GRAPHING

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 4.8

Difficulty level: Hard.

What students should definitely get: The main concerns in graphing a function, how to figure out what needs figuring out. It is important for students to go through all the graphing examples in the book and do more hands-on practice. Transformations of graphs. Quickly graphing constant, linear, quadratic graphs.

What students should hopefully get: How all the issues of symmetry, concavity, inflections, periodicity, and derivative signs fit together in the grand scheme of graphing. The qualitative characteristics of polynomial function and rational function graphs, as well as graphs involving a mix of trigonometric and polynomial functions.

Weird feature: Ironically, there are very few pictures in this document. The naive explanation is that I didn't have time to add many pictures. The more sophisticated explanation is that since the purpose here is to review how to graph functions, having actual pictures drawn perfectly is counterproductive. Please keep a paper and pencil handy and sketch pictures as you feel the need.

Executive summary

0.1. **Symmetry yet again.** Words...

(1) All mathematics is the study of symmetry (well, not all).
(2) One interesting kind of symmetry that we often see in the graph of a function is *mirror symmetry* about a vertical line. This means that the graph of the function equals its reflection about the vertical line. If the vertical line is $x = c$ and the function is f, this is equivalent to asserting that $f(x) = f(2c - x)$ for all x in the domain, or equivalently, $f(c + h) = f(c - h)$ whenever $c + h$ is in the domain. In particular, the domain itself must be symmetric about c.
(3) A special case of mirror symmetry is the case of an *even function*. An even function is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(4) Another interesting kind of symmetry that we often see in the graph of a function is *half-turn symmetry* about a point on the graph. This means that the graph equals the figure obtained by rotating it by an angle of π about that point. A point (c, d) is a point of half-turn symmetry if $f(x) + f(2c - x) = 2d$ for all x in the domain. In particular, the domain itself must be symmetric about c. If f is defined at c, then $d = f(c)$.
(5) A special case of half-turn symmetry is an odd function, which is a function having half-turn symmetry about the origin.
(6) Another symmetry is *translation symmetry*. A function is *periodic* if there exists $h > 0$ such that $f(x + h) = f(x)$ for all x in the domain of the function (in particular, the domain itself should be invariant under translation by h). If a smallest such h exists, then such an h is termed the period of f.
(7) A related notion is that of a function with *periodic derivative*. If f is differentiable for all real numbers, and f' is periodic with period h, then $f(x + h) - f(x)$ is constant. If this constant value is k, then the graph of f has a two-dimensional translational symmetry by (h, k) and its multiples.

Cute facts...

(1) Constant functions enjoy mirror symmetry about every vertical line and half-turn symmetry about every point on the graph (can't get better).

(2) Nonconstant linear functions enjoy half-turn symmetry about every point on their graph. They do not enjoy any mirror symmetry because they are everywhere increasing or everywhere decreasing.
(3) Quadratic (nonlinear) functions enjoy mirror symmetry about the line passing through the vertex (which is the unique absolute maximum/minimum, depending on the sign of the leading coefficient). They do not enjoy any half-turn symmetry.
(4) Cubic functions enjoy half-turn symmetry about the point of inflection, and no mirror symmetry. Either the first derivative does not change sign anywhere, or it becomes zero at exactly one point, or there is exactly one local maximum and one local minimum, symmetric about the point of inflection.
(5) Functions of higher degree do not necessarily have either half-turn symmetry or mirror symmetry.
(6) More generally, we can say the following for sure: a nonconstant polynomial of even degree greater than zero can have at most one line of mirror symmetry and no point of half-turn symmetry. A nonconstant polynomial of odd degree greater than one can have at most one point of half-turn symmetry and no line of mirror symmetry.
(7) If a function is continuously differentiable and the first derivative has only finitely many zeros in any bounded interval, then the intersection of its graph with any vertical line of mirror symmetry is a point of local maximum or local minimum. The converse does not hold, i.e., points where local extreme values are attained do *not* usually give axes of mirror symmetry.
(8) If a function is twice differentiable and the second derivative has only finitely many zeros in any bounded interval, then any point of half-turn symmetry is a point of inflection. The converse does not hold, i.e., points of inflection do *not* usually give rise to half-turn symmetries.
(9) The sine function is an example of a function where the points of inflection and the points of half-turn symmetry are the same: the multiples of π. Similarly, the points with vertical axis of symmetry are the same as the points of local extrema: odd multiples of $\pi/2$.
(10) For a periodic function, any translate by a multiple of the period of a point of half-turn symmetry is again a point of half-turn symmetry. (In fact, any translate by a multiple of half the period is also a point of half-turn symmetry).
(11) For a periodic function, any translate by a multiple of the period of an axis of mirror symmetry is also an axis of mirror symmetry. (In fact, translation by multiples of half the period also preserve mirror symmetry).
(12) A polynomial is an even function iff all its terms have even degree. Such a polynomial is termed an *even polynomial*. A polynomial is an odd function iff all its terms have odd degree. Such a polynomial is termed an *odd polynomial*.
(13) Also, the derivative of an even function (if it exists) is odd; the derivative of an odd function (if it exists) is even.

Actions ...

(1) Worried about periodicity? Don't be worried if you only see polynomials and rational functions. Trigonometric functions should make you alert. Try to fit in the nicest choices of period. Check if smaller periods can work (e.g., for $\sin^2$, the period is π). Even if the function in and of itself is not periodic, it might have a periodic derivative or a periodic second derivative. The sum of a linear function and a periodic function has periodic derivative, and the sum of a quadratic function and a periodic function has a periodic second derivative.
(2) Want to milk periodicity? Use the fact that for a periodic function, the behavior everywhere is just the behavior over one period translates over and over again. If the first derivative is periodic, the increase/decrease behavior is periodic. If the second derivative is periodic, the concave up/down behavior is periodic.
(3) Worried about even and odd, and half-turn symmetry and mirror symmetry? If you are dealing with a quadratic polynomial, or a function constructed largely from a quadratic polynomial, you are probably seeing some kind of mirror symmetry. For cubic polynomials and related constructions, think half-turn symmetry.
(4) Use also the cues about even and odd polynomials.

0.2. **Graphing a function.** Actions ...

(1) To graph a function, a useful first step is finding the domain of the function.

(2) It is useful to find the intercepts and plot a few additional points.
(3) Try to look for symmetry: even, odd, periodic, mirror symmetry, half-turn symmetry, and periodic derivative.
(4) Compute the derivative. Use that to find the critical points, the local extreme values, and the intervals where the function increases and decreases.
(5) Compute the second derivative. Use that to find the points of inflection and the intervals where the function is concave up and concave down.
(6) Look for vertical tangents and vertical cusps. Look for vertical asymptotes and horizontal asymptotes. For this, you may need to compute some limits.
(7) Connect the dots formed by the points of interest. Use the information on increase/decrease and concave up/down to join these points. To make your graph a little better, compute the first derivative (possibly one-sided) at each of these points and start off your graph appropriately at that point.

Subtler points... (see the "More on graphing" notes for an elaboration of these points; not all of them were covered in class):

(1) When graphing a function, there may be many steps where you need to do some calculations and solve equations and you are unable to carry them out effectively. You can skip some of the steps and come back to them later.
(2) If you cannot solve an equation exactly, try to approximate the locations of roots using the intermediate value theorem or other results such as Rolle's theorem.
(3) In some cases, it is helpful to graph multiple functions together, on the same graph. For instance, we may be interested in graphing a function and its second and higher derivatives. There are other examples, such as graphing a function and its translates, or a function and its multiplicative shifts.
(4) A graph can be used to suggest things about a function that are not obvious otherwise. However, the graph should not be used as conclusive evidence. Rather, the steps used in drawing the graph should be retraced and used to give an algebraic proof.
(5) We are sometimes interested in sketching curves that are not graphs of functions. This can be done by locally expressing the curve piecewise as the graph of a function. Or, we could use many techniques similar to those for graphing functions.
(6) For a function with a piecewise description, we plot each piece within its domain. At the points where the definition changes, determine the one-sided limits of the function and its first and second derivatives. Use this to make the appropriate open circles, asymptotes, etc.

1. Graphing in general

The goal of this lecture is to make you more familiar with the tools and techniques that can be used to graph a function. The book has a list of points that you should keep in mind. The list in the book isn't complete – there are a number of additional points that tend to come up for functions of particular kinds, but it is a good starting point. But in this lecture, we'll focus on something more than just the techniques – we'll focus on the broad picture of why we want to draw graphs and what information about the function we want the graph to convey. Working from that, we will be able to reconstruct much of the book's strategy.

1.1. **Graphs – utility, sketching and plotting.**

The graph of a function f on a subset of the real numbers is the set of points in $\mathbb{R}^2$ (the plane) of the form $(x, f(x))$, where x is in the domain of f. The graph of f gives a geometric description of f, and it completely determines f. For a given $x = x_0$, $f(x_0)$ is the y-coordinate of the unique point of the graph that is also on the line $x = x_0$.

Graphs are useful because they allow us to see many things about the function at the same time, and enable us to use our visual instincts to answer questions about the function. It is usually easy to look at the graph and spot, without precise measurement, phenomena such as periodicity, symmetry, increase, decrease, discontinuity, change in direction, etc. Thus, the graph of a function, *if correctly drawn*, is not only equivalent in information content to the function itself, it makes that information content much more easy to read.

The problem is with the caveat *if correctly drawn*. The domains of most of the functions we consider are unions of intervals, so they contain infinitely many points. *Plotting the graph* in a complete sense would involve evaluating the function at these infinitely many points. In practice, *graph plotting* works by dividing the domain into very small intervals (say, of length 10^{-3}), calculating the values of the function (up to some level of accuracy, say 10^{-4}) at the endpoints of the intervals, and then drawing a curve that passes through all the graph points thus obtained. This last joining step is typically done using straight line segments.[1]

Unfortunately, although softwares such as Mathematica are good for plotting graphs, we humans would take too long to do the millions of evaluations necessary to plot graphs. However, we have another asset, which is our brains. We need to use our brains to find some substitute for plotting the graph that still gives a reasonable approximation of the graph and *captures the qualitative characteristics that make the graph such an informative representation of the function*. The process that we perform is called *graph sketching*.

A sketch of a graph is good if any information that is visually compelling from the sketch (without requiring precise measurement) is actually *correct* for the function. In other words, a good sketch may mislead people into thinking that $f(2) = 2.4$ while it is actually 2.5, but it should not make people think that f is increasing on the interval $(2, 3)$ if it is actually decreasing on the interval.[2]

1.2. **The domain of a function.** The domain of a function is easy to determine from its graph. Namely, the domain is the subset of the x-axis obtained by orthogonally projecting the graph onto the x-axis. In other words, it is the set of possible x-coordinates of points on the graph.

So, the first step in drawing the graph is finding the domain. We consider two main issues here:

(1) Sometimes, the domain may contain an open interval without containing one or both of the endpoints of that interval. In other words, there may be points in the boundary of the domain but not in the domain. In such cases, try to determine the limits (left and/or right, as applicable) of the function at these boundary points. If finite, we have open circles. If equal to $+\infty$ or $-\infty$, we have vertical asymptotes.
(2) In cases where the domain of the function stretches to $+\infty$ and/or $-\infty$, determine the limit(s). Any finite limit thus obtained corresponds to a horizontal asymptote.

Intercepts and a bit of plotting. So that the graph is not completely wrong, it is helpful to make it realistic using a bit of plotting. The book suggests computing the x-intercepts and the y-intercept.

The x-intercepts are the points where the graph intersects the x-axis, i.e., the points of the form $(x, 0)$ where $f(x) = 0$. There may be zero, one, or more than one x-intercepts. The y-intercept is the unique point where the graph intersects the y-axis, i.e., the point $(0, f(0))$. Note that if 0 is not in the domain of the function, then the y-intercept does not make sense.

In addition to finding the intercepts, it may also be useful to do a bit of plotting, e.g., finding $f(x)$ for some values of x, or finding solutions to $f(x) = y$ for a few values of y. The intercepts are the bare minimum of plotting. They're important to compute mainly because the values of the intercepts are visually obvious and it would be misleading to people viewing the graph if these values were obtained wrong.

1.3. **Symmetry/periodicity.** Another thing that is visually obvious from the graph is *patterns of repetition*. There are two kinds of patterns of repetition that we are interested in:

(1) *Periodicity*: The existence of $h > 0$ such that $f(x + h) = f(x)$ for all x in $\mathbb{R}$. Periodicity is graphically visible – the shape of the graph repeats after an interval of length h. Note that we can talk of periodicity even for functions that are not defined for all real numbers, as long as it is true that the domain itself is invariant under the addition of h. For instance, tan has a period of π.
(2) *Symmetry: even and odd*: An even function ($f(x) = f(-x)$ for all $x \in \mathbb{R}$) exhibits a particular kind of symmetry: symmetry about the y-axis. An odd function ($f(x) = -f(-x)$ for all $x \in \mathbb{R}$) exhibits *half-turn symmetry* about the origin. Both these properties are geometrically visible. Note that we can talk of even and odd for functions not defined for all real numbers, as long as the domain is symmetric about 0. For instance, $f(x) := 1/x$ is odd and $f(x) := 1/x^2$ is even.

There are somewhat more sophisticated versions of this:

[1]If you have seen computer graphics in the old days where computer memory and processing speed was limited, you would have seen that computer renderings of geometric figures such as circles was done using small line segments. As we improve the resolution, the line segments become smaller and smaller until our eyes cannot make out the difference.

[2]This does raise an interesting point, which is that the reason why sketches seem adequate even when inaccurate is because of our limited observational power – the correctly plotted graph would not look too different in terms of compelling visual information, and hence, we find the sketch good enough for our purposes.

(1) Periodicity with shift: This happens when there exists $h > 0$ and $k \in \mathbb{R}$ such that $f(x+h) = f(x)+k$ for all $x \in \mathbb{R}$. Thus, the graph of f repeats after an interval of length h, but it is shifted vertically by k. Note that the case of shift 0 is precisely the case where f itself is periodic. If f is also differentiable, this is equivalent to the derivative being periodic.

    A function is periodic with shift if and only if it is the sum of a periodic function and a linear function. The breakup as a sum is unique up to constants. The periodic function part can be thought of as representing the seasonal trend and the linear function part can be thought of as representing the secular trend.
(2) Half turn symmetry about axes other than the y-axis.
(3) Mirror symmetry about points other than the origin.

With the exception of *periodicity with shift*, all the other notions are discussed in detail in the second set of lecture notes on functions (Functions: A Rapid Review (Part 2)) so we will not repeat that discussion. Since the mirror symmetry and half turn symmetry material was not covered in class at the time, we'll take a short detour in class to cover that material.

1.4. **First derivative.** The next step in getting a better picture of the function is to use the derivative. The derivative helps us find the intervals on which the function is increasing and decreasing, the critical points, and other related phenomena. We shall return in some time to the application of this information to graph-sketching.

1.5. **Second derivative.** If the function is twice differentiable (at most points) the second derivative is another useful tool. We can use the second derivative to find intervals where the function is concave up, intervals where the function is concave down, and inflection points of the function. Combining this with information about the first derivative, we can determine intervals where the function is increasing and concave up (i.e., increasing at an increasing rate), increasing and concave down (i.e., increasing at a decreasing rate), decreasing and concave up (i.e., decreasing at a decreasing rate), or decreasing and concave down (i.e., decreasing at an increasing rate).

1.6. **Classifying and understanding points of interest.** Some of the cases of interest are:

(1) Point of discontinuity: Separately compute the left-hand limit, right-hand limit and value. If either one-sided limit is $\pm\infty$, we have a vertical asymptote. If a one-sided limit equals the value, the graph has a closed circle. If a one-sided limit exists but does not equal the value, the graph has an open circle.
(2) Critical point where the function is continuous and not differentiable: Determine whether the left-hand derivative and right-hand derivative individually exist. If so, determine the values of these derivatives. If the left-hand and right-hand derivatives do not exist as finite values, try determining the left-hand limit and right-hand limit of the derivative. If the limit of the derivative is $+\infty$ from both sides or $-\infty$ from both sides, we have a vertical tangent at the point. if the limit of the derivative is $+\infty$ from one side and $-\infty$ from the other side, we have a vertical cusp at the point. In all cases, determine the value of the function at the point.
(3) Critical points where the derivative of the function is zero: Determine whether this is a point of local maximum, a point of local minimum, a point of inflection, or none of these. In any case, determine the value of the function at the point.
(4) Point of inflection: Determine the value of the function as well as the value of the first derivative at the point. Also, determine whether the graph switches from concave up to concave down or concave down to concave up at the point.

Critical points, and phenomena related to the first derivative, are usually geometrically compelling, so it is important to focus on getting them right so as not to paint a misleading picture. The precise location of points of inflection is less geometrically compelling, except when such a point is also a critical point. Generally, it is geometrically clear that there exists an inflection point in the interval between two points, because the graph is concave up at one point and concave down at the other. However, the precise location of the critical point may be hard to determine. Thus, getting the precise details of inflection points correct is desirable but not as basic as getting the critical points correct.

1.7. **Sketching the graph.** We first plot the points of interest and values (including $\pm\infty$, corresponding to vertical asymptotes), as well as the horizontal asymptotes for points at infinity. Here, *points of interest* includes the critical points and inflection points, intercept points, and a few other points added in to get a preliminary plot. In addition to plotting the graph points (which is the pair $(x, f(x))$ where x is the point of interest in the domain), it is also useful to compute the one-sided derivatives at each of the points of interest, and draw a short segment of the tangent line (or half-line, if only one-sided derivatives exist) corresponding to that.

Next, we use the increase/decrase and concave up/down information, as well as the tangent half-lines, to make the portions of the graph between these points of interest. This is the step that involves some guesswork. The idea is that because we are sure that the main qualitative characteristics (increase versus decrease, concave up versus concave down) are correct, errors in further shape details are not a big problem.

Since these actual shapes are the result of guesswork, it is particularly important that the issues of symmetry and periodicity be taken into account while sketching. For a periodic function, it is better to have a *somewhat less accurate shape repeated faithfully in each period* than a number of different-looking shapes in different periods. Similar remarks apply for symmetry and even/odd functions.

The book has a number of worked out examples, and you should go through them. To keep your homework set of manageable size, I haven't included graph sketching problems in the portion of the homework to be submitted. But I have recommended a few graph sketching problems from the book's exercises and you should try these (and others if you want) and can check your answer against a graphing calculator or software.

2. Graphing particular functions

Here we discuss various simple classes of functions and how they can be graphed. For functions in these well behaved classes, we do not need to go through the entire rigmarole for graphing.

2.1. **Constant and linear functions.** We begin by looking at the constant function $f(x) := k$. This function is soporific, because you know the graph of the function is a straight horizontal line, the derivative of the function is zero everywhere, it is constant everywhere. Every point is a local minimum and a local maximum in the trivial sense. The limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are also both equal to k.

Next, we look at the linear function $f(x) := ax + b$ where $a \neq 0$. This function has graph a straight line. The tangent line at any point on the graph is the same straight line. The slope of the straight line is a. If $a > 0$, the function is increasing everywhere, and if $a < 0$, the function is decreasing everywhere. The derivative is the constant a and the second derivative is 0.

If $a > 0$, then $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. If $a < 0$, then $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$.

2.2. **Quadratic functions.** Consider the function $f(x) := ax^2 + bx + c$, where $a \neq 0$. This is a quadratic function. The derivative function $f'(x)$ is equal to $2ax + b$, the second derivative $f''(x)$ is the constant function $2a$, and the third derivative is 0 everywhere. In other words, the slope of the tangent line to the graph of this function is not constant, but it is changing at a constant rate.

The graph of this function is called a *parabola*. We describe the graph separately for the cases $a > 0$ and $a < 0$.

In the case $a > 0$, we have $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. The function attains a local as well as an absolute minimum at the point $x = -b/2a$, and the value of the minimum is $(4ac - b^2)/4a$. The point $(-b/2a, (4ac - b^2)/4a)$ is termed the *vertex* of the parabola. f is decreasing on the interval $(-\infty, -b/2a]$ and increasing on the interval $[-b/2a, \infty)$. Also, the graph of f is symmetric (i.e., a *mirror symmetry*) about the vertical line $x = -b/2a$.

In the case $a < 0$, we have $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -\infty$. The function attains a local as well as an absolute maximum at the point $x = -b/2a$, and the value of the maximum is $(4ac - b^2)/4a$. The point $(-b/2a, (4ac - b^2)/4a)$ is termed the *vertex* of the parabola. The function is increasing on the interval $(-\infty, -b/2a]$ and decreasing on the interval $[-b/2a, \infty)$.

Finally, note the following about the existence of zeros, based on cases about the sign of the discriminant $b^2 - 4ac$:

(1) Case $b^2 - 4ac > 0$ or $b^2 > 4ac$: In this case, there are two zeros, and they are located symmetrically about $-b/2a$. If $a > 0$, the function f is positive to the left of the smaller root, negative between the roots, and positive to the right of the larger root. If $a < 0$, the function f is negative to the left of the smaller root, positive between the roots, and negative to the right of the larger root.
(2) Case $b^2 - 4ac = 0$ or $b^2 = 4ac$: In this case, $-b/2a$ is a zero of multiplicity two. The vertex is thus a point on the x-axis with the x-axis a tangent line to it. Note that if $a > 0$, the parabola lies in the upper half-plane and if $a < 0$, the parabola lies in the lower half-plane.
(3) Case $b^2 - 4ac < 0$ or $b^2 < 4ac$: In this case there are no zeros. If $a > 0$, the parabola lies completely in the upper half-plane, and if $a < 0$, the parabola lies completely in the lower half-plane.

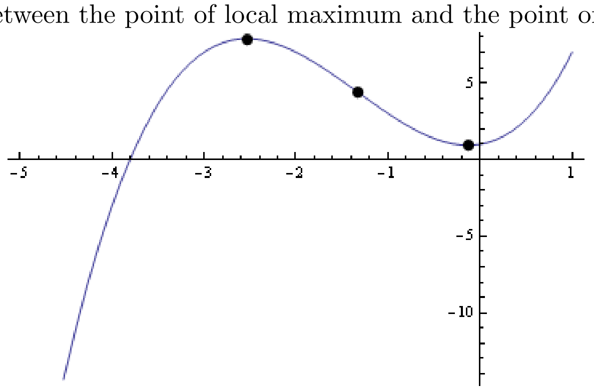
2.3. **Cubic functions.** We next look at the case of a cubic polynomial, $f(x) := ax^3 + bx^2 + cx + d$, where $a \neq 0$. We carry out the discussion assuming $a > 0$. In case $a < 0$, maxima and minima get interchanged and the sign of infinities on limits get flipped.[3]

So let's discuss the case $a > 0$. We have $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$. Notice that, by the intermediate-value theorem, the cubic polynomial takes all real values. The derivative of the function is $f'(x) = 3ax^2 + 2bx + c$, the second derivative is $f''(x) = 6ax + 2b$, the third derivative is $f'''(x) = 6a$ and the fourth derivative is zero. This means that not only is the slope changing, but it is changing at a changing rate, but the rate at which that rate is changing isn't changing (yes, you read that right).

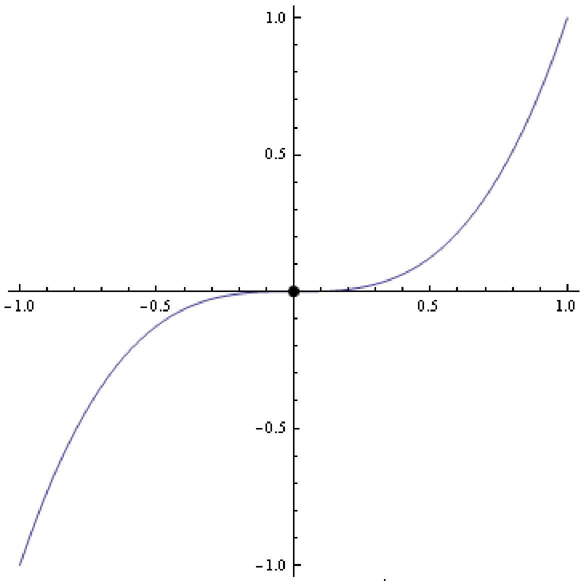
We now try to determine where the function has local maxima and minima, and where it is increasing or decreasing. For this, we need to first find the critical points. The critical points are solutions to $f'(x) = 0$. The discriminant of the quadratic polynomial f' is $4b^2 - 12ac$. We make three cases based on the sign of the discriminant.

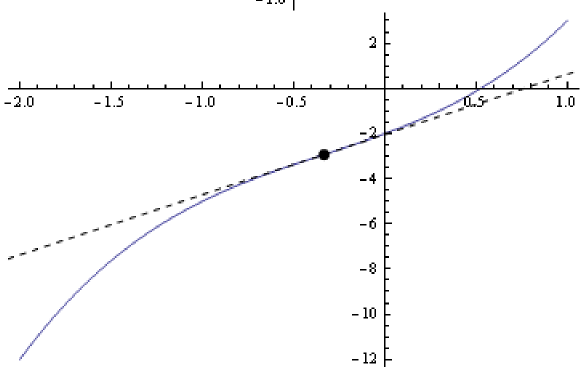
(1) $4b^2 - 12ac > 0$, or $b^2 > 3ac$: In this case, there are two critical points, given by the two solutions to the quadratic equation. We also see that, since $a > 0$, f' is positive to the left of the smaller root, negative between the two roots, and positive to the right of the larger root. Thus, f is increasing from $-\infty$ to the smaller root, decreasing between the two roots, and increasing from the larger root to ∞. The smaller root is thus a point of local maximum and the larger root is a point of local minimum.
(2) $4b^2 - 12ac = 0$, or $b^2 = 3ac$: In this case, there is one critical point, namely $-b/3a$. The function is increasing all the way through, so although this is a critical point, it is neither a local maximum nor a local minimum. In fact, it is a point of inflection, where both the first and the second derivative become zero.
(3) $4b^2 - 12ac < 0$, or $b^2 < 3ac$: In this case, the function has no critical points and is increasing all the way through.

Any cubic polynomial enjoys a half-turn symmetry about the point $(-b/3a, f(-b/3a))$, i.e., the graph is invariant under a rotation by π about this point. This center of half-turn symmetry is also the unique point of inflection for the graph. In the case that $b^2 > 3ac$, the point of half-turn symmetry is the exact midpoint between the point of local maximum and the point of local minimum.



[3] Another way of thinking of it is that we can first plot the graph by taking out a minus sign on the whole expression, then flip it about the x-axis.





2.4. **Polynomials of higher degree.** Here are some general guidelines to understanding polynomials of higher degree:

(1) The limits at $\pm\infty$ are determined by whether the polynomial has even or odd degree, and the sign of the leading coefficient. Positive leading coefficient and even degree mean a limit of $+\infty$ on both sides. Negative leading coefficient and even degree mean a limit of $-\infty$ on both sides. Positive leading coefficient and odd degree mean a limit of $+\infty$ as $x \to \infty$ and $-\infty$ as $x \to -\infty$. Negative leading coefficient and odd degree mean a limit of $+\infty$ as $x \to -\infty$ and $-\infty$ as $x \to \infty$.
(2) The points where the function could potentially change direction are the zeros of the first derivative. For a polynomial of degree n, there are at most $n - 1$ of these points. For such a point, we can use the first-derivative test and/or second-derivative test to determine whether the point is a point of local maximum, local minimum, or a point of inflection. *Note that for polynomial functions, any critical point must be a point of local maximum, local minimum, or a point of inflection.* There are no other possibilities for polynomial functions, because the number of times the first and/or second derivative switch sign is finite, hence we cannot construct all those weird counterexamples involving oscillations when dealing with polynomial functions.
(3) Between any two zeros of the polynomial there exists at least one zero of the derivative (this follows from Rolle's theorem). This can help us bound the number of zeros of a polynomial using information we have about the number of zeros of the derivative of that polynomial.
(4) A polynomial of odd degree takes all real values, and in particular, intersects every horizontal line at least once.
(5) A polynomial of even degree and positive leading coefficient has an absolute minimum value, and takes all values greater than or equal to that absolute minimum value at least once. A polynomial

of even degree and negative leading coefficient has an absolute maximum value, and takes all values less than or equal to that absolute maximum value at least once.

2.5. **Rational functions: the many concerns.** A lot of things are going on with rational functions, so we need to think about them more carefully than we thought about polynomials.

Graphing the function requires putting these pieces together, each of which we have dealt with separately:

(1) Determine where the rational function is positive, negative, zero, and not defined.
(2) At the points where the rational function is not defined, determine the left-hand and right-hand limits. In most cases, these limits are $\pm\infty$. The exceptions are for cases such as the $FORGET$ function, defined as $FORGET(x) = x/x$, which is not defined at $x = 0$, but has a finite limit at that point. *These exceptions only occur in situations where the rational function as originally expressed is not in reduced form.*
(3) Determine the limits of the rational function at $\pm\infty$. Note that this depends on how the degrees of the numerator and denominator compare and the signs of the leading coefficients.
(4) Consider the derivative f', and do a similar analysis on the derivative. The regions where the derivative is positive are the regions where f is increasing. The regions where the derivative is negative are the regions where f is decreasing.
(5) Consider the second derivative f'', and do a similar analysis on this. Use this to find the regions where f is concave up and the regions where f is concave down.

We combine all of these to draw the graph of f. We can also use all this information to determine where the function attains its local maxima and local minima.

2.6. **Piecewise functions.** Let's now deal with functions that are piecewise polynomial or rational functions. We'll also use this occasion to discuss general strategies for handling functions with piecewise definitions.

First, we need a clear piecewise definition, i.e., a definition that gives a polynomial or rational function expression on each part of the domain. The original definition may not be in that form. Here are some things we need to do:

(1) Whenever the whole expression, or some component of it, is in the absolute value, we make cases based on whether the expression whose absolute value is being evaluated is positive or negative. The transitions usually occur either at points where the expression is not defined, or at points where the absolute value is zero.
(2) Whenever the expression involves something like $\max\{f(x), g(x)\}$, then we make cases based on whether $f(x) > g(x)$ or $f(x) < g(x)$. The transition occurs at points where $f(x) = g(x)$ or at points where one or both of f and g is undefined.

Once we have the definition in piecewise form, we can differentiate, with the rule being to use the formula for differentiating in each piece where we have the expression. If the function is continuous at the points where the definition changes, we can use these formal expressions to calculate the left-hand derivative and right-hand derivative. We can then combine all this information to get a comprehensive picture of the function.

2.7. **A max-of-two-functions example.** *Note*: This or a very similar example appeared in a past homework. You might want to revisit that homework problem.

Consider $f(x) := \max\{x - 1, \frac{x}{x+1}\}$. We first need to get a piecewise description of f. For this, we need to determine where $x - 1 > x/(x + 1)$ and where $x - 1 < x/(x + 1)$. This reduces to determining where $(x^2 - x - 1)/(x + 1)$ is positive, zero, and negative.

The expression is positive on $((1 + \sqrt{5})/2, \infty) \cup (-1, (1 - \sqrt{5})/2)$, negative on $((1 - \sqrt{5})/2, (1 + \sqrt{5})/2) \cup (-\infty, -1)$, zero at $(1 \pm \sqrt{5})/2$, and undefined at -1. Thus, we get that:

$$f(x) = \{ \begin{matrix} x - 1, & x \in ((1 + \sqrt{5})/2, \infty) \cup (-1, (1 - \sqrt{5})/2) \\ \frac{x}{x+1}, & x \in (-\infty, -1) \cup [(1 - \sqrt{5})/2, 1 + \sqrt{5}/2] \end{matrix}$$

Next, we want to determine the limits of f as $x \to \pm\infty$. Since the definition for $x > (1 + \sqrt{5})/2$ is $x - 1$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x - 1 = \infty$. On the other hand, the definition for $x < -1$ is $x/(x + 1)$, so

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x/(x+1)$. This is a rational function where the numerator and denominator have equal degrees, and the leading coefficients are both 1, so the limit as $x \to -\infty$ is 1.

Next, we want to find out the left-hand limit and right-hand limit at the point $x = -1$. The definition from the left side is $x/(x+1)$. The denominator approaches 0 from the left side and the numerator approaches a negative number, so the quotient approaches $+\infty$. The right-hand limit is $\lim_{x \to -1} x - 1 = -2$.

Next, let us try to determine where the function is increasing and decreasing. For this, we need to differentiate the function on each interval.

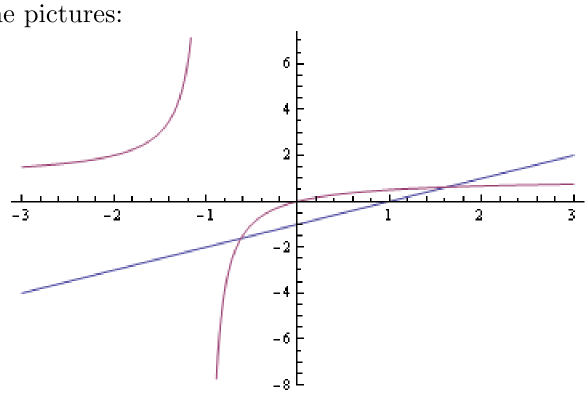
On the intervals $(-\infty, -1)$ and $[(1-\sqrt{5})/2, (1+\sqrt{5})/2]$, f is equal to $x/(x+1)$. The derivative is thus $1/(x+1)^2$, which is positive everywhere, and hence, in particular, on this region. Thus, f is increasing on $(-\infty, -1)$ as well as on $[(1-\sqrt{5})/2, (1+\sqrt{5})/2]$. On the intervals $(-1, (1-\sqrt{5})/2)$ and $(1+\sqrt{5}/2, \infty)$, f is defined as $x - 1$. The derivative is 1, so f is increasing on these intervals as well. In fact, since f is continuous at $1 + \sqrt{5}/2$, f is increasing on $[(1+\sqrt{5})/2, \infty)$. Combining all this information, we obtain that f is increasing on $(-\infty, -1)$ and on $(-1, \infty)$.

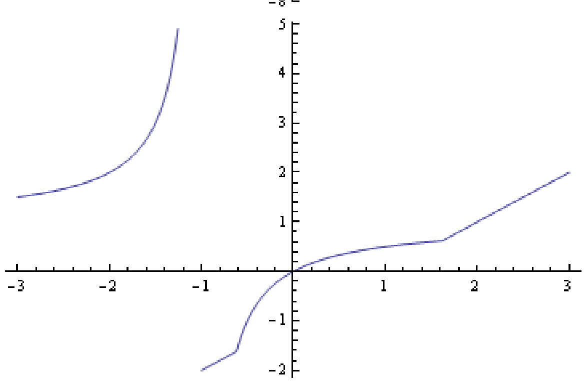
We can now understand and graph f better. As $x \to -\infty$, $f(x) \to 1$, and as $x \to -1^-$, $f(x) \to +\infty$. Thus, on the interval $(-\infty, -1)$, f increases from 1 to ∞. Since the right-hand limit at -1 is -2 and the limit at ∞ is ∞, we see that on the interval $(-1, \infty)$, f increases from -2 to ∞. There are two intermediate points where the definition changes: $(1 \pm \sqrt{5})/2$. From -1 to $(1-\sqrt{5})/2$, f increases from -2 to $(-1-\sqrt{5})/2$ in a straight line. Between $(1-\sqrt{5})/2$ and $(1+\sqrt{5})/2$, f increases from $(-1-\sqrt{5})/2$ to $(-1+\sqrt{5})/2$, but not in a straight line. From $(1+\sqrt{5})/2$ onward, f increases in a straight line again.

The critical points are $(1 \pm \sqrt{5})/2$. Neither of these is a local minimum or a local maximum. There is no absolute maximum, because the left-hand limit at -1 is ∞, so the function takes arbitrarily large positive values.

The function does not take arbitrarily small values. In fact, a lower bound on the function is -2. Despite this, the function has no absolute minimum, because -2 arises only as the right-hand limit at -1 and not as the value of the function at any specific point.

Note that more careful graphing of the function would also take into account concavity issues. Here are the pictures:





2.8. **Trigonometric functions.** Trigonometric functions are somewhat more difficult to study because, unlike the case of polynomials and rational functions, there could be infinitely many zeros.

One technique that is sometimes helpful when dealing with periodic functions is to concentrate on the behavior in an interval the length of one period, draw conclusions from there, and then use that to determine what happens everywhere. A very useful fact here is that f is a periodic function with period p, then f' (wherever it exists) also has period p. Similarly, the points and values of local maxima, local minima, absolute maxima and absolute minima all repeat after period p. In particular, in order to find the absolute maximum or absolute minimum, it suffices to find the absolute maximum or absolute minimum over a closed interval whose length is one period.

Consider, for instance, the function $f(x) := \sin x \cos x$. Since both sin and cos have a period of 2π, f repeats after 2π (so the period divides 2π). So, it suffices to find maxima and minima over the interval $[0, 2\pi]$. At the endpoints the value is 0. The derivative of the function is $\cos^2 x - \sin^2 x = \cos(2x)$. For this to be zero, we need $2x$ to be an odd multiple of $\pi/2$, so $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. We can use the second-derivative test to see that the points $\pi/4, 5\pi/4$ are points of local maximum and the points $3\pi/4, 7\pi/4$ are points of local minimum. The value of the local maximum is $1/2$ and the value of the local minimum is $-1/2$.

(It turns out that the function $\sin x \cos x$ has a period of π, and can also be thought of as $(1/2)\sin(2x)$.)

2.9. **Mix of polynomial and trigonometric functions.** When the function is a mix involving polynomial and trigonometric functions, it is not usually periodic, nor is it a polynomial, so we need to do some ad hoc work.

For instance, consider the function $f(x) := x - 2\sin x$. The derivative is $f'(x) = 1 - 2\cos x$. Note that although f is not periodic, f' is periodic, so in order to find out where $f' > 0$, $f' = 0$, and $f' < 0$, we can restrict attention to the interval $[-\pi, \pi]$.

We have $f'(x) < 0$ for $x \in (-\pi/3, \pi/3)$, $f'(x) = 0$ for $x \in \{-\pi/3, \pi/3\}$, and $f'(x) > 0$ for $x \in (\pi/3, \pi) \cup (-\pi, -\pi/3)$.

Translating this by multiples of 2π, we obtain that $f'(x) < 0$ for $x \in (2n\pi - \pi/3, 2n\pi + \pi/3)$ for n an integer, $f'(x) = 0$ for $x \in \{2n\pi - \pi/3, 2n\pi + \pi/3\}$, and $f'(x) > 0$ at other points. Thus, f keeps shifting between increasing and decreasing.

On the other hand, for the function $f(x) := 2x - \sin x$, the derivative is $f'(x) = 2 - \cos x$. This is always positive, so f is increasing.

2.10. **Functions involving square roots and fractional powers.** For functions involving squareroots or other fractional powers, we first need to figure out the domain. Then, we use the usual techniques to handle things.

Consider, for instance, the function:

$$f(x) := \sqrt{x} + \sqrt{1 - x}$$

The domain of this function is the set of values of x for which both $\sqrt{x}$ and $\sqrt{1-x}$ is defined. This turns out to be the set $[0, 1]$, since we need both $x \geq 0$ and $1 - x \geq 0$. We can differentiate f to get:

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{1-x}}$$

Note that although f is defined on the closed interval $[0, 1]$, f' is defined on the *open* interval $(0, 1)$ – it is not defined at the endpoints. In fact, the right-hand limit at 0 is $+\infty$ and the left-hand limit at 1 is $-\infty$.

Next, we want to determine where $f'(x) = 0$. Solving this, we get $x = 1/2$. Thus, $x = 1/2$ is a critical point. We also see that for $x < 1/2$, $\sqrt{x} < \sqrt{1-x}$, so the reciprocal $1/2\sqrt{x}$ is greater than the reciprocal $1/2\sqrt{1-x}$. Thus, the expression for $f'(x)$ is greater than 0. On the other hand, to the right of $1/2$, $f'(x) < 0$. Thus, f' is positive to the left of $1/2$ and negative to the right of $1/2$, yielding that f is increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$. Thus, f attains a unique absolute maximum at $1/2$, with value $\sqrt{2}$.

A more complicated version of the coffee shop problem. Remember the coffee shop problem, where there are two coffee shops located at points $a < b$ on a two-way street, and our task was to construct the function that describes distance to the nearest coffee shop. Let's now look at a somewhat different version, where the coffee shops are both located off the main street.

Suppose coffee shop A is located at the point $(0,1)$ and coffee shop B is located at the point $(2,2)$, and our two-way street is the x-axis. The goal is similar to before: write as a piecewise function the distance from the nearest coffee shop.

Define $p(x)$ as the distance from A and $q(x)$ as the distance from B. Then, we have $p(x) = \sqrt{x^2+1}$ and $q(x) = \sqrt{(x-2)^2+4} = \sqrt{x^2-4x+8}$. Our goal is to write down explicitly the function $f(x) := \min\{p(x), q(x)\}$.

In order to do this, we need to consider the function $p(x) - q(x)$ and determine where it is positive, zero and negative. Define $g(x) := p(x) - q(x)$. Then, for $g(x) = 0$, we need:

$$\sqrt{x^2+1} = \sqrt{x^2-4x+8}$$

Squaring both sides and simplifying, we obtain that $x = 7/4$. Since g is continuous, we can see that it has constant sign to the left of $7/4$ (which turns out to be negative, as we see by evaluating at 0) and constant sign to the right of $7/4$ (which turns out to be positive, as we see by evaluating at 2). Thus, our expression for f is given by:

$$f(x) = \{ \begin{array}{rl} \sqrt{x^2+1}, & x \in (-\infty, 7/4] \\ \sqrt{x^2-4x+8}, & x \in (7/4, \infty) \end{array}$$

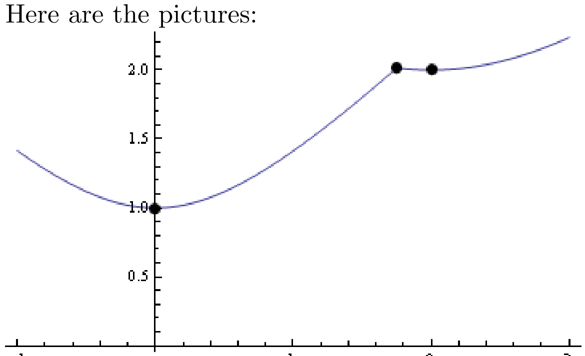
We can now use this to calculate f'. $f'(x) = x/\sqrt{x^2+1}$ to the left of $7/4$ and $(x-2)/\sqrt{x^2-4x+8}$ to the right of $7/4$. At the point $7/4$, the left-hand derivative is $7/\sqrt{65}$ and the right-hand derivative is $-1/\sqrt{65}$. The function is not differentiable at $7/4$.

Next, we want to determine where $f' > 0$, $f' = 0$ and $f' < 0$. For $x < 7/4$, we see that $f'(x) < 0$ for $x \in (-\infty, 0)$, $f'(0) = 0$, and $f'(x) > 0$ for $x \in (0, 7/4)$. For $x > 7/4$, we see that $f'(x) < 0$ for $x \in (7/4, 2)$, $f'(2) = 0$, and $f'(x) > 0$ for $x \in (2, \infty)$ (this should again be clear by looking at the picture geometrically. The distance to coffee shop A decreases till we get to the same x-coordinate as A, then it increases. At some point, B starts becoming closer, whence the distance to B starts decreasing, till we reach the point with the same x-coordinate as B, and then it starts increasing).

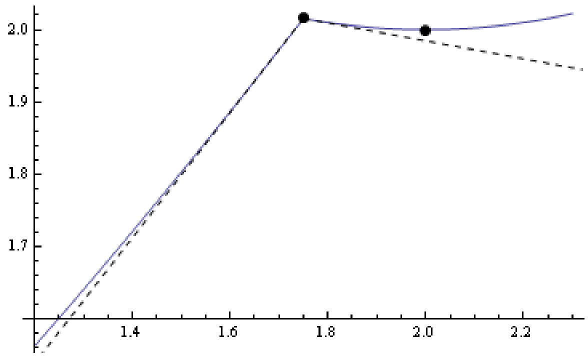
Thus, f is decreasing on $(-\infty, 0]$, increasing on $[0, 7/4]$, decreasing on $[7/4, 2]$, and increasing on $[2, \infty)$. The critical points are 0, $7/4$, and 2. There are local minima at 0 (with value 1) and 2 (with value 2) and a local maximum at $7/4$ (with value $\sqrt{65}/4$). The limits at $\pm\infty$ are both ∞. Thus, there is no absolute maximum, but the absolute maximum occurs at 0, and it has value 1.

Notice that although the picture here is qualitatively somewhat similar to the case where both coffee shops are on the x-axis, there are also some small differences – the graph never touches the x-axis, and the function is differentiable with derivative zero at two of the three critical points.

Here are the pictures:



Hre is the picture zoomed in (note: axes not centered at origin) near the value $x = 7/4$.

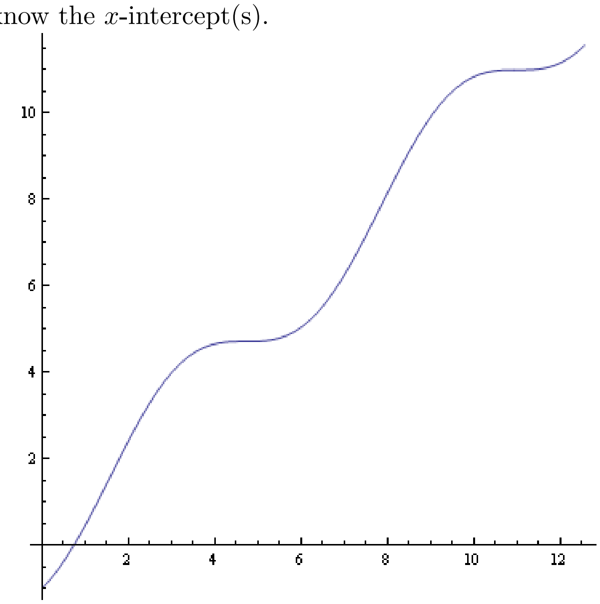


3. Subtle issues

3.1. **Equation-solving troubles.** In some cases, it is not computationally easy to do each of the suggested steps. For instance, we may not have any known method for solving $f(x) = 0$ for the given function f. Similarly, we may not have any known method for solving $f'(x) = 0$ or $f''(x) = 0$.

In cases where we do not have exact solutions, what we should do is try to find the number of solutions and the intervals in which these solutions lie, to as close an approximation as possible. Two useful tools in this are the *intermediate-value theorem* and *Rolle's theorem*.

For instance, consider the function $f(x) := x - \cos x$. f is an infinitely differentiable function, and its derivative, $1+\sin x$, is periodic with period 2π. Thus, the graph of f repeats after 2π, with a vertical upward shift of 2π. We can further find that f is increasing everywhere, because $1+\sin x \geq 0$ for all x, with equality occurring only at isolated points. $f''(x) = \cos x$, so f is concave up on $(-\pi/2, \pi/2)$ and its 2π-translates, and f is concave down on $(\pi/2, 3\pi/2)$ and its 2π-translates. The inflection points of f are precisely the odd multiples of $\pi/2$. The x-intercept is -1.We thus have a fairly complete picture of f, except that we do not know the x-intercept(s).



Although we do not know the x-intercept(s) precisely, we have some qualitative information. First, there can be at most one x-intercept, because f is increasing on $\mathbb{R}$. The intermediate-value theorem now reveals that the x-value must be somewhere between 0 ($f(0) = -1$) and $\pi/2$ ($f(\pi/2) = \pi/2$). In other words, the zero occurs in the segment between the y-intercept and the first inflection point after that. This is fine for a rough visual guide, but for a more accurate graph, we might like to narrow the location of the zero further. We can narrow it down further to $(\pi/6, /\pi/4)$ using elementary trigonometric computations. Further narrowing is best done with the aid of a computer.

Note that even if we did not bother about knowing the x-intercept before sketching the graph, our graph sketch would have been quite okay and would in fact have *suggested* the location of the x-intercept. This is an example of a general principle: *Often, even if we are computationally unable to handle all the suggested steps for graph-sketching, a preliminary sketch based on the steps we could successfully execute gives enough valuable hints.* The moral of the story is to not be discouraged about not executing a few steps and instead to do as much as possible with the steps already executed, and then seek alternative ways of tackling the recalcitrant steps.

See also Example 5 in the book.

3.2. **Graphing multiple functions together.** In many situations, it is necessary to be able to graph multiple functions together. This is sometimes necessary to compare and contrast these functions. Some examples include:

(1) Graphing a function and its first, second and higher derivatives together: This is often visually useful in discerning patterns about the function, and helps with rapid switching between the global and local behavior of a function.
(2) Graphing a function and another function obtained by scaling or shifting it: For instance, it may be helpful to graph $f(x)$ and $g(x) := f(x+h)$ on the same graph. This allows for easy visual insight into how the value of f changes after an interval of length h.
(3) Graphing two functions to determine their intersection points, angles of intersection, etc.

When graphing multiple functions together, the procedure is similar to that when graphing a single function, but the following additional point needs to be kept in mind: It is important to make sure that any visually obvious inferences made about the comparison of values of the functions are correct. For instance, it is important to get right which function is bigger where. The ideal way to do this is to find precisely the points of intersection – however, that may not be possible because the equation involved cannot be solved. Nonetheless, try to bound the locations of intersection points in small intervals using the intermediate value theorem. (Note that for functions obtained as derivatives, we can use Rolle's theorem and the mean value theorem.)

3.3. **Transformations of functions/graphs.** Also, if the two functions are related in terms of a transform, it is important that the geometric picture suggested by the transform is the correct one. Here are some examples:

(1) Suppose we have two functions f and g where $g(x) := f(x+h)$. Then, the graph of g should be the graph of f shifted left by h. If h is negative, it is the graph of f shifted right by $-h = |h|$.
(2) Suppose we have $g(x) := f(x) + C$. Then, the graph of g equals the graph of f shifted upward by C. If C is negative, it is the graph of f shifted downward by $-C = |C|$.
(3) Suppose $g(x) := f(\alpha x)$. Then, the graph of g should be the graph of f shrunk along the x-dimension by a factor of α. If α is negative, then this shrinking is a composite of a shrinking by $|\alpha|$ and a flip about the y-axis.
(4) Suppose $g(x) := \alpha f(x)$. Then the graph of g should be the graph of f expanded along the y-dimension by a factor of α. If α is negative, this involves an expansion by $|\alpha|$ and a flip about the x-axis.

3.4. **Can a graph be used to prove things about a function?** Yes and no. Remember that the way we drew the graph was using algebraic information about the function. So anything we deduce from the graph, we could directly deduce from that algebraic information, without drawing the graph.

The importance of graphs is that *they suggest good guesses that may not be obvious simply by looking at the algebra.* In other words, they allow visual and spatial intuition to complement the formal, symbolic intuition of mathematics. However, once the guess is made, it should be possible to justify without resort to the graph. Such justifications may use theorems such as the intermediate value theorem, Rolle's theorem, the extreme value theorem, and the mean value theorem. *In cases where things suggested by the graph cannot be verified algebraically, it is possible that some unstated and unjustified assumption was made while drawing the graph.*

3.5. **Sketching curves that are not graphs of functions.** Some curves are not in the form of functions, and cannot be expressed in that form because there are multiple y-values for a given x-value. To sketch such curves, we follow similar guidelines, but there are some changes:

(1) There is no clear concept of domain. However, it is still useful to determine the possible x-values for the curve and the possible y-values for the curve. This allows us to bound the curve in a rectangle or strip. For instance, consider the curve $x^4 + y^4 = 16$. Then, the x-value is in the interval $[-2, 2]$ and the y-value is in the interval $[-2, 2]$.
(2) We can use the techniques of sketching graphs of functions by breaking the curve down into graphs of functions. For instance, the curve $x^4 + y^4 = 16$ can be broken down as a union of graphs of two functions: $y = (16 - x^4)^{1/4}$ and $y = -(16 - x^4)^{1/4}$. We can sketch both graphs using the techniques of graph-sketching (in fact, it suffices to sketch the first graph and then construct the second graph as the reflection of the first graph about the x-axis).
(3) In cases where this separation is not easy to do, we can still try to draw the graph using the general techniques: use implicit differentiation to find the first derivative and second derivative, determine the critical points, local extreme values, points of inflection, regions of increase and decrease, regions of concave up and concave down, and so on.

3.6. **Piecewise descriptions, absolute values and max/min of two functions.** To graph a function explicitly given in piecewise form, we need to keep in mind the following things:

(1) Within the domain of each definition, plot the graph of the function the usual way.
(2) At the points where the definition changes, determine the one-sided limits, one-sided limits of first derivatives, and one-sided limits of second derivatives. These points are likely candidates for discontinuity of the function, likely candidates for discontinuity of the derivative, and likely candidates for discontinuity of the second derivative of the function.
(3) Piece this information together to draw the overall graph. Use open circles, closed circles etc. to mark clearly the limits at the points of definition changes.

In some cases, it is helpful to draw the graphs of each of the pieces over *all real values* and then pick out the requisite pieces from the relevant domains of definition.

If a function is defined as the maximum of two functions or the minimum of two functions, or in terms of absolute values, then we can first express it as a piecewise function and then graph it. Alternatively, we can graph both the functions (taking care of the points of intersection) and then use a combination of visual insight and algebra to graph the maximum and/or minimum of the two functions.

4. Addenda

4.1. **Addendum: Plotting graphs using Mathematica.** It is possible to plot the graph of a function using Mathematica. Doing a few such plots can help reinforce your intuition about the shape of graphs.

The Mathematica syntax is:

```
Plot[f[x],{x,a,b}]
```

This plots the graph of $f(x)$ for $x \in [a, b]$.

For instance:

```
Plot[x^2,{x,0,1}]
```

plots the graph of x^2 for $x \in [0, 1]$.

The command:

```
Plot[x - Sin[x],{x,-3*Pi,3*Pi}]
```

plots the graph of the function $x - \sin x$ on the interval $[-3\pi, 3\pi]$. Note that it is not possible to graph a function from $-\infty$ to ∞, so we have to stay content with finite plots.

It is also possible to plot the graphs of multiple functions together. For instance:

```
Plot[{Sin[x],(Sin[x])^2},{x, -Pi,Pi}]
```

This plots the graphs of the fnuctions sin and $\sin^2$ on the interval $[-\pi, \pi]$. To learn more, see the Mathematica documentation on the Plot function.

We can also use Mathematica to find where a function is positive, zero, and negative. You can use the Solve, Reduce, and FindRoot functions in Mathematica:

(1) The Solve function only solves equalities, and may not find all solutions. It also uses formal methods, so may not find the solutions numerically. However, it will give a formal solution saying π instead of $3.14\ldots$, for instance).
(2) The Reduce function is more powerful. It solves both equalities and inequalities, and finds all solutions. Like Solve, it only works for certain kinds of functions where these analytical and formal methods can be applied.
(3) The FindRoot function can be used to find points where a function is zero numerically. It is applicable to functions that involve a mixture of algebra and trigonometry. However, since it uses numerical methods, it may not give the exactly correct answer (for instance, it may compute 0.998 instead of 1).

For instance, we can do:

```
Reduce[x^3 - x - 6 > 0,x]
```

and find that the solution set to this is $x > 2$.

For something non-algebraic, we can find the roots:

```
FindRoot[x - Cos[x],{x,1}]
```

find a solution to $\cos x = x$. Note that Solve and Reduce do not work here because of the mixture of algebra and trigonometry. See the documentation on Solve, Reduce, and FindRoot.

We can also find the derivative of a function. First, define the function, e.g.:

```
f[x_] := x - Sin[x]
```

We can then refer to the derivative of f as f' and the second derivative as f''. Thus, we can do:

```
Reduce[{f''[x] > 0,-Pi < x, x < Pi},x]
```

This finds all solutions to $f''(x) > 0$ for x in the open interval $(-\pi, \pi)$.

These commands allow us to execute most of the computational aspects needed for graph-sketching using Mathematica.

Addendum: using a graphing software or graphing calculator. When using a graphing software or graphing calculator to plot the graph of a function, please make sure you zoom in and out enough to make sure that you are not fooled because of the scale chosen by the calculator. For instance, plotting the graph of $x^2 \sin(1/x)$ using a graphing software makes it seem like it crosses the x-axis at only finitely many points. However, zooming in closer to zero shows a lot of oscillation close to zero, and the more you zoom in, the more oscillation you see. Thus, it is important to use graphing software as a complement rather than a substitute for basic mathematical common sense.