INFORMAL INTRODUCTION TO LIMITS

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 2.1, parts of Sections 2.4.

Corresponding material in homework problems: Homework 2, Routine problems 1–4, 7–9.

Difficulty level: Easy to moderate, assuming you have seen some intuitive concept of limits before.

Covered in class?: Probably not. We may go over some small part of this quickly before covering $\epsilon - \delta$ definitions of limits.

Things that students should definitely get: To define limits, you need to get really really close. There are two directions from which to approach a real number: left and right. The notation for limits and one-sided limits. The intuitive meaning of the existence of limits and of continuity,, one-sided continuity, continuity on intervals. The notions of removable and jump discontinuity.

Things that students should hopefully get: The allusion to why taking limits for functions on a plane is harder because of multiple directions of approach.

Executive summary

Words ...

1. (1) On the real line, there are two directions from which to approach a point: the *left* direction and the *right* direction.
2. (2) For a function f, $\lim_{x \to c} f(x)$ is read as "the limit as x approaches c of $f(x)$. Equivalently, as x approaches c, $\lim_{x \to c} f(x)$ is the value that $f(x)$ approaches.
3. (3) $\lim_{x \to c} f(x)$ makes sense only if f is defined *around* c, i.e., both to the immediate left and to the immediate right of c.
4. (4) We have the notion of the *left hand limit* $\lim_{x \to c^-} f(x)$ and the *right hand limit* $\lim_{x \to c^+} f(x)$. The *limit* $\lim_{x \to c} f(x)$ exists if and only if (both the left hand limit and the right hand limit exist and they are both equal).
5. (5) f is termed *continuous* at c if c is in the domain of f, the limit of f at c exists, and $f(c)$ equals the limit. f is termed *left continuous* at c if the left hand limit exists and equals $f(c)$. f is termed *right continuous* at c if the right hand limit exists and equals $f(c)$.
6. (6) f is termed *continuous* on an interval I in its domain if f is continuous at all points in the interior of I, continuous from the right at any left endpoint in I (if I is closed from the left) and continuous fromthe left at any right endpoint in I (if I is closed from the right).
7. (7) A *removable discontinuity* for f is a discontinuity where a two-sided limit exists but is not equal to the value. A *jump discontinuity* is a discontinuity where both the left hand limit and right hand limit exist but they are not equal.

Actions: See the procedure in the last subsection on computing limits for polynomial and rational functions.

Note: These notes cover only the informal and intuitive concept of limits that you should be familiar with, and do not include the $\epsilon - \delta$ definitions. The $\epsilon - \delta$ definitions are covered in subsequent notes which we will go through very carefully in class.

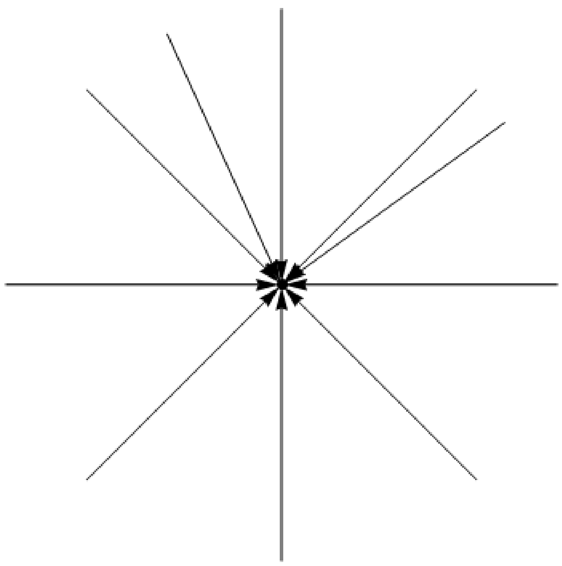
1. Intuitive conception of limits

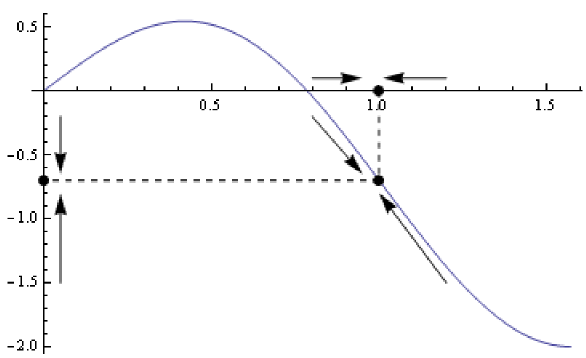
1.1. **The real numbers and two-sidedness of approach.** The first thing you need to know is that in order to understand limits, you really need to appreciate the real numbers. There's something particularly interesting about the real numbers: you can get *really really close* to a real number without equaling it. That's not something you can do with more sparse sets such as the integers.

It is this ability to sneak really close to something without being equal to it that allows us to talk of limits. This is something you should keep in mind – we'll get back to it later again when we talk of another kind of limit in 153 – the limit of a sequence. That's a very different but in some ways remarkably similar notion, but we'll come to it in due course.

Now, in the picture, I sneaked up on this number from one side – the left side. But I could sneak up to it from another side – the right side. This two-sidedness makes things pretty interesting. By the way, this is one advantage of dealing with a line – there are only two directions to worry about. Imagine, just imagine, if you were dealing with a plane. Then there would be the left side, the right side, the up side, the down side, this side, that side – too many! Luckily for us, we can postpone all those headaches for multivariable calculus, which is beyond the scope of the 150s. So we focus right now on this simple real line.

Here's the kind of picture you can avoid thinking about for now:





1.2. **Beginning and verbal gymnastics.**

So let's be really abstract. Suppose f is a function from a subset of the reals to a subset of the reals and c is a real number. What we would like to know is the answer to this question: as x gets really close to c, what does $f(x)$ get close to? If $f(x)$ is heading towards a specific destination, that's called its limit, and it has the notation:

$$\lim_{x \to c} f(x)$$

This is read as "the limit as x approaches c of $f(x)$". An equation such as:

$$\lim_{x \to c} f(x) = b$$

can be read in two ways: "the limit as x approaches c of $f(x)$ is b" or "as x approaches c, $f(x)$ approaches b". By the way, some people say *tends to* instead of *approaches*. Some people say *goes to* and those who're living at the point c may say *comes to*.

Now, let's take some examples. Suppose $f(x) = x$. So f is what is called the *identity function*. It is like a mirror that gives back what is put into it. Well, what then is $\lim_{x\to 0} f(x)$? Well, $f(x) = x$, so this is $\lim_{x\to 0} x$. So this reads like a word puzzle: "as x tends to 0, what does x tend to?" Of course, 0. In fact, more generally,

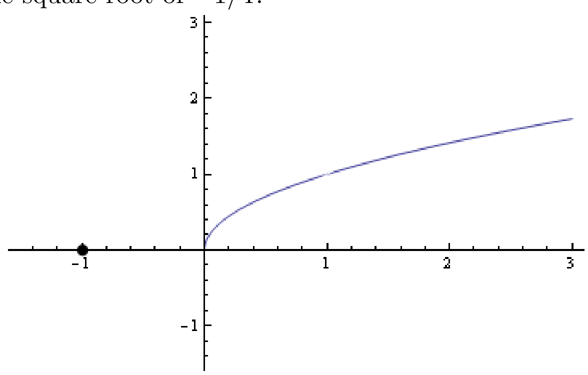
$$\lim_{x\to c} x = c$$

1.3. **Beyond our reach: can't limit to what you can't approach.** Okay, here's the next question: what is $\lim_{x\to -1} \sqrt{x}$. By the way, remember that $\sqrt{x}$ is defined as the nonnegative square root. So what is this limit? In other words, as x approaches, gets really really close to -1, what does $\sqrt{x}$ approach?

Verbal gymnastics doesn't work the same way as it does for the previous limit, so this one requires some serious thought. Okay, to simplify matters, let's first tackle the right side and then the left side.

Okay, let's try the right side. Let's start from far off. What is the square root of 4? It's 2. What is the square root of 3? $\sqrt{3}$ is approximately $1.732\ldots$. Square Root of 2 is $1.414\ldots$, square root of 1 is 1, square root of 0 is 0. Hmmm. So the square root seems to be decreasing. So you might guess right now that the limit is some negative number.

But to check this guess, you need to go down the negative aisle. And because there's a paucity of integers, we need to use fractional numbers. So let's try some negative number between 0 and -1. Say $-1/4$. What's the square root of $-1/4$?



It doesn't *have a square root*. It's a negative number. In fact, the domain of the square root function is the nonnegative reals, the interval $[0, \infty)$. And that's bad. Which means that as we get even a little close to -1, we cannot evaluate the function from the right side. So the limit from the right side doesn't make sense.

The limit from the left side doesn't make sense either, because the function isn't defined *anywhere* to the left of -1.

What's the message to take from this? It makes sense to talk of the limit of a function at a point, if the function is defined at places very close to the point. If it isn't, it's like, as some people say, putting "lipstick on a pig." You can take the limit of a function that doesn't exist, and it still doesn't exist.

1.4. **One-sided limits.** There are two further notions, that are mirror images of each other: the *left hand limit* and the *right hand limit*. The left hand limit at a is denoted as $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

The left hand limit of a function is the limit as you approach the domain value from the left side. The right hand limit of a function is the limit as you approach the domain value from the right side. If a function is defined on both the left and the right side of a point, there are five possibilities:

(1) Neither the left hand limit nor the right hand limit exist.
(2) The left hand limit exists but the right hand limit does not exist.
(3) The left hand limit does not exist but the right hand limit exists.
(4) Both the left hand and the right hand limit exist, but they are not equal.

(5) Both the left hand and the right hand limit exist, and they are equal. In this case, we say that the function *has a limit* and the limit is equal to both these values.

Phew! What a wide range of possibilities! But you should be happy that there are only two directions of approach: left and right. If and when you study multivariable calculus, you'll be studying functions on a plane, where you have not two, but *infinitely many* directions. If computing limits from two directions is a headache, computing limits from infinitely many directions is like enduring torture for eternity.

Things are a little different for values that are at extreme ends of the domain. For instance, think about the function $f : [0, 1] \to [0, 1]$ given by $f(x) = \sqrt{1 - x^2}$. Now, at the point -1, a left hand limit doesn't make sense because the function is not defined to the left of -1. So, only the right hand limit does. Similarly at the point 1, the right hand limit doesn't make sense but the left hand limit does.

There's a little confusion about conventions in what I'm going to say, but I'll just stick with the book on this one: if the point c is at the boundary of the domain and so the function isn't defined on one side, the book says that talking of the limit at c, or writing $\lim_{x \to c} f(x)$, is not meaningful. However, we can talk of the one-sided limit from the side that the function is defined. You may see a different convention at other places, but we'll just stick to the book for now. That means that if the point is at the boundary of the domain, you should clearly specify a one-sided limit instead of just taking *the limit.*

2. Continuity

2.1. The concept of continuity. Suppose f is a function and it is defined *around* a point a, i.e., f is defined at the point a and is also defined in some open interval containing a. Then f is continuous at a if the limit of f exists at a and equals $f(a)$. This means that the left hand limit and the right hand limit of f exist at a and are equal to $f(a)$. In symbols:

$$f \text{ continuous at } a \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

If f is defined at a and on the *immediate right* of a, then we say that f is *right continuous* or *continuous from the right* at a if the right hand limit of f at a equals $f(a)$. In symbols:

$$f \text{ right continuous at } a \iff \lim_{x \to a^+} f(x) = f(a)$$

If f is defined at a and on the *immediate left* of a, then we say that f is *left continouus* or *continuous from the left* at a if the left hand limit of f at a equals $f(a)$. In symbols:

$$f \text{ left continuous at } a \iff \lim_{x \to a^-} f(x) = f(a)$$

2.2. Continuity on an interval. Suppose I is an interval (open, closed, half-open half-closed, possibly infinite in one or both directions). A function f whose domain contains I is termed *continuous* on I if f is continuous for all *interior* points of I (i.e., all points of I that are not at the boundary of I) and is continuous from the appropriate side at all boundary points. We consider all cases below:

(1) If $I = (a, b)$, f needs to be continuous at all points of I.
(2) If $I = [a, b]$, f needs to be continuous at all points of (a, b), right continuous at a, and left continuous at b.
(3) If $I = [a, b)$, f needs to be continuous at all points of (a, b) and right continuous at a.
(4) If $I = (a, b]$, f needs to be continuous at all points of (a, b) and left continuous at b.
(5) If $I = (a, \infty)$ or $(-\infty, b)$ or $(-\infty, \infty)$, f needs to be continuous at all points of I.
(6) if $I = [a, \infty)$, f needs to be continuous at all points of (a, ∞) and right continuous at a.
(7) If $I = (-\infty, b]$, f needs to be continuous at all points of $(-\infty, b)$ and left continuous at b.

3. Plumbing leaks

3.1. Filling in the hole in the FORGET function. I hope you remember the *FORGET* function that we defined a little earlier:

$$FORGET(x) = \frac{x}{x}$$

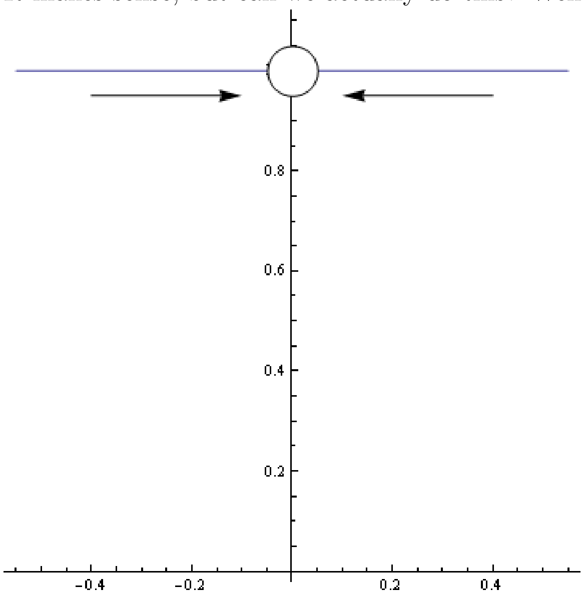
When I defined this function, we discussed that the function is *not* defined at zero. Why? Because at 0, when we plug in, we get a 0 in the numerator and a 0 in the denominator. Zero in the denominator is bad! This expression makes no sense. So forget about evaluating this function at 0.

However, it definitely makes sense to ask whether the *limit* exists at 0:

$$\lim_{x \to 0} FORGET(x)$$

Why does it make sense? Because the function is defined at all points other than 0, it is defined at all points that are close to 0 but not equal to it. It's defined at all points to the left of 0 and at all points to the right of 0. The *only* point where it is not defined is 0. So, it makes sense to evaluate the limit at 0.

It makes sense, but can we actually do this? Well, let's use the graph.



Okay, so we see that the function is 1 to the left of zero, 1 to the right of zero. What should the limit be? If there's any justice in the world, it should be 1. And it is.

Let's see this mathematically:

$$FORGET(x) = \frac{x}{x}, \qquad x \neq 0$$

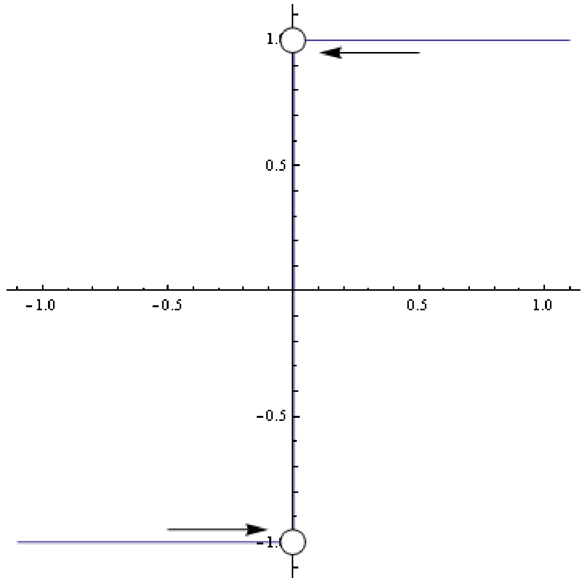
Thus:

$$\lim_{x \to 0} FORGET(x) = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

Now you may say: *why can we cancel now?* The reason why we can cancel x in that step above is that now, since we are *only approaching* 0 and *are not equal to it*, x can be canceled. And that is the beauty of limits. The thing that gives you trouble *at* a point doesn't give you any trouble *near* the point, and because you are sneaking up from nearby rather than evaluating at the point, you evade trouble. It's the roundabout maneuver when a direct assault fails.

3.2. **Looking back at the signum function.** The signum function, denoted sgn, is the function defined as $x/|x|$ when $x \neq 0$. Under some conventions, it is considered undefined at $x = 0$. Under other conventions, its value at 0 is defined to be 0.

The signum function is continuous – in fact, locally constant – at all nonzero points in the domain. At the point 0, it takes the value -1 everywhere on the left and it takes the value 1 everywhere on the right. Thus, the left hand limit is -1 and the right hand limit is 1. Thus, the function jumps from the value -1 to 1 at 0.



The signum function differs from the FORGET function in this important respect: for the FORGET function, we could *fix* or *remove* the discontinuity at 0 by filling in the value 1, because the *limit at* 0 *exists*. However, for the signum function, there is no way of fixing the discontinuity at 0 because the limit does not exist, which happens in turn because the left and right hand limits differ.

3.3. **Kinds of discontinuities.** We now consider some important kinds of *discontinuities* for functions, i.e., situations where a function f is defined aound a point c but is not continuous at c. Here are two important kinds of discontinuities:

(1) *Removable discontinuities* are discontinuities where the limit of f at c exists but is not equal to $f(c)$. There could be two reasons for this: either $f(c)$ is not defined (as for the FORGET function) or it is defined but is not equal to the limit.
(2) *Jump discontinuities* are discontinuities where both the left hand limit and the right hand limit exist and are finite, but they are not equal. Note that the value of the function at c can be changed to make the function continuous from the left at c. It can also be changed to make the function continuous from the right at c. But we cannot choose a value $f(c)$ such that both things happen simultaneously. An example is the signum function.

These are not the only possibilities. There are also infinite discontinuities (where the left hand limit or the right hand limit or both is/are $\pm\infty$) and oscillatory discontinuities. We will return to this topic later.

Aside:Left and right are based on source, not target. The left hand limit is the limit where the approach is *from* the left, but the direction in which the approach happens is *toward* the right. In other words, the choice of hand is based on the direction *from* which approach is being made rather than the direction in which the approach is happening.

A similar convention is followed when specifying the direction of winds. A *northern* wind is a wind that blows *from* north to south. If you took geography in school or read weather forecasts, you should be familiar with this.

[I have a guess as to the reason for choosing this convention, but it's purely speculative, so I won't include it here.]

4. Our nice cocooned world

Living on Planet Earth in an age of affluence, we are used to taking niceties for granted. Of course, if we consider the whole world throughout history, poverty is more the default condition of humans than affluence.

In the same way, the functions you've been dealing with so far are nice and sweet. For all their complications and complexities, they don't throw tantrums. In this course, we'll continue to deal, for the most part, with nice functions, but we'll explore the more rugged terrain every once in a while to appreciate our good fortune and the boundaries of our understanding. For now, let's review how good we have it and how to handle the occasional hiccup.

4.1. **Limits of polynomial and rational functions.** All polynomial functions are continuous, so the limit of a polynomial function at a point equals the value of the polynomial function at that point. In other words, if f is a polynomial function and c is a number, then $\lim_{x \to c} f(x) = f(c)$.

For rational functions, we evaluate the limit of a rational function f at a point c using the following rules:

(1) First, try to evaluate the numerator and the denominator at the point. If the denominator is nonzero at the point, the limit equals the value. If the denominator evaluates to 0 at the point, and the numerator evaluates to something nonzero at the point, then the limit is not defined. If both the numerator and the denominator evaluate to 0 at the point, then *more work is needed.*

(2) If both the numerator and the denominator are 0 at the point c, then there is a factor of $x - c$ in both the numerator and the denominator. Cancel this factor. This is permissible because we are only doing this cancellation *near* the point c, not *at* the point c. Keep doing such cancellations till there are no common factors of $x - c$ in the numerator and the denominator, and go back to step (1).

Here are some examples:

The function $f(x) = (x^2 - 3x + 2)/(x - 3)$ has $\lim_{x \to 1} f(x) = 0/(-2) = 0$.

The function $f(x) = (x^2 - 3x + 2)/(x - 1)$ has $\lim_{x \to 1} f(x) =??$ Well, evaluation gives 0 for both the numerator and the denominator, so we cancel the $(x - 1)$ factor, and we get:

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x - 2)}{x - 1} = \lim_{x \to 1} x - 2 = 1 - 2 = -1$$

Similarly, if $f(x) = (x - 1)/(x^2 - 3x + 2)$, we have:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{1}{x - 2} = \frac{1}{1 - 2} = -1$$

And here's another example:

$$\lim_{x \to 1} \frac{x - 3}{x^2 - 3x + 2}$$

This limit is not defined, because the numerator approaches -2 and the denominator approaches 0.