FUNCTIONS: A RAPID REVIEW (PART 2)

MATH 152, SECTION 55 (VIPUL NAIK)

Difficulty level: Easy to moderate. Some of the symmetry concepts (half turn symmetry and mirror symmetry) and some of the proof techniques are likely to be new to students. The rest should be straightforward.

Covered in class?: This will roughly correspond to material covered on Wednesday September 28. Most of the trickier aspects of this will be covered in class, but many small points will be omitted due to time constraints. Hence, it is recommended that you read through these notes either before or after lecture.

Corresponding material in the book: Sections 1.6, 1.7. However, some ideas (including mirror symmetry and half turn symmetry) are not covered in the book. In some other cases, ideas covered later in the book are introduced at this early stage since you are already familiar with calculus.

Corresponding material in homework problems: Homework 1, routine problems 7–8, advanced problems 2–3.

Things that students should get immediately: Definitions of various notions of pointwise combination of functions, scalar multiples of functions, and compositions of functions. Definitions of even, odd, and periodic functions.

Things that students should get with effort: Definitions and graphical interpretations of half turn symmetry and mirror symmetry. Proof techniques related to showing even, odd, and periodic.

Executive summary

For first time reading, skip to the next section.

Words ...

(1) Given two functions f and g, we can define pointwise combinations of f and g: the sum $f+g$, the difference $f-g$, the product $f \cdot g$, and the quotient f/g. For the sum, difference, and product, the domain is the intersection of the domains of f and g. For the quotient, the domain is the intersection of the domain of f and the set of points where g takes a nonzero value.
(2) Given a function f and a real number α, we can consider the scalar multiple αf.
(3) Given two functions f and g, we can try talking of the composite function $f \circ g$. This is defined for those points in the domain of g whose image lies in the domain of f.
(4) One interesting kind of symmetry that we often see in the graph of a function is *mirror symmetry* about a vertical line. This means that the graph of the function equals its reflection about the vertical line. If the vertical line is $x = c$ and the function is f, this is equivalent to asserting that $f(x) = f(2c - x)$ for all x in the domain, or equivalently, $f(c+h) = f(c-h)$ whenever $c+h$ is in the domain. In particular, the domain itself must be symmetric about c.
(5) A special case of mirror symmetry is the case of an *even function*. An even function is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(6) Another interesting kind of symmetry that we often see in the graph of a function is *half-turn symmetry* about a point on the graph. This means that the graph equals the figure obtained by rotating it by an angle of π about that point. A point (c, d) is a point of half-turn symmetry if $f(x) + f(2c - x) = 2d$ for all x in the domain. In particular, the domain itself must be symmetric about c. If f is defined at c, then $d = f(c)$.
(7) A special case of half-turn symmetry is an odd function, which is a function having half-turn symmetry about the origin. By definition, the domain of an odd function is symmetric about $\mathbb{R}$. An odd function, if defined at 0, takes the value 0 at 0.

(8) A function f defined on $\mathbb{R}$ is periodic if there exists $h > 0$ such that $f(x+h) = f(x)$ for every $x \in \mathbb{R}$. If there is a smallest $h > 0$ satisfying this, such a h is termed the *period*. Constant functions are periodic but have no period. The sine and cosine functions are periodic with period 2π.

Actions ...

(1) To prove that a function is periodic, try to find a h that *works* for every x. To prove that a function is periodic but has no period, try to show that there are arbitrarily small $h > 0$ that work.
(2) To prove that a function is even or odd, just try proving the corresponding equation for all x. Nothing but algebra.
(3) If a function is defined for the positive or nonnegative reals and you want to extend the definition to negatives to make it even or odd, extend it so that the formula is preserved. So define $f(-x) = f(x)$, for instance, to make it even.

1. Ways of creating new functions from old

1.1. **Pointwise combinations of functions.** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are functions. By the way, before we proceed, a clarification on notation and terminology. When I say $f : A \to B$ is a function, then the *domain* of the function is A. However, the *range* of the function need not be B. All that notation means is that the range of the function is a *subset* of B. It might be equal to B, but there's no guarantee. And the reason why we allow this kind of latitude is that it makes it a lot easier to write things down if we do not need to calculate the exact range all the time. And, by the way, the set B is termed the *co-domain.*

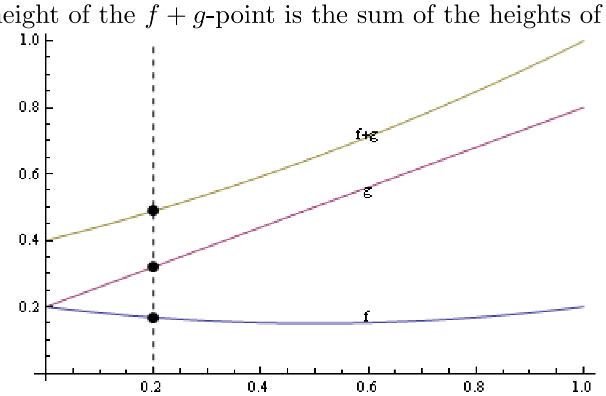
What does $f + g$ mean? So the first thing you might say is: "How can we add functions? I thought we could add only numbers." And the answer is that we don't yet know how to make sense of it, but once we do, it seems intuitive.

So remember, to define $f + g$ as a function, we need to describe where it sends x. So $(f + g)(x)$ is defined as follows:

$$(f + g)(x) := f(x) + g(x)$$

This is the sum of the two functions, and if you stick around in the world of mathematics, you'll hear people say that the sum is defined *pointwise*. What this really means is that to add two functions, what we do is add the *values* of the functions at each *point.*

Here's a picture showing two functions f and g and their sum $f + g$. Note that for each vertical line, the height of the $f + g$-point is the sum of the heights of the f-point and the g-point:



Now, I assumed that the functions are both defined from $\mathbb{R}$ to $\mathbb{R}$. And so, what is the domain of the function $f + g$? Well, it is $\mathbb{R}$ again, because as you can see from the definition, since you can evaluate f and g at a point, you can also evaluate $f + g$ at that point. And, by the way, I use the word *point* where I actually mean *real number* – secretly, I'm thinking of real numbers as points on the number line.

What if f is a function defined on a smaller domain (i.e., a subset of $\mathbb{R}$) and g on another smaller domain (i.e., another subset of $\mathbb{R}$)? In that case, $f + g$ is defined on the *intersection* of dom(f) and dom(g). Why the intersection? Because to evaluate $f + g$ at a point, you need to evaluate f at the point *and* g at the point, and then add those values. And to be able to evaluate *both*, the input should be in the domain of both functions.

We similarly define:

$$(f - g)(x) := f(x) - g(x)$$
$$(f \cdot g)(x) := f(x)g(x)$$
$$(f/g)(x) := f(x)/g(x)$$

For the case of the difference and product, the domain is the intersection of the domains. For the ratio, or quotient, we need to be a little more careful: the domain of the new function is inside the intersection of the domains of f and g, but there's a caveat: we need to exclude points at which $g = 0$.

1.2. **Scalar multiples of functions.** Suppose f is a function and α is a real number. The function αf is defined as:

$$(\alpha f)(x) := \alpha f(x)$$

For instance, $2f$ is the function that sends x to $2f(x)$, while $-f$ is the function sending x to $-f(x)$.

1.3. **Composition of functions.** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are functions. Then $f \circ g$ is defined as the following function:

$$(f \circ g)(x) := f(g(x))$$

$f \circ g$ is termed the *composition* of the functions f and g. Orally, we say "f composed with g". Note that the function written on the right is the one the we apply *first*, so in the case of function composition, we work from right to left. This can be potentially confusing.

We can also define the composite of two functions when their domains are subsets of $\mathbb{R}$. The domain of the composite $f \circ g$ is that subset of $\text{dom}(g)$ whose image under g lies inside $\text{dom}(f)$. There is a more precise way of expressing this, but it will take us too far afield, so we will skip it.

1.4. **Are there other ways of creating new functions?** Yes, but we will see them later. The most significant of these are differentiation, integration, and taking inverse functions.

1.5. **Why do ways of creating new functions from old matter?** First, of course, ways of creating new functions from old help us create new functions from old. However, just as new food recipes are of little interest to those unenthusiastic about cooking and eating, new function recipes may seem pointless to those unenthusiastic about playing with functions. There is a deeper reason.

The point is that these ways of creating new functions from old are *already in use when we think of and create new functions*. By *explicitly identifying* the various recipes used to create new functions from old, we hope to get a better mental model of functions that *already exist*. Both pointwise combination and composition are implicitly used all the time without our even knowing it. Making them explicit is like writing down an explicit recipe for a dish that we've already been cooking and eating.

We will better be able to understand a new phenomenon for *all functions* when we are able to break the process of such understanding into two steps: (i) understanding the phenomenon for a list of basic building block functions, (ii) understanding how the phenomenon interacts with the recipes for creating new functions from old. For instance:

(1) In order to learn how to differentiate functions, we do two things: (i) learn formulas for differentiating a list of basic functions (e.g., derivatives of power functions, trigonometric functions, etc.) (ii) learn formulas for the derivative of a new function created by a recipe from other functions, in terms of those other functions and their derivatives (e.g., derivatives of sums, differences, scalar multiples, product rule, quotient rule, and chain rule).
(2) In order to learn how to integrate functions, we do two things: (i) learn integration formulas for a list of basic functions (ii) learn procedures for integrating complicated functions in terms of their basic building blocks (unfortunately, the rules for product and composition are not straightforward, making integration a much more messy and also much more interesting business).

(3) To prove that all functions in a particular collection satisfy a property such as continuity or differentiability, it suffices to prove the property for the basic building blocks of the collection, and then to prove that the various ways of building new functions from old within the collection preserve the property.

2. Symmetries of functions

2.1. **Even and odd functions.** Let's first discuss the concept of even function and odd function for globally defined functions, i.e., functions defined for *all* real numbers.

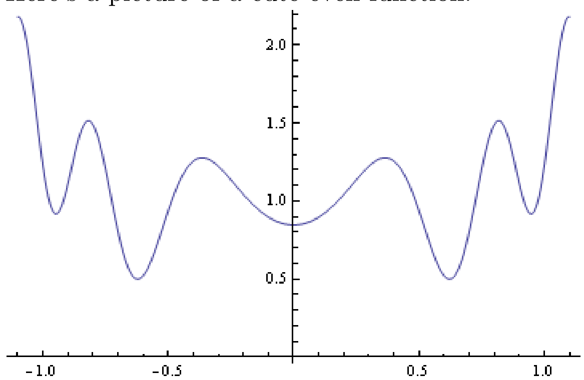
By the way, as I pointed out earlier, when I say $f : \mathbb{R} \to \mathbb{R}$, you should *not* assume that the range is $\mathbb{R}$. I just mean a globally defined function that takes real values.

So we say that f is an *even function* if:

$$f(x) = f(-x) \ \forall \ x \in \mathbb{R}$$

So, what does this mean from the point of view of its graph? Well, it turns out that this is equivalent to saying that the graph is symmetric about the y-axis.

Here's a picture of a cute even function:

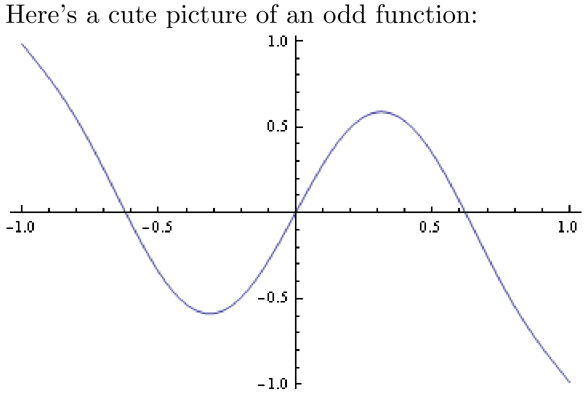


We say that f is an *odd function* if:

$$f(-x) = -f(x) \ \forall \ x \in \mathbb{R}$$

This is equivalent to saying that the graph has a *rotational* symmetry about the origin. If you rotate the graph by π (that's $180\,^\circ$) you get back to the original thing.

Here's a cute picture of an odd function:



The notion of even and odd function also makes sense for functions whose domain is not the whole real numbers, but rather, is a subset of the real numbers. The notion makes sense only when the domain is *symmetric* about 0, i.e., whenever x is in the domain of the function, so is $-x$. Some examples of domains symmetric about 0 are: intervals of the form $[-a, a]$, intervals of the form $(-a, a)$, intervals of the form $(-a, a) \setminus \{0\}$, intervals of the form $[-a, a] \setminus \{0\}$, the set of all integers $\mathbb{Z}$, the set of all rational numbers $\mathbb{Q}$, a union of intervals of the form $(-b, -a) \cup (a, b)$, and many more.

2.2. **Mirror symmetry.** We say that a function f possesses mirror symmetry about the line $x = c$ if the domain of f is symmetric about c and, for all $x \in \text{dom}(f)$, we have:

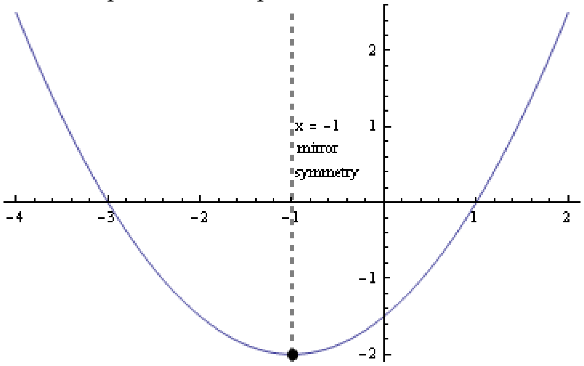
$$f(x) = f(2c - x)$$

Equivalently, for all $h > 0$, $c + h \in \text{dom}(f)$ if and only if $c - h \in \text{dom}(f)$, and if so, then:

$$f(c + h) = f(c - h)$$

Even functions are a special case: they have mirror symmetry about the y-axis.

Here's a picture of a quadratic function that has mirror symmetry about the line $x = -1$.



2.3. **Half turn symmetry.** We say that a function f possesses half turn symmetry about the point (c, d) if the domain of f is symmetric about c and, for all $x \in \text{dom}(f)$, we have:

$$f(x) + f(2c - x) = 2d$$

Equivalently, for all $h > 0$, $c + h \in \text{dom}(f)$ if and only if $c - h \in \text{dom}(f)$, and if so, then:

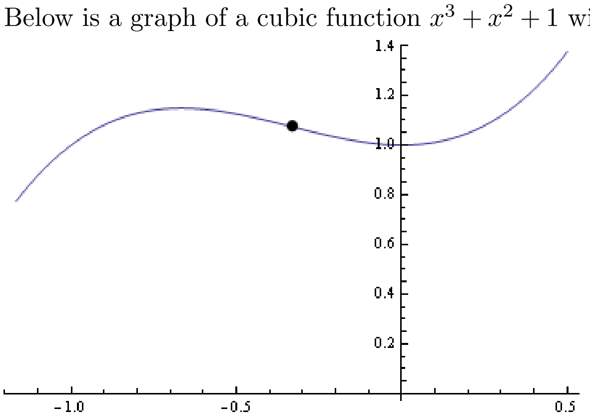
$$f(c + h) + f(c - h) = 2d$$

In other words, the point (c, d) is the midpoint between $(c + h, f(c + h))$ and $(c - h, f(c - h))$.

If $c \in \text{dom}(f)$, then we are forced to have $d = f(c)$.

Odd functions are a special case with the point of half turn symmetry about the origin $(0, 0)$.

Below is a graph of a cubic function $x^3 + x^2 + 1$ with half turn symmetry about the point $(-1/3, 29/27)$.



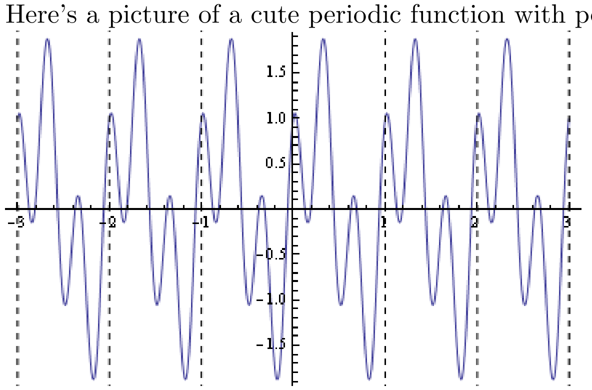
2.4. **Periodic functions.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. We say that f is a *periodic function* if there exists a $h > 0$ such that:

$$f(x + h) = f(x) \ \forall \ x \in \mathbb{R}$$

The *period* (more correctly called the *fundamental period*) of f is the smallest $h > 0$ for which the above holds (for all $x \in \mathbb{R}$).

The trigonometric functions are examples of periodic functions. For instance, sin and cos have period 2π. What about the $\sin^2$ function? Well, 2π works, but it isn't the smallest thing that works. The smallest h that works is π.

Here's a picture of a cute periodic function with period 1:



2.5. **Other notions of symmetry.** There are many other notions of symmetry for functions that we will encounter as we start drawing graphs. The most significant of these is the periodic + linear symmetry, which is observed for functions that can be expressed as a sum of a periodic function and a linear function. More on this later.

2.6. **Why do notions of symmetry matter?** Notions of symmetry are important for a number of reasons, including the following:

(1) For functions which possess symmetry, graphing the function can be a lot easier since the symmetry allows us to fill in the graph at many points based on a small part.
(2) Symmetry allows us to deduce properties about derivatives of the function.
(3) Symmetry allows us to deduce properties about definite integrals. Often, definite integrals can be computed using symmetry properties even though antiderivatives are hard or impossible to express explicitly.

3. Proving and reasoning involving these functions

3.1. **Proof positive: showing something to be even, odd, or periodic.** To show that a function f is even, we start with a *generic* x, compute $f(x)$ and $f(-x)$, and show that both are equal.

To show that a function f is odd, we start with a *generic* x, compute $f(x)$ and $f(-x)$, and show that the results are negatives of each other.

Showing that a function f is periodic is somewhat trickier. f is defined to be periodic if there exists $h > 0$ such that $f(x+h) = f(x)$ for all x in the domain of f. Thus, to show that f is periodic, we first need to find a value of h that works. After we have chosen a specific numerical value of h, we then pick a *generic* x and show that $f(x+h) = f(x)$.

In logic notation, periodicity states that:

$$\exists h > 0 \text{ such that } \forall\, x \in \mathrm{dom}(f), f(x+h) = f(x)$$

The $\exists$ stands for an existential quantifier and the $\forall$ stands for a universal quantifier. For existentially quantified variables, we need to come up with a specific value that "works" while for universally quantified variables, we need to show that every value works, which we do by picking a *generic* value.

3.2. **Relation between symmetry and creation of new functions.** Here are some important facts that can be proved using the techniques mentioned in the previous subsection:

(1) The set of even functions is closed under addition, subtraction, scalar multiples, pointwise multiplication, and pointwise division (where defined). All constant functions are even. [Sidenote: In mathematical jargon, we say that even functions form an algebra.]
(2) The set of odd functions is closed under addition, subtraction, and scalar multiples. It is also closed under composition.

(3) A product of two odd functions is even.
(4) A product of an even function and an odd function is odd.
(5) A composite $f \circ g$ where g is even is also even.
(6) If f is even and g is odd, then the composite $f \circ g$ is even.
(7) If f_1 and f_2 are periodic functions with periods h_1 and h_2 such that h_1/h_2 is a rational number, then $f_1 + f_2$, $f_1 - f_2$, and $f_1 \cdot f_2$ are all periodic functions.
(8) If f and g are functions such that g is periodic, so is $f \circ g$.

3.3. **Negative proofs: not even, not odd, not periodic.** To show that a function f is not even, it suffices to find just one counterexample, i.e., to find one value of x such that both x and $-x$ are in the domain of f but $f(-x) \neq f(x)$. A similar technique works for showing that f is not odd.

Let's look at an example Consider the function:

$$f(x) := x^2 - x + 1$$

Claim. f is not an even function.

Proof. If f were an even function, then we would have, for every $x \in \mathbb{R}$, that $f(x) = f(-x)$. Thus, to show that f is not an even function, it suffices to find one value of x at which $f(x) \neq f(-x)$.

In fact, the value $x = 1$ suffices:

$$f(1) = 1, \qquad f(-1) = 3$$

So clearly $f(1) \neq f(-1)$. $\square$

Claim. f is not an odd function.

Proof. If f were an odd function, then we would have, for every $x \in \mathbb{R}$, that $f(x) = -f(-x)$. Thus, to show that f is not an odd function, it suffices to find one value of x at which $f(-x) \neq -f(x)$.

In fact, the value $x = 1$ suffices:

$$f(1) = 1, \qquad f(-1) = 3$$

So clearly $f(-1) \neq -f(1)$. $\square$

Showing that a function is not periodic is trickier. Recall that f being periodic is equivalent to the following:

$$\exists h > 0 \text{ such that } \forall\, x \in \operatorname{dom}(f), f(x+h) = f(x)$$

Showing this to be false would entail showing that there is *no value* of h that works in the above. Equivalently, we need to show that every value of h fails. Thus, we want to show the following:

$$\forall h > 0, \exists x \in \operatorname{dom}(f) \text{ such that } f(x+h) \neq f(x)$$

Note that the $\exists$ quantifier gets replaced by a $\forall$ quantifier and the $\forall$ quantifier gets replaced by a $\exists$ quantifier. This is a universal feature of logical negation, and shall be crucial to a clear understanding of $\epsilon - \delta$ proofs that we will encounter soon in this course. Let's now show that the function $f(x) := x^2 - x + 1$ is not a periodic one.

Claim. f is not a periodic function.

The proof technique we use here is what is called *proof by contradiction*. What we do is start out by assuming that f is a periodic function and then do some work and show that we have come up with something that is obviously false.

Proof. Suppose f were a periodic function. By the definition of periodic function, there exists $h > 0$ such that:

$$f(x+h) = f(x) \; \forall \; x \in \mathbb{R}$$

Simplifying this, we obtain that:

$$\begin{aligned}
(x+h)^2 - (x+h) + 1 &= x^2 - x + 1 \\
\implies (x+h)^2 - x^2 + x - (x+h) &= 0 \\
\implies 2xh + h^2 - h &= 0 \\
\implies h(2x + h - 1) &= 0 \\
\implies 2x + h - 1 &= 0 \text{ (using } h > 0\text{, so } h \neq 0) \\
\implies x &= \frac{1-h}{2}
\end{aligned}$$

Thus, there is *exactly* one value of x, namely $x = (1-h)/2$, such that $f(x+h) = f(x)$. Thus, it is certainly not true that $f(x+h) = f(x)$ for *all* $x \in \mathbb{R}$, and we have the desired contradiction. So, f is not a periodic function. ∎

3.4. **Extending the domain with even/odd/periodic constraint.** Given a function f defined on the nonnegative reals, there is a unique way of extending the domain of f to all reals to obtain an even function. Similarly, if in addition $f(0) = 0$, there is a unique way of extending the domain of f to all reals to obtain an odd function.

For $x < 0$, the even way of extending *defines* $f(x)$ as equal to $f(-x)$, and the odd way of extending defines $f(x)$ as equal to $-f(-x)$. Graphically, for even functions, the part of the graph of the function for $x < 0$ is obtained by reflecting about the y-axis the part of the graph of the function for $x > 0$. For odd functions, the $x < 0$ part of the graph is obtained from the $x > 0$ part of the graph by performing a half turn about the origin.

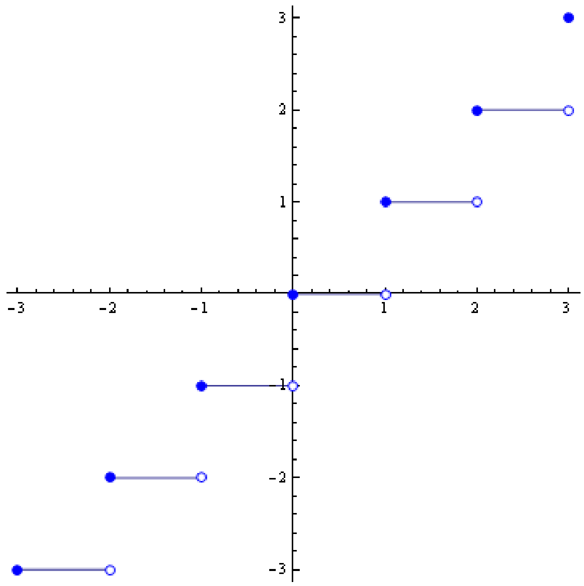
Similarly, given a function defined on a closed interval $[a, b]$ such that $f(a) = f(b)$, we can extend f uniquely to a periodic function for which $h = b - a$ works.

4. More offbeat functions

4.1. **Greatest integer function and fractional part function.** The *greatest integer function*, denoted by $[]$, is defined as follows. For $x \in \mathbb{R}$, the greatest integer function of x, denoted $[x]$, is defined as the greatest integer less than or equal to x. Thus, $[3] = 3$, $[\pi] = 3$, $[0.6] = 0$, $[\sqrt{47}] = 6$, $[-\sqrt{2}] = 2$, $[-7/3] = 3$, and so on.

The greatest integer function is a *piecewise constant function* or *step function* and it has a discontinuity at every integer, with an upward step size of 1. The greatest integer function is also termed the *floor function*.

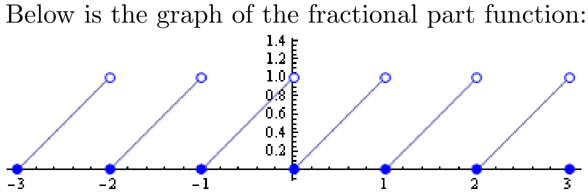
Here is the graph of the greatest integer function:



Closely related to the greatest integer function is the *fractional part function.* The fractional part of x, denoted $\{x\}$, is defined as $x - [x]$. Thus, the fractional part is 3.42 is 0.42, while the fractional part of -0.42 is 0.58.

The fractional part function is piecewise linear, with discontinuities at every integer. Between consecutive integers n and $n + 1$, the function rises linearly from 0 to 1, but just when it is about to reach 1, it slips back down to 0 to start all over again.

Below is the graph of the fractional part function:



4.2. **Functions defined differently for rationals and irrationals.** For the piecewise definitions of functions that we have seen so far, the *pieces* are intervals or unions of intervals, and thus there are points at the boundaries between the pieces where the function can be thought of as *changing* definition. There is a much more messy kind of piecewise definition, where the pieces do not look like intervals or unions of intervals, but are instead scattered across the domain.

One example is where the pieces are taken to be the rational numbers and irrational numbers respectively. both the rational numbers and irrational numbers are dense in the real numbers – in other words, every nonempty open interval in the reals contains both rational and irrational numbers.

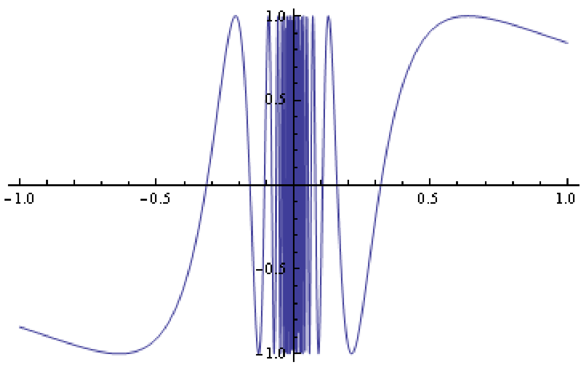
For instance, the *Dirichlet function* is defined as:

$$f(x) := \{ \begin{array}{ll} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{array}$$

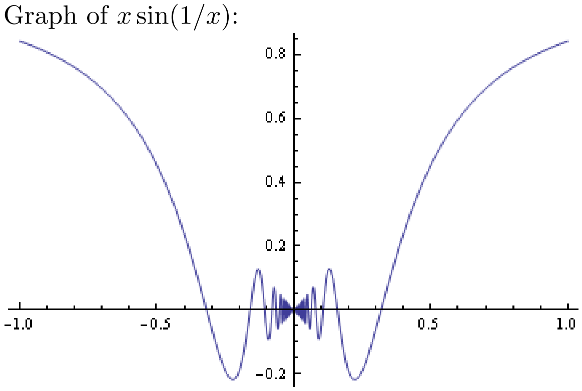
There are some variants of this where there is one constant value (not necessarily 1) for the rational numbers and another constant value (not necessarily 0) for the irrational numbers. There are also other variants that you will see as we explore continuity and differentiability further.

4.3. **The topologist's sine curve.** We will also be looking at the functions $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$, $x^3\sin(1/x)$ throughout the course. Graphs of these functions are given below.

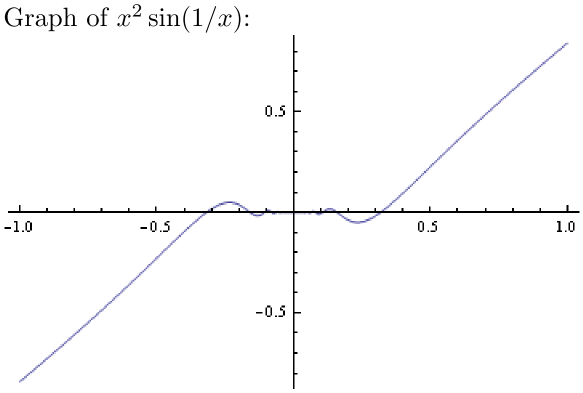
Graph of $\sin(1/x)$:



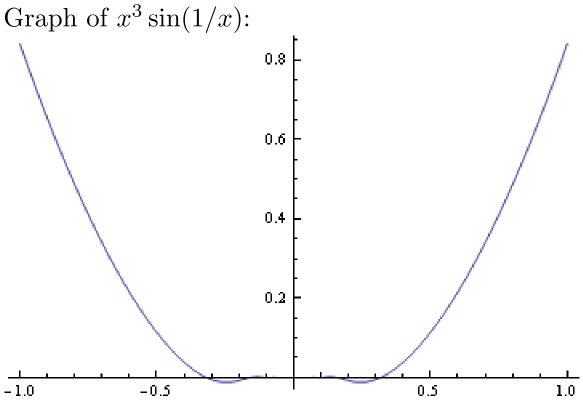
Graph of $x \sin(1/x)$:



Graph of $x^2 \sin(1/x)$:



Graph of $x^3 \sin(1/x)$:



4.4. **Why do we care about weird functions?** The greatest integer function and fractional part function have applications to real world situations, particularly when those real world situations have integer constraints. For instance, you can only buy and sell integer quantities of some commodity.

However, the rational-irrational dichotomy functions and the topologist's sine curve have very few practical applications as functions. Their main utility is to provide *examples that allow us to test the soundness of definitions and notions of continuity and differentiability*. Most of the natural examples of functions are too nice for us to test whether our definitions of continuity and differentiability can rough it out. You can think of them as the equivalent of high school bullies who make you a strong person, as long as you don't cave in to them.