$, to $-\infty$, different infinities from different sides, oscillatory between finite bounds, oscillatory between infinite bounds, etc.

An indeterminate form does *not* mean that we can throw up our hands. To the contrary, indeterminate form means that the limit *needs more work*. This extra work typically involves understanding more about the nature of the *specific functions* f and g near the point of approach. For rational functions, we try to cancel common factors between the numerator and denominator. There are more general approaches such as trigonometric limits, l'Hopital's rule and power series, which we will see later in the course.

Intuitively, what matters is: does the numerator go to 0 more quickly, does the denominator go to 0 more quickly, or do they both go to 0 at roughly the same rate. We will explore this theme in mind-numbing theme later in 152 and even more in 153.

1.7. **Lonely denominator blow-ups: undefined limit.** If $\lim_{x \to c} f(x) = L \neq 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} f(x)/g(x)$ is not defined. In other words, the limit *does not exist*. When we later study infinity as a limit, we will consider in more detail whether the (one-sided) limit exists as an infinity. Note that if a limit is infinite, we still say that the limit "does not exist."

1.8. **One-sided and two-sided limits.** So far, in all the situations where we have been saying that *the limit exists*, we mean that the *two-sided* limit exists. Recall that we have that $\lim_{x \to c} f(x) = L$ if f is defined in an open interval about c (except possibly at c) and if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x such that $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

Let's also recall what it means to say that the *left-hand limit exists*. We say that $\lim_{x \to c^-} f(x) = b$ if f is defined to the immediate left of c, and if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x such that $0 < c - x < \delta$, we have $|f(x) - b| < \epsilon$.

Basically, what's happening is that now, we need only a one-sided trap for δ, i.e., a trap of the form $(c - \delta, c)$ rather than a trap of the form $c - \delta, c + \delta)$.

Similarly, we say that $\lim_{x \to c^+} = b$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x such that $0 < x - c < \delta$, we have $|f(x) - b| < \epsilon$.

Now, it turns out that all the results we proved *so far* about limits also hold for one-sided limits from either side. So, for instance, we have the following for left-hand limits.

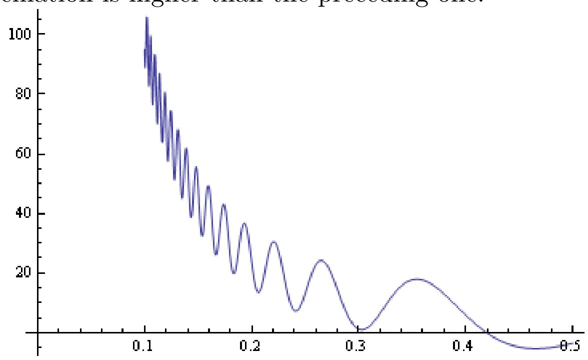
(1) If the left-hand limit $\lim_{x \to c^-} f(x)$ exists, it is unique.
(2) If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^-} g(x)$ exist, then $\lim_{x \to c^-}(f(x) + g(x))$ exists and equals the sum of the individual limits.
(3) If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^-} g(x)$ exist, then $\lim_{x \to c^-}(f(x) + g(x))$ exists and equals the difference of the individual limits.
(4) If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^-} g(x)$ exist, then $\lim_{x \to c^-}(f(x)g(x))$ exists and equals the product of the individual limits.
(5) If $\lim_{x \to c^-} f(x)$ exists and α is a real number, then $\lim_{x \to c^-} \alpha f(x) = \alpha \lim_{x \to c^-} f(x)$.

Analogous results hold for right-hand limits.

1.9. **Infinity as a limit.** In the discussion so far, when we said that the limit *exists*, we meant that it *exists* and is *finite*. And that's the way it'll continue to be. Nonetheless, since the book makes some mentions of infinity, and since you may have seen these concepts when going through intuitive introductions to limits, let me briefly describe the role of infinity. We will master these definitions formally a little later in the course. For now, the coverage is (relatively) informal.

When we're thinking in terms of limits, then we say that the limit is ∞, or $+\infty$, when the number is getting larger and larger and not bouncing back to very small values. This doesn't mean it can't oscillate – it can. But rather, for every number N you pick, there exists a $\delta > 0$ such that, if $0 < |x - c| < \delta$, then $f(x) > N$. In other words, it eventually gets above any barrier *and stays above*. And if that's the case, we say that the limit is $+\infty$.

For instance, the picture below is of a function that approaches $+\infty$ as $x \to 0$ – because of page-fitting considerations, the function has been plotted only till $x = 0.1$ – but the function oscillates. However, each oscillation is higher than the preceding one.



Similarly, there is the notion of the limit being $-\infty$.

Now, what happens with functions, and the classic example is the function $1/x$, but you'll see this for other functions, is that the left-hand limit approaches infinity from one direction and the right-hand limit approaches infinity from the other direction. So in this case, for the function $f(x) := 1/x$, the left-hand limit is $-\infty$ and the right-hand limit is $+\infty$. By the way, neither limit *exists* in the sense that we will use the word, because neither limit is finite. But a two-sided limit doesn't even exist as an infinity, because the two sides are approaching two different infinities.

On the other hand, for the function $g(x) := 1/x^2$, both the left-hand limit and the right-hand limit are $+\infty$.

And generalizing from these examples, we can see some general rules emerging:

(1) If $\lim_{x \to c} f(x)$ is a positive number and $\lim_{x \to c} g(x) = 0$, but the approach is *from the positive direction*, then $f(x)/g(x) \to \infty$ as $x \to c$. Analogous observations apply to negative numbers.
(2) If $\lim_{x \to c} f(x)$ is a positive number and $\lim_{x \to c} g(x) = 0$, but the approach is *from the positive direction*, then $f(x)/g(x) \to \infty$ as $x \to c$. Analogous observations apply to negative numbers.

We'll talk more about infinities as limits, and consider more complicated cases, a little later in the course.

1.10. **Proof that limit of sums is sum of limits.** [**Note**: We will probably not go over this proof in class in full detail, and you are not expected to know this proof for any of your tests. However, I strongly suggest that you at least try to understand this proof temporarily. The ideas involved here are extremely useful for understanding some of the more advanced limits stuff that we will see in 153.]

We'll now discuss how to prove the statement that, as an English phrase, would read: "the sum of the limits is the limit of the sums". In other words, we are trying to prove that if $\lim_{x \to c} f(x) = L$ and if $\lim_{x \to c} g(x) = M$, then $\lim_{x \to c} f(x) + g(x) = L + M$.

So the way to think about it is that f comes close to L and g comes close to M, so doesn't $f + g$ comes close to $L + M$? Yes, it does. But to make that precise, what we need to do is to loosen the definition of what it means to come close.

You've probably heard of *rounding errors*. For instance, you may say that 1.4 rounds off to 1 and 2.3 rounds off to 2. But when you add the two numbers, you get 3.7, and 3.7 rounds off to 4, rather than $1 + 2 = 3$.

So, the upshot is that just because a is close to a' and b is close to b', doesn't necessarily mean that $a+b$ is just as close to $a'+b'$. However, even if it isn't *as close*, it is still close. The point is that when you add things, the *margins of error add*.

So the way we use this idea is to make sure that our margins of error for the functions f and g are both so small that when you add them up, you still get a small margin of error.

Let's now flesh out the proof details. We need to show that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c}(f(x) + g(x))$ exists. Let's call $L = \lim_{x \to c} f(x)$ and $M = \lim_{x \to c} g(x)$.

Let's discuss this. What we need to do is to show that, for every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|(f(x) + g(x)) - (L + M)| < \epsilon$. Here's how we do this.

Since f is continuous, consider the value $\epsilon/2$. There exists a value $\delta_1 > 0$ such that, if $0 < |x - c| < \delta_1$, we have $|f(x) - L| < \epsilon/2$. What we're doing here is using $\epsilon/2$ as the value of ϵ for f.

Similarly, for g, we use $\epsilon/2$ again. So, there exists a value $\delta_2 > 0$ such that, if $0 < |x - c| < \delta_2$, we have $|g(x) - M| < \epsilon/2$.

Now consider $\delta = \min\{\delta_1, \delta_2\}$. Then, if $0 < |x - c| < \delta$, we have $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$, so:

$$|f(x) - L| < \epsilon/2 \tag{*}$$

$$|g(x) - M| < \epsilon/2 \tag{**}$$

We thus get, using the triangle inequality and (*) and (**):

$$|(f(x) + g(x)) - (L + M)| \le |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon$$

2. Continuity theorems

We now proceed to a discussion of theorems on continuity. Most of these are analogous to corresponding theorems about limits.

2.1. Recall the definition.

It may be worthwhile recalling the definitions of limit and continuity side by side. Let's do that.

We say that $\lim_{x \to c} f(x) = L$ if f is defined in an open interval about c (except possibly at c itself) and, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

We say that f is continuous at c if f is defined in an open interval about c (including at the point c) and, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every x satisfying $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Some differences: for the definition of limit, we do not require the function to be defined *at* the point c, but for the definition of continuity, we do. The L that we use for the definition of continuity is the value $f(c)$. Also, we can drop the $0 <$ part in the definition of continuity.

2.2. Theorems about continuity.

The definition we gave above was for a function being continuous *at a point*, and we can use that definition, along with the limit theorems, to prove that continuity at a point is preserved by sums, scalar multiplies, differences, products, and, if the denominator function is not zero, ratios. Explicitly (Theorem 2.4.2, Page 84):

If f and g are functions that are both continuous at a point $c \in \mathbb{R}$, then:

(1) $f + g$ is continuous at c.
(2) $f - g$ is continuous at c.
(3) $f \cdot g$ is continuous at c.
(4) If $g(c) \neq 0$, then f/g is continuous at c.

Further, if f is continuous at c, then αf is continuous at c for any real number α.

Why are these statements true? Essentially, they are all immediate corollaries of the corresponding statements for limits. For instance, if you take for granted the theorem that $\lim_{x \to c}(f+g)(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$, then given that f and g are both continuous at c, you can substitute the values and get $\lim_{x \to c}(f + g)(x) = f(c) + g(c) = (f + g)(c)$. The same proof idea works for differences and scalar multiples.

2.3. **Composition of continuous functions.** We'll next state a result, which is Theorem 2.4.4 of the book, about the composition of continuous functions. You don't need to know the proof of this statement.

The results is that if f and g are functions such that g is continuous at c and f is continuous at $g(c)$, then the composite function $f \circ g$ is continuous at c. And remember the way the composition works – the function written on the right is the one that is applied first. So here's this element c, and we first apply g, and we get to $g(c)$. Then we apply f, and we get to $f(g(c)) = (f \circ g)(c)$.

So I hope you see why it is important to require that f is continuous at $g(c)$ and g is continuous at c. In particular, it does not matter whether f is continuous at c, because the input that is fed into f is not c but $g(c)$.

2.4. **One-sided continuity.** In a previous lecture, we defined one-sided continuity. Left-continuity means that the left-hand limit exists and equals the value of the function at the point. Right-continuity means that the right-hand limit exists and equals the value of the function at the point.

And, as you might have guessed, the results we stated about sum, difference, scalar multiples, product, and quotient of functions being continuous extends to one-sided continuity. It turns out, interestingly, that the results about composition do *not* necessarily hold in the one-sided sense. In fact, one of your quiz questions on Friday of the first week was exactly about this issue. The main reason is that, for $f \circ g$ to have the required one-sided continuity at c, we need that $g(x)$ approach $g(c)$ from the correct direction.

Some weaker versions can be salvaged: for instance, if f and g are both left-continuous functions, and g is an increasing function, then $f \circ g$ is also left-continuous. There are other weaker versions too, which we will not get into here.

2.5. **Continuity theorems also hold on intervals.** So far, we have stated the continuity theorems at individual points. From these, we can deduce continuity theorems on intervals. Specifically, if I is an interval (of whatever sort), and f and g are continuous functions on I, then:

(1) $f + g$ is continuous on I.
(2) $f - g$ is continuous on I.
(3) $f \cdot g$ is continuous on I.
(4) f/g is continuous at those points of I where it is defined, i.e., where g takes nonzero values.

The interval version of the result for composition is a little trickier, because to deduce the continuity of $f \circ g$, we need f to be continuous, not on the original domain of g, but on the range of g which feeds into the domain of f. Here is one formulation.

Suppose I and J are intervals in $\mathbb{R}$, f is a continuous function on J, and g is a continuous function on I such that the range of g is contained in J. Then $f \circ g$ is a continuous function on I.

2.6. **Most functions you've seen are continuous.** So this is the time to pause and review and think: "which of the functions that we have seen are continuous, and do we have the tools to make sure?"

Well, with the tedious $\epsilon - \delta$ definition of limits, we actually proved that the constant functions are continuous, and that the function $f(x) = x$ is continuous. And we did this basically by showing that the limit at any point equals the value at the point. And now, we know that we can multiply things together, multiply by scalars, and add. And if you think for a moment, you'll see that that shows that polynomial functions are continuous.

For instance, the polynomial $2x^3 - 3x^2 + x + 1$ is the sum of the functions that send x to $2x^3$, $-3x^2$, x, and 1 respectively. Each of these functions itself is a product of multiple copies of the function sending x to itself, multiplied by some scalar. And each step of the construction/deconstruction of a polynomial preserves continuity. So we see why/how polynomial functions are continuous.

Rational functions, which are functions obtained as the ratio of two polynomials, are not necessarily globally defined, because the denominator may blow up at some point. However, the ratio results that we have established show that, at all the places where the denominator does not blow up, the rational function is continuous. Hence, with rational functions, we are in the position that *wherever the function is defined, it is continuous.*

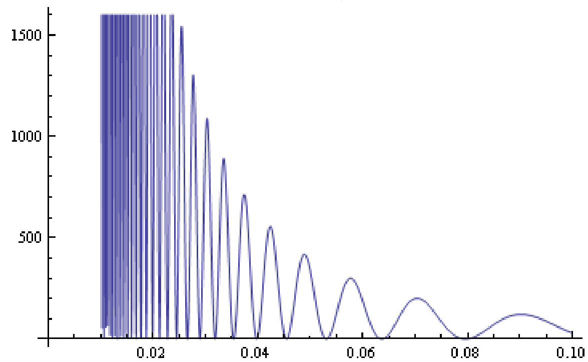
Another thing you should know is that the trigonometric functions sin and cos are continuous. And, since the trigonometric function tan is defined as the ratio of these, tan is again continuous at all the points where it is defined – the points where it is not continuous are the points where it is not defined, which are the points where cos takes the value 0.

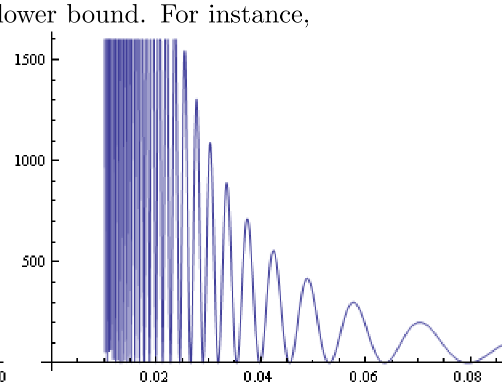
2.7. **Partying wild with freaky functions.** There are two directions of approach on the real line: left and right. And hence we can talk of the left-hand limit and the right-hand limit. And it is important that the real line has two directions.

So you're happy, because there are only two directions of approach, and the function comes in nicely form both sides – but that's not really the case. We saw the example of $\sin(1/x)$, frolicking back and forth cheerfully and indifferent to the coming crash at 0. And then in the homework you've been plagued by things like the Dirichlet function that is defined separately on the rationals and irrationals. And what these things show is that even the functions defined on what appears to be a straight line can show incredible diversity and jumping up and down.

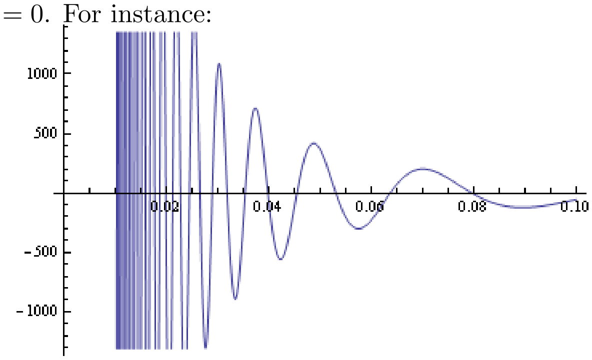
So let's think a bit about what can happen when a function is defined around a point but a one-sided limit is not defined. Here are some possibilities:

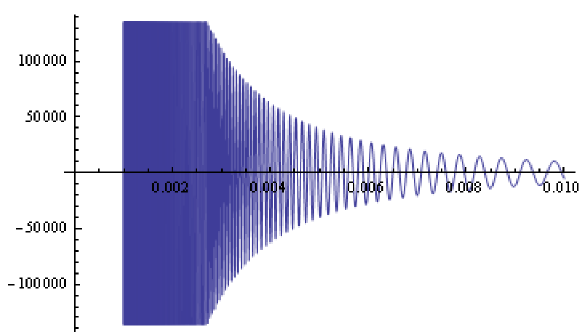
(1) As x approaches c, the function *oscillates* or jumps around, so it doesn't settle down, but it is still bounded. Some examples of this are the $\sin(1/x)$ function and the Dirichlet function.
(2) As x approaches c, the function heads for $+\infty$. This means that whatever height you set, the function eventually crosses that height and stays above. For instance, $1/x^2$, for $c = 0$, from either side. Or $1/x$ from the right side.
(3) As x approaches c, the function heads for $-\infty$. For instance, $-1/x^2$, for $c = 0$. Or $1/x$ from the left side.
(4) As x approaches c, the function oscillates between $+\infty$ and some finite lower bound. For instance, $\sin^2(1/x)/x^2$, for $c = 0$.





(5) As x approaches c, the function oscillates between $+\infty$ and $-\infty$. For instance, $\sin(1/x)/x^2$, for $c = 0$. For instance:





This is just scratching the surface. And to complicate matters further, we could have different behavior from the left and from the right. So you see that there's realy a wild party going on here.

3. Three important theorems

3.1. **The pinching theorem.** This theorem is also called the *squeeze theorem* or the *sandwich theorem*. Here's what it says:

If f, g, h are functions defined in a neighborhood of c, with the property that close to c, we have $f(x) \leq g(x) \leq h(x)$ for all x, and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ for all x, then we also have $\lim_{x\to c} g(x) = L$.

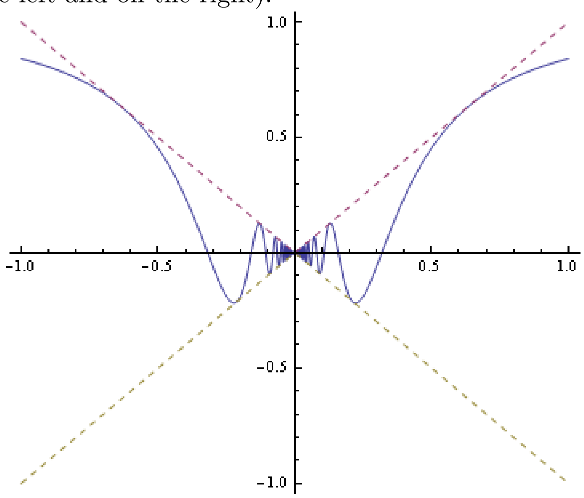
Analogous results hold for the left-hand limits and right-hand limits.

Basically, what this says is that if a function is trapped between two functions, both of which are approaching a particular value, then the function trapped in between also approaches that same value.

Here are some applications of this theorem.

(1) Recall the function in Homework 2: a funcion g defined as $g(x) = x$ for rational values of x and $g(x) = 0$ for irrational values of x. We want to show that $\lim_{x\to 0} g(x) = 0$. We first argue this for the right-hand limit. On the right, if we define $f(x) := 0$ and $h(x) := x$, then $f(x) \leq g(x) \leq h(x)$, and both f and h approach 0 at 0. Hence, g also approaches 0 at 0. On the left, we have $h(x) \leq g(x) \leq f(x)$, and again, since both f and h approach 0, so does g.

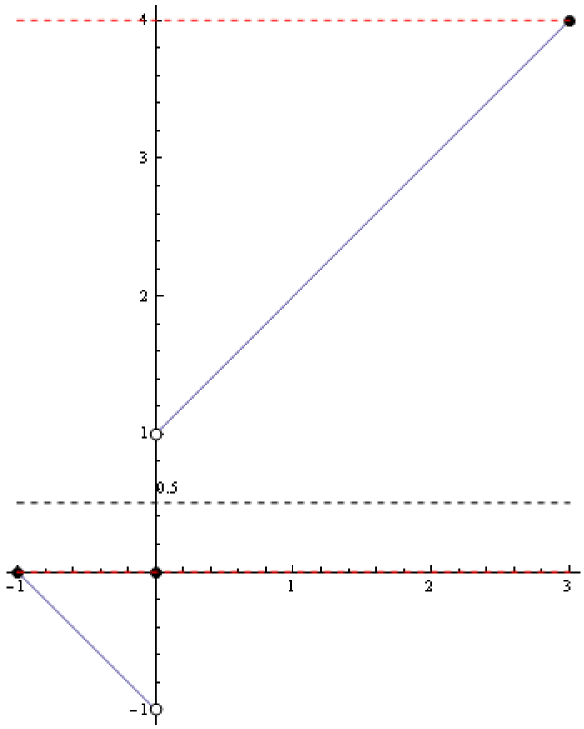
(2) Another example is the function $g(x) := x\sin(1/x)$. This is different from the $\sin(1/x)$ example that we saw earlier, because with this new function, the coefficient x causes a *damping* in the amplitude of the oscillations. To show that $\lim_{x\to 0} g(x) = 0$, we can use the pinching theorem, by squeezing g between the functions $f(x) = x$ and $h(x) = -x$ (again, the pinching will occur in different ways on the left and on the right).



Intermediate-value theorem. The intermediate-value theorem says that if f is a continuous function on a closed interval $[a, b]$, with $f(a) = c$ and $f(b) = d$, f takes every possible value between c and d. If $c < d$, this would mean that the range of f contains $[c, d]$. If $c > d$, this would mean that the range of f contains $[d, c]$.

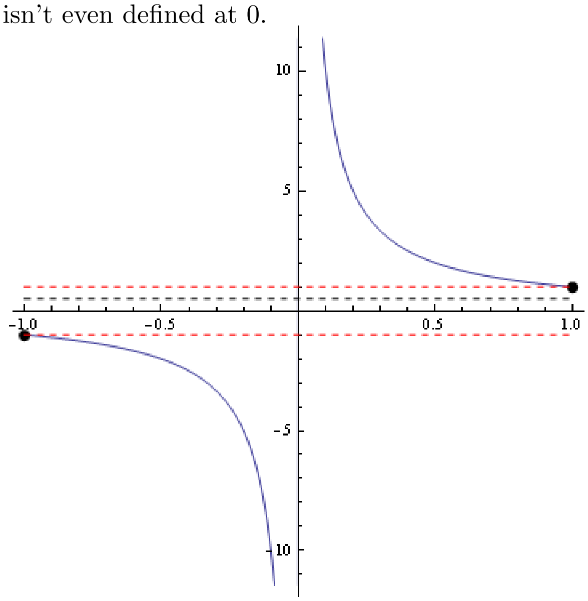
Note that f may take other values as well – for instance, it could go really high somewhere in between and then come back down, but what this theorem tells us is that it takes *at least* all the values between c and d.

The important thing here is that the function needs to be *continuous*. And the significance of continuity is that the function cannot suddenly *jump* from one place to the other – it has to go through all the intermediate steps. The graph below, for instance, plots the function $f(x) := (x+1)\operatorname{sgn}(x)$ on the interval $[-1, 3]$, where we define sgn as the signum functions, which is -1 on negative numbers, 0 at 0, and 1 on positive numbers. Note that $f(-1) = 0$ and $f(3) = 4$, but f does not take the value $1/2$ (which is between $f(-1)$ and $f(3)$) anywhere on its domain.



Here's an important caveat. If the function isn't continuous, we may get wrong conclusions by applying the intermediate-value theorem.

For instance, consider the function $f(x) = 1/x$. Then, $f(-1) = -1$ and $f(1) = 1$. Hence, a naive application of the intermediate-value theorem would suggest that there exists $x \in [-1, 1]$ such that $f(x) = 0$. But this is nonsense – $1/x$ can never be equal to 0. So, something went wrong in our application of the intermediate-value theorem. What went wrong? Just the fact that $1/x$ is not continuous on $[-1, 1]$ – in fact, it isn't even defined at 0.



Another wrong application would be to say that since $f(-1) = -1$ and $f(1) = 1$, there exists $x \in [-1, 1]$ such that $f(x) = 1/2$. This is wrong, because the only x for which $f(x) = 1/2$ is $x = 2$, which is not in the interval $[-1, 1]$. Again, the reason we went astray is that the function $f(x) = 1/x$ is not continuous on $[-1, 1]$.

3.2. **Justifying the existence of the square root function.** One way in which the intermediate-value theorem gets used, and that is to justify the existence of inverse functions. We'll discuss this in greater generality probably toward the end of the course.

Suppose we needed to justify the existence of $\sqrt{3}$. In other words, we needed to show the existence of a positive number x such that $x^2 = 3$. Here's how the intermediate-value theorem can be used. Consider the function $f(x) := x^2$. This function is everywhere continuous. Now, $f(1) = 1$ and $f(2) = 4$. Hence, by the intermediate-value theorem there exists some $x \in [1, 2]$ such that $f(x) = 3$.

3.3. **"Solving" equations using the intermediate-value theorem.** When you looked at an equation, your first urge was probably to solve it. And you solved it by manipulating stuff, applying the formula for the root of a quadratic, etc., etc. You had a toolkit of methods to solve equations, and you tried to faithfully apply this toolkit.

But there are times when you cannot find precise expressions for the solutions of equations. For instance, if I write a polynomial equation of degree 5, and ask you to solve it, may be you try a few values and then give up, but there's no general formula you can plug in to find the solutions (in fact, mathematicians have actually *proved* that general formulas do not exist – a proof you will probably not see unless you choose to major in mathematics). So, that's bad news. And similarly, if I write $\cos x = x$ and ask you to solve that, you have no mathematical way of finding a solution.

However, even if we cannot find exact solutions, we may be able to determine whether solutions exist, and even narrow down the interval in which they exist. And one tool in doing this is the intermediate-value theorem.

So consider the equation $\cos x = x$. The first thing you do is to take the difference, which in this case is $\cos x - x$. This is continuous, so the intermediate-value theorem applies, so to show that this difference is zero somewhere, it is enough to show that there's some place where it's positive and some place where it's negative. Well, let's draw the graphs of the functions x and $\cos x$ to get a bit if the intuition.

So looking at the graphs, you see that it's likely that a solution exists somewhere between 0 and $\pi/2$, because that's where the function x overtakes the function $\cos x$. Let's try to see this using the intermediate-value theorem. At 0, we have $\cos x - x = 1$, and at $\pi/2$, we have $\cos x - x = -\pi/2$. So, the function $\cos x - x$ goes from a positive value 1 to a negative value $-\pi/2$, which means that at some point in between, the function must be zero, and that's the point where $\cos x = x$.

Now, we can actually narrow down the value where $\cos x = x$ a little further, by trying to evaluate $f(x)$ for other values of x. For instance, we see, by evaluation, that $f(\pi/3) = 1/2 - \pi/3 < 0$, so the intermediate-value theorem tells us that $f(x) = 0$ for some $x \in [0, \pi/3]$. Next, we try $f(\pi/4) = 1/\sqrt{2} - \pi/4 < 0$, so, in fact, $f(x) = 0$ for some $x \in [0, \pi/4]$. And we can narrow things down still further by checking that $f(\pi/6) > 0$, so in fact, there is a solution to $f(x) = 0$ for $x \in [\pi/6, \pi/4]$.

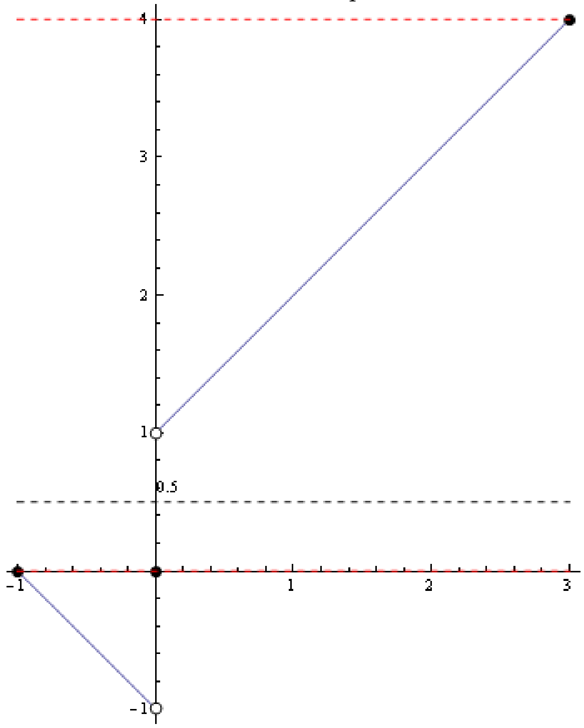
So, we see that we can use the intermediate-value theorem, along with evaluating the function at mutliple points, to narrow down pretty well where a function is zero. If you're more interested in these techniques, you should read more about the *bisection method* – it is discussed on Page 102 under the header *Project 2.6.*

3.4. **Extreme-value theorem.** Another theorem, that you should know, though you will not have the opportunity to apply it much right now, is the extreme-value theorem. This states that if f is a continuous real-valued function on a closed interval $[a, b]$, then f attains its maximum and its minimum, and of course, by the intermediate-value theorem, all the values in betweem. So if M is the maximum and m is the minimum of f, then the range of f is an interval of the form $[m, M]$.

Okay, there are many parts to this theorem, so let's understand it part-by-part. What does it mean to say that the function *attains its maximum*? Basically, we mean that there is some value M in the range of f that is the largest value in the range of f. And similarly, *attains its minimum* happens when there is some value in the range of f that is the smallest value in the range of f.

So, may be you're thinking that every function shoul attain its maximum and minimum. That's not true. In fact, a function on an interval stretching to infinity, or a function defined on an open interval, need not attain a maximum or minimum. For instance, the function $1/x$ on the interval $(0, \infty)$ doesn't attain either a maximum or minimum – it tends to (but doesn't reach) ∞ on one side and tends to (but doesn't reach) 0 on the other side.

Also, a discontinuous function defined on a closed interval need not attain a maximum and minimum. For instance, the function $(x+1)\,\mathrm{sgn}(x)$ on $[-1,3]$, which we discussed a little while ago, does not attain a minimum value because of the open circle at the low point:



So okay, we are somehow using something about continuity – may be it isn't clear what, but something, and we're also using something about closed intervals, to say that there is a maximum and minimum. What about the next part of the statement, which says that the range is precisely the stuff that's in between? Well, everything in the range has to be between the maximum and minimum – that's the definition of maximum and minimum. But why does everything between the maximum and minimum have to be in the range? Well, you can think of that as the intermediate-value theorem in action again.