

## FORMAL DEFINITION OF LIMIT

MATH 152, SECTION 55 (VIPUL NAIK)

**Difficulty level:** Hard. Full attention needed.

**Covered in class?:** Yes. But it is strongly recommended that you read this, as well as the book, preferably *prior* to the lecture.

**Corresponding material in the book:** Section 2.2.

**Corresponding material in homework problems:** Homework 2 advanced problems 1–6 and 10.

**Things that students should get immediately:** The definition of limit is tricky but there is a “method behind the madness”. The subtleties in the definition are largely to avoid problems with functions that fluctuate too much.

**Things that students should get with effort:** The full definition of limit in terms of  $\epsilon$ s and  $\delta$ s. Ideally, write the definition for both generic functions and specific functions, and be able to clearly identify the bounding task that is needed. Also, be able to execute  $\epsilon - \delta$  proofs for constant, linear, quadratic functions and for functions that are piecewise of these types.

**Things that students should hopefully get:** The approach to showing that certain limits do not exist.

### EXECUTIVE SUMMARY

Words ...

- (1)  $\lim_{x \rightarrow c} f(x) = L$  if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for every  $x \in \mathbb{R}$  satisfying  $0 < |x - c| < \delta$  (in other words,  $x \in (c - \delta, c) \cup (c, c + \delta)$ ), we have  $|f(x) - L| < \epsilon$  (in other words,  $f(x) \in (L - \epsilon, L + \epsilon)$ ).
- (2) What that means is that however small a trap (namely  $\epsilon$ ) the skeptic demands, the person who wants to claim that the limit does exist can find a  $\delta$  such that when the  $x$ -value is  $\delta$ -close to  $c$ , the  $f(x)$ -value is  $\epsilon$ -close to  $L$ .
- (3) The negation of the statement  $\lim_{x \rightarrow c} f(x) = L$  is: there exists  $\epsilon > 0$  such that for every  $\delta > 0$  there exists  $x \in \mathbb{R}$  such that  $0 < |x - c| < \delta$  but  $|f(x) - L| \geq \epsilon$ .
- (4) The statement  $\lim_{x \rightarrow c} f(x)$  doesn't exist: for every  $L \in \mathbb{R}$ , there exists  $\epsilon > 0$  such that for every  $\delta > 0$  there exists  $x \in \mathbb{R}$  such that  $0 < |x - c| < \delta$  but  $|f(x) - L| \geq \epsilon$ .
- (5) We can think of  $\epsilon - \delta$  limits as a game. The skeptic, who is unconvinced that the limit is  $L$ , throws to the prover a value  $\epsilon > 0$ . The prover must now throw back a  $\delta > 0$ . Then, the skeptic provides a value of  $x$  within a  $\delta$ -distance of  $c$ . If the  $f(x)$ -value is within an  $\epsilon$ -distance of  $L$ , the prover wins. Otherwise, the skeptic wins.  $L$  being the limit means that the prover has a winning strategy, i.e., the prover has a way of picking, for any  $\epsilon > 0$ , a value of  $\delta > 0$  suitable to that  $\epsilon$ .
- (6) The function  $f(x) = \sin(1/x)$  is a classy example of a limit not existing. The problem is that, however small we choose a  $\delta$  around 0, the function takes all values between  $-1$  and  $1$ , and hence refuses to be confined within small  $\epsilon$ -traps.
- (7) We say that  $f$  is continuous at  $c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ .

Actions...

- (1) If a  $\delta$  works for a given  $\epsilon$ , then every smaller  $\delta$  works too. Also, if a  $\delta$  works for a given  $\epsilon$ , the same  $\delta$  works for any larger  $\epsilon$ .
- (2) Constant functions are continuous, we can choose  $\delta$  to be anything. In this  $\epsilon - \delta$  game, the person trying to prove that the limit does exist wins no matter what  $\epsilon$  the skeptic throws and no matter what  $\delta$  is thrown back.
- (3) For the function  $f(x) = x$ , it's continuous, and  $\delta = \epsilon$  works.

- (4) For a linear function  $f(x) = ax + b$  with  $a \neq 0$ , it's continuous, and  $\delta = \epsilon/|a|$  works. That's the largest  $\delta$  that works.
- (5) For a function  $f(x) = x^2$  taking the limit at a point  $p$ , the limit is  $p^2$  (the function is continuous) and  $\delta = \min\{1, \epsilon/(1 + |2p|)\}$  works. It isn't the best, but it works.
- (6) For a function  $f(x) = ax^2 + bx + c$ , taking the limit at a point  $p$ , the limit is  $f(p)$  (the function is continuous) and  $\delta = \min\{1, \epsilon/(|a| + |2ap + b|)\}$  works. It isn't the best, but it works.
- (7) If there are two functions  $f$  and  $g$  and  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = L$ , and  $h$  is a function such that  $h(x) = f(x)$  or  $h(x) = g(x)$  for every  $x$ , then  $\lim_{x \rightarrow c} h(x) = L$ . The  $\delta$  that works for  $h$  is the minimum of the  $\delta$ s that work for  $f$  and  $g$ . This applies to many situations: functions defined differently on the left and right of the point, functions defined differently for the rationals and the irrationals, functions defined as the max or min of two functions.

## PEP TALK

In this really important lecture, we're going to try and understand the formal definition of limit. This formal definition is really tricky to understand and it is sort of like a "rite of passage", like in some places the boys have to kill a tiger to become men. So it's the same way – this is when all the wishy-washy precalculus stuff ends and proper college calculus begins. And this definition has a lot of subtleties, and I hope you took the time to read the book and try to understand the definition.

### 1. RUGGED TERRAIN

**1.1. The topologist's sine curve.** First, let's recall the graph of the sine curve. This is a nice function – it is a periodic function with period  $2\pi$ , and it is not just continuous, it is very smooth, waving smoothly. As good a function as you can get.

We now consider a slightly different function, which is the function:

$$f(x) := \sin(1/x)$$

Can you think of  $f$  as a composite of two functions? Indeed,  $f = \sin \circ g$ , where  $g(x) = 1/x$ . So what are the points where  $f$  is defined? Well, what could be the problem? The first problem could be that the function  $g(x) = 1/x$  isn't defined. And that happens at  $x = 0$ . So 0 is a problem point. Once we've done the  $1/x$  part, what next? Well, we need to take  $\sin$  of that, which isn't a problem, because  $\sin$  is defined for all real numbers. Thus, the domain of the function is all nonzero real numbers, or  $\mathbb{R} \setminus \{0\}$ , also written as  $(-\infty, 0) \cup (0, \infty)$ .

So when I first saw this function, I was like: *ouch!* What does it even mean to be such a function? What would the graph of such a function look like? Well, there are ways to compose graphs pictorially, but we don't want to go into those right now. So we'll just do some *ad hoc* stuff.

First, let's do the positive side. Suppose  $x \rightarrow +\infty$ . What happens to  $1/x$ ? Well, it approaches 0, from the right side. And as you can see from the graph of  $x$ , that means it goes to zero. Now in the graph of  $\sin$ , you see a very quick, almost straight line descent to zero. But when you are seeing this for  $1/x$ , this is almost painfully slow, so this is how it is going to look – it is going to sort off go more and more horizontal. By the way, in this picture, the  $x$ -axis is called a *horizontal asymptote*. We'll talk about asymptotes a little later in the course.

So the graph of  $\sin(1/x)$  reaches 1, here, at  $x = 2/\pi$ . And as you know,  $2/\pi$  is around  $7/11$ , so it is less than 1. So that part from 0 to  $\pi/2$  in the  $\sin$  graph gives rise to this part from  $2/\pi$  to  $\infty$ . That really small part in the  $\sin$  graph becomes this really huge part out here.

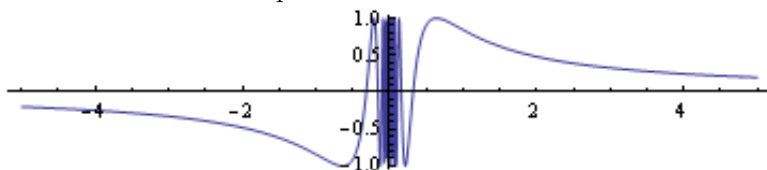
And all the other oscillations in the  $\sin$  graph get compressed into the little region between 0 and  $2/\pi$ . So the graph falls to 0 here at  $1/\pi$ , then, it falls to  $-1$  at  $2/3\pi$ , then comes back to zero at  $1/2\pi$ , and then the oscillations are faster than ever before. And as you get closer and closer to zero from the right, it is almost like it's madly just going up and down between  $-1$  and  $1$ .

What about the negative side? Well, if you wanted to build your skills, you would do the whole thing again, but we can save some time by making an observation. The sine function is an odd function, and the function sending  $x$  to  $1/x$  is an odd function, so the composite is also an odd function. Or another way of

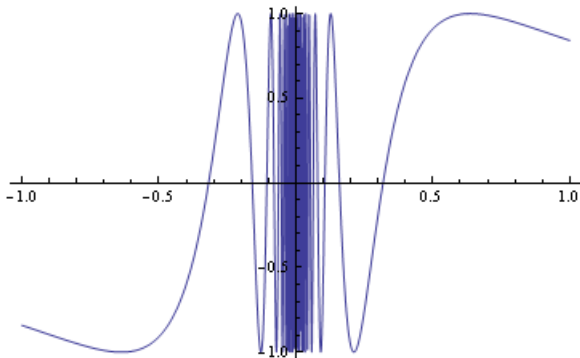
thinking about this is that  $\sin(1/(-x)) = \sin(-1/x) = -\sin(1/x)$ . So, what's going to happen is that the graph on the left is just obtained from the graph on the right using two flips, or a half turn about the origin.

Okay, now this is the graph that you need to look at and remember. By the way, some mathematicians have a name for the graph, or more specifically, the  $x > 0$  part; they call it the *topologist's sine curve*.

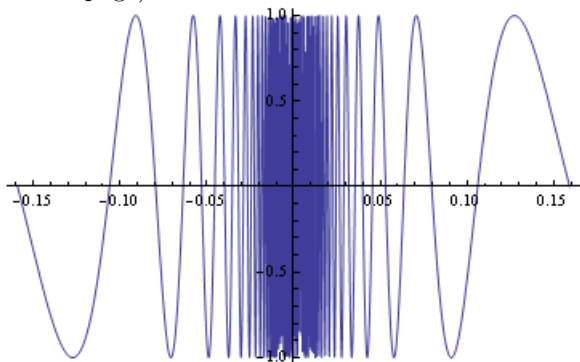
Here's the zoomed out picture:



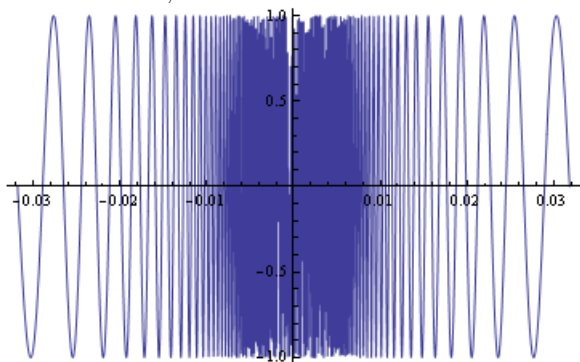
Here's a somewhat intermediate picture where we restrict the domain to  $[-1, 1]$ .



Here is the picture zoomed in further (note: the  $x$  and  $y$  axes are scaled differently to make the picture fit in the page):



And here it is, zoomed in even further:



**1.2. One definition of limit.** We first discuss a definition of limit that is wrong but it is wrong in an interesting way. And although it is wrong, it could still be a meaningful definition and is a useful concept in some cases.

Well, let's try to build this wrong definition. We want to interpret the sentence:

$$\lim_{x \rightarrow c} f(x) = L$$

So one way of thinking of this is: as  $x$  gets arbitrarily close to  $c$ ,  $f(x)$  gets arbitrarily close to  $L$ . And so here's one guess:

Wrong definition of limit:

For every  $\epsilon > 0$  and every  $\delta > 0$ , there exists  $x$  such that  $0 < |x - c| < \delta$  and  $|f(x) - L| < \epsilon$ .

Okay, let's try to interpret this. This is saying that if you pick a really small interval  $(c - \delta, c + \delta)$  around  $c$  on the  $x$ -axis, and a really small interval  $(L - \epsilon, L + \epsilon)$  around  $L$  on the  $f(x)$ -axis, then you get this rectangle. And what my definition is saying is that you'll have some point of the curve in this rectangle (but is not the point  $(c, L)$  itself). So what this definition is saying is that however small a rectangle you make around the point  $(c, L)$ , you will have some points of the form  $(x, f(x))$ .

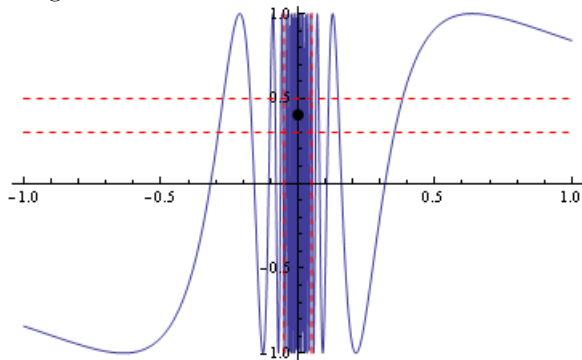
This seems like a reasonable description, because it says that you have these points that are really close. And this is certainly satisfied for most of the functions you have seen. But this is not the correct definition.

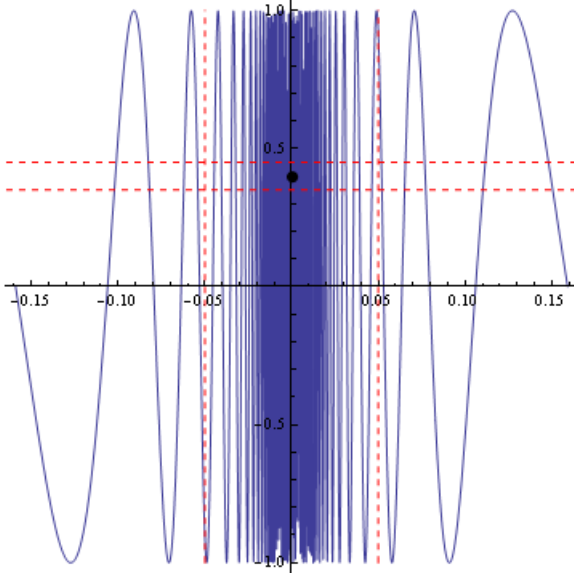
**1.3. Back to the topologist's sine curve.** So to understand what is wrong with this definition, we need to look again at the topologist's sine curve. And we need to look at this curve at the point 0, and ask: what is the limit at 0?

Well, I claim that by the definition I gave, the limit at 0 can be any number between  $-1$  and  $1$ . Why? Well, think about what is happening to the function close to zero on the positive side. It is madly oscillating. This means that however small an interval around 0 you take for the  $x$ -value, the  $\sin(1/x)$  function takes all possible values between  $-1$  and  $1$ . So, however small a rectangle you draw at any point, you're going to get points that are in there.

So the thing with the function is that its rapid oscillation is creating a problem: by our definition, we get all points in the interval  $[-1, 1]$  as the limit. But that's not the way we would like limits to behave – we want the limit to be unique, and it should be a reasonable description of where the function *tends to*. If this definition says that the limit at 0 could be both 0 and  $1/2$  and  $-1/2$  and  $1$ , and this jars your intuition, it is this definition that you need to throw out.

Here's a graphical illustration, where we are trying to study the approach of  $\sin(1/x)$  to  $0.4$  as  $x$  approaches 0. Note that however small a rectangle we make around the point, it contains lots of points of the  $\sin(1/x)$  graph. However, the  $\sin(1/x)$  graph is all over the place, so the function is never *trapped* within a small rectangle.





**1.4. Challenging!** So I know that you're not happy with me – I just tried to shove down your throat a wrong definition and you spent the time trying to understand it and then I told you that this is the wrong definition. So, this was just to give you a flavor that getting the definition is challenging, and a definition that might seem right could be riddled with holes. I know this is exhausting, so I'll skip right to the correct definition and explain why it works. And then after some time, I suggest you come back and compare the correct definition and the wrong definition and try to understand exactly what the difference is.

The way I think about the definition is in terms of a *cage* (or *trap*). And the reason why we need this notion of a cage or trap is precisely to avoid these kinds of oscillations that give rise to multiple limits. So, here is the formal definition:

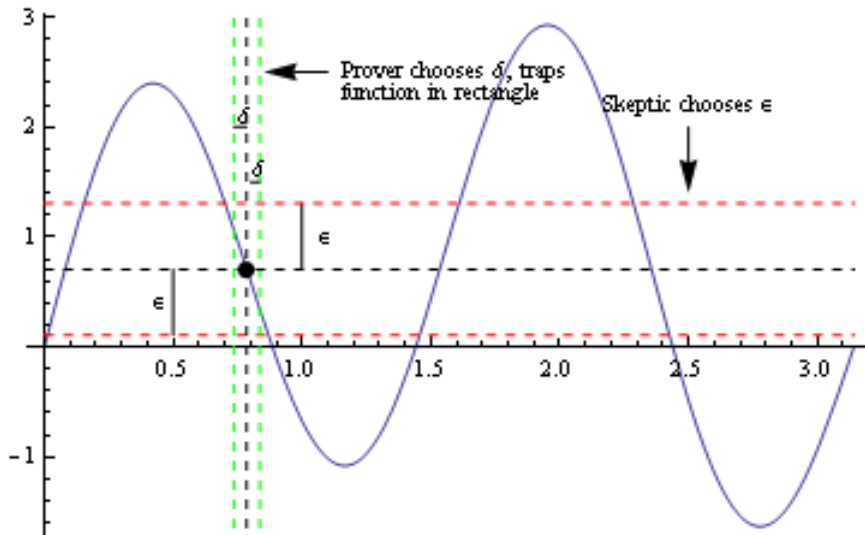
We say that  $\lim_{x \rightarrow c} f(x) = L$  (as a two-sided limit) if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for every  $x$  such that  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ .

That's quite a mouthful. Let's interpret it graphically. What it is saying is that: "for every  $\epsilon$ " so we consider this region  $(L - \epsilon, L + \epsilon)$ , so there are these two horizontal bars at heights  $L - \epsilon$  and  $L + \epsilon$ . Next it says, there exists a  $\delta$ , so there exist these vertical bars at  $c + \delta$  and  $c - \delta$ . So we have the same rectangle that we had in the earlier definition.

What is different this time is that we not only demand that the graph have a few points in that rectangle, but rather, that it lies completely inside the rectangle. And this is the crucial difference between this definition and the previous definition, that allowed sometimes-in-sometimes-out graphs. Because, here we are saying that the condition holds for *all*  $x$  such that  $0 < |x - c| < \delta$ . Which is the same as saying that the condition holds on the interval  $(c - \delta, c + \delta)$  minus the point  $\{c\}$  itself. Or, you can think of that set as  $(c - \delta, c) \cup (c, c + \delta)$ . And we're insisting that the condition hold for all things, not just that there exists a point here or a point there.

The other main difference from the earlier definition is that in this definition, the value of  $\delta$  depends on  $\epsilon$ . So here is another way of thinking of this definition that I find useful. Suppose I claim that as  $x$  tends to  $c$ ,  $f(x)$  tends to  $L$ , and you are skeptical. So you throw me a value  $\epsilon > 0$  as a challenge and say – can I trap the function within  $\epsilon$ ? And I say, yeah, sure, because I can find a  $\delta > 0$  such that, within the ball of radius  $\delta$  about  $c$ , the value  $f(x)$  is trapped in an interval of size  $\epsilon$  about  $L$ . So basically you are challenging me: can I create an  $\epsilon$ -cage? And for every  $\epsilon$  that you hand me, I can find a  $\delta$  that does the job of this cage.

Here's a pictorial illustration:



**1.5. Formal description as a game.** We make the discussion above somewhat more formal by encoding it as a game. Consider the assertion:

$$\lim_{x \rightarrow c} f(x) = L$$

where specific values are provided for  $c$  and  $L$  and for the function  $f$ . Suppose, further, that  $f$  is defined at all points on the immediate left and the immediate right of  $c$  (otherwise, the game would be meaningless).

The game has two players, a *prover*, whose goal is to show that the limit statement above is correct, and a *skeptic* (in higher mathematics jargon, this person is called a *verifier*, but I think you'll find *skeptic* a more intuitive term) who is far from convinced and wants to raise the best counter-arguments. The game has three moves:

- The skeptic chooses an  $\epsilon > 0$  (the subtext being the interval  $(L - \epsilon, L + \epsilon)$ ), effectively telling the prover: “try to trap the function with  $\epsilon$  of  $L$  if you can.”
- The prover chooses a  $\delta > 0$  (the subtext being the interval  $(c - \delta, c + \delta)$ , excluding the point  $c$ ), effectively telling the skeptic: “here’s a trap that works.”
- The skeptic chooses a value of  $x$  such that  $0 < |x - c| < \delta$ , (i.e., within the set  $(c - \delta, c + \delta) \setminus \{c\}$ ), challenging the prover at that specific  $x$ .

Once these moves are complete: we compute  $|f(x) - L|$ . If it is less than  $\epsilon$ , the prover wins. Otherwise, the skeptic wins.

We say that  $\lim_{x \rightarrow c} f(x) = L$  is *true* if the prover has a *winning strategy*. In other words, *no matter what* choice the skeptic makes for  $\epsilon$ , the prover has a (smart) choice of  $\delta$  such that *no matter what* value of  $x$  the skeptic chooses in  $(c - \delta, c) \cup (c, c + \delta)$ ,  $f(x)$  lies in the interval  $(L - \epsilon, L + \epsilon)$ .

If, in contrast, the *skeptic* has a winning strategy, then we declare the statement to be false.

The key take-away is that in order to show a limit statement to be true, we need to devise a winning strategy for the prover in the above game. Note that the winning strategy must work against an extremely smart skeptic, not merely against a skeptic who makes a silly choice of  $\epsilon$ .

**1.6. One-sided limits.** The definition of limit we have given is:

$\lim_{x \rightarrow c} f(x) = L$  if, for every  $\delta > 0$ , there exists  $\delta > 0$  such that, for every  $x$  such that  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \delta$ .

This definition is fine when the function is defined on both sides. Note the way we are using *both sides* of  $c$ , because when we say  $0 < |x - c| < \delta$ , we are including the  $\delta$ -interval on the left side  $(c - \delta, c)$  and the right side  $(c, c + \delta)$ .

For the right hand limit, we want to restrict  $x$  to the interval  $(c, c + \delta)$  on the right side, and for the left hand limit, we want to restrict  $x$  to the interval  $(c - \delta, c)$ .

Here's how we define the *left hand limit*:  $\lim_{x \rightarrow c^-} f(x) = L$  if, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, for all  $x$  satisfying  $0 < c - x < \delta$ , we have  $|f(x) - L| < \epsilon$ .

Note that  $0 < c - x < \delta$  is the same as saying that  $x \in (c - \delta, c)$ .

Here's how we define the *right hand limit*:  $\lim_{x \rightarrow c^+} f(x) = L$  if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x$  satisfying  $0 < x - c < \delta$ , we have  $|f(x) - L| < \epsilon$ .

Note that  $0 < x - c < \delta$  is the same as saying that  $x \in (c, c + \delta)$ .

## 2. STRATEGY STOCKPILE

We now discuss strategies for showing that a particular limit exists and has a particular value, and hence, for showing that a function is continuous.

**2.1. What we need to do.** Recall the way of thinking of a limit in terms of a *cage* or a *trap*:

We say that  $\lim_{x \rightarrow c} f(x) = L$  (as a two-sided limit) if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for every  $x$  such that  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ .

This is quite a mouthful, so let's slow down and try to understand what it means. Let's interpret it graphically. What it is saying is that: "for every  $\epsilon$ " so we consider this region  $(L - \epsilon, L + \epsilon)$ , so there are these two horizontal bars at heights  $L - \epsilon$  and  $L + \epsilon$ . Next it says, there exists a  $\delta$ , so there exist these vertical bars at  $c + \delta$  and  $c - \delta$ . And what we're saying is that if the  $x$ -value is trapped between the vertical bars  $c - \delta$  and  $c + \delta$  (but is not equal to  $c$ ), the  $f(x)$ -value is trapped between  $L - \epsilon$  and  $L + \epsilon$ .

The important thing to note here is that the value of  $\delta$  depends on the value of  $\epsilon$ . As I said earlier, we can think of this as a game, where I (as the prover) am trying to prove to you that the limit  $\lim_{x \rightarrow c} f(x) = L$  and you are a skeptic who is trying to catch me out. So you throw  $\epsilon$ s at me, and challenge me to show that I have a  $\delta$  to trap that  $\epsilon$ . And if I have a winning strategy, that enables me to find a  $\delta$  for every  $\epsilon$  that you throw at me, then yes, the limit is equal to  $L$ .

**2.2. The winning strategy for constant functions.** So the real question is: can I obtain a winning strategy? And what would such a strategy be? It would be some procedure, some function, that takes as input a value of  $\epsilon$  and outputs a value of  $\delta$  in terms of that  $\epsilon$ . Now I know that's a mouthful, so let's look at some simple examples.

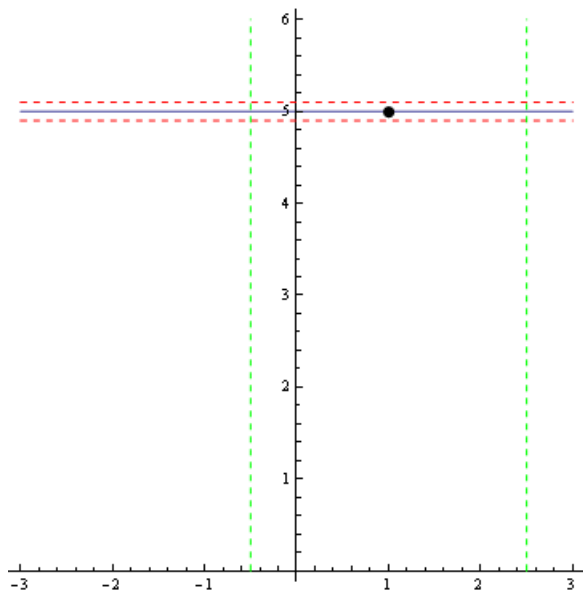
We'll take two kinds of functions for which the limit is particularly easy to compute: *constant functions* and the identity function. Let's first look at a constant function that sends every real number to  $k$ . So let me call this  $f$ . So  $f$  is a function with the property that  $f(x) = k$  for all  $x \in \mathbb{R}$ . So, what can we say about the graph of  $f$ ? Well, it is this horizontal line at height  $k$ . So far, so good.

Let's look at a point  $c$ , and try to calculate  $\lim_{x \rightarrow c} f(x)$ . Which is the same as trying to compute  $\lim_{x \rightarrow c} k$ . The guess, from looking at the graph, is that the limit equals  $k$ . So how do we show this in terms of the  $\epsilon - \delta$  definition?

Okay, lost? No problem. Sometimes, when unwinding mathematical expressions, you (and even I) get lost. That's not the time to give up. Rather, it is the time to refocus and go back to the original definition and work things out again.

So we want to show that  $\lim_{x \rightarrow c} f(x) = k$ . In other words, we want to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|f(x) - k| < \epsilon$ . By the way, that  $k$  that appeared at the end there is because we're claiming the limit is  $k$ .

Before we unravel that (and it's a bit of an anti-climax once we do), let's just think of this graphically. We are saying that for every  $\epsilon > 0$ , so we are thinking of the region between the horizontal bars at heights  $k - \epsilon$  and  $k + \epsilon$ . And then we want to say that there exists a  $\delta > 0$  (that we haven't determined) so we are thinking of the vertical bars at  $c - \delta$  and  $c + \delta$ , so we have this rectangle. And we have to choose  $\delta$  such that that part of the graph lies inside that rectangle.



But the picture makes it clear that we can choose just about any  $\delta > 0$ ! So in the case of the constant function, we can choose just about anything and not run into trouble. So, let's see that from the algebra.

We want to show that there exists a  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|f(x) - k| < \epsilon$ . but what's  $|f(x) - k|$ ? it is  $|k - k| = 0$ , and so is less than  $\epsilon$ , so the condition is tautologically satisfied for all  $x$ . So any  $\delta$  will do.

Another way of thinking of this is that in this case, I don't need a trap at all – any trap will do because the function is already at the right place all along! Also, remember that the kind of examples that gave us headaches were things where the function changed a lot from point to point, and constant functions change as little as possible – they don't change at all.

Or, thinking of it in terms of the two-person game where you (the skeptic) throw me  $\epsilon$ s and I throw back  $\delta$ s that work, this is a really easy game. Whatever you throw at me, I can throw back anything at you. This is a great game for me – no matter how smart you are or how stupid I am, I always win.

I urge you to go through this example carefully and understand it thoroughly. It's also Example 5 on page 67 of the book.

**2.3. The winning strategy for the identity function.** Okay, now when I asked you to calculate the limit of the identity function, you might remember I said it's like a word puzzle: "as  $x$  approaches  $c$ , what does  $x$  approach?" Well, obviously  $c$ . So now we want to do the fancy  $\epsilon - \delta$  version of that same argument.

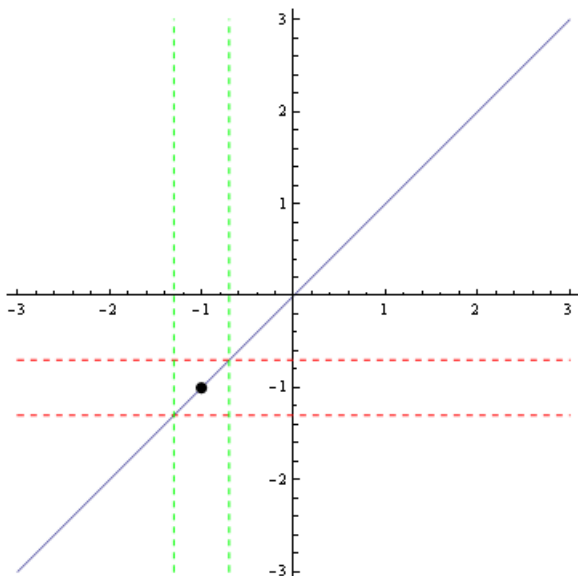
Let's unwind the definition. And by the way, the main thing, once you have got the correct definition, is to be able to carefully apply it by interpreting it correctly. That isn't easy but it isn't impossible. In fact, at some stage you should be able to take a definition that you're seeing for the first time and apply it to a given problem well.

So define  $g(x) = x$ . So we want to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $0 < |x - c| < \delta$ , then  $|g(x) - c| < \epsilon$ . And by the way, the second  $c$  came because we're claiming that the limit is  $c$ . And by the way, since  $g(x) = x$ , I can rewrite this as follows:

I want to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $0 < |x - c| < \delta$ , then  $|x - c| < \epsilon$ . Hmm. So what  $\delta$  has the property that whenever  $0 < |x - c| < \delta$ , then  $|x - c| < \epsilon$ ? Again, the answer is almost tautological: set  $\delta = \epsilon$ .

For the game where the function is the identity function, I can choose the identity function as my winning strategy, in the sense that whatever  $\epsilon$  you throw at me, I throw back the same value to you for  $\delta$ . We can also see this graphically:





I urge you to go through this example carefully and understand it thoroughly. It's also Example 5 on page 67 of the book.

**2.4. The winning strategy for linear functions: motivation.** Okay, let's up the ante a little bit, getting more abstract without actually making our function a lot harder. Suppose we have the function:

$$f(x) := ax + b$$

So this  $f$  is a *linear function*. And we want to determine the limit  $\lim_{x \rightarrow c} f(x)$ . Here, the numbers  $a$  and  $b$  are unknown constants, i.e., in a specific problem situation, their values will be known.

The first thing you can do is draw the graph, and this graph is a straight line. The slope of that straight line depends on the value of  $a$ . So if  $a$  is positive, this is a south-west to north-east line, and if  $a$  is negative, this is a north-west to south-east line. And, if  $a = 0$ , the line is flat. And the parameter  $b$  tells you further where the line is placed – different  $b$ s give different lines that are all parallel to each other.

Your guess would be that the limit at  $c$  is  $f(c)$ , which is sort of pictorially clear, because a straight line looks very continuous. And you'll notice that this generalizes both the constant function and the identity function: the constant function would be the case  $a = 0$ , and the identity function would be the case  $a = 1$  and  $b = 0$ . Since we already settled the constant function case, we'll assume  $a \neq 0$ .

So, what again do I want to show?  $\lim_{x \rightarrow c} f(x) = f(c)$ . And so, I need to find, for every  $\epsilon$ , a suitable  $\delta$  such that something holds. What thing?

So, what I want is to find, for every  $\epsilon > 0$ , a value  $\delta > 0$  such that for  $0 < |x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ . We now try to substitute the actual expression for  $f$  in that last inequality. So let's do this simplification on the side.

$$f(x) - f(c) = (ax + b) - (ac + b) = a(x - c)$$

So, what we want is the following: for every  $\epsilon > 0$ , find a value  $\delta > 0$  such that for  $0 < |x - c| < \delta$ , we have  $|a(x - c)| < \epsilon$ .

Okay, now this looks reasonable, but it isn't tautological as the previous examples were. We need to do some thinking, and there is a real logical impasse here. So let's look at the part:

$$|x - c| < \delta$$

What does this allow us to say about  $|a(x - c)|$ ? Well, we can try multiplying the above inequality by  $|a|$ , and we get:

$$|a||x - c| < |a|\delta$$

And, note that there isn't any sign change because  $|a| > 0$  (as we assumed  $a \neq 0$ ). And, using the absolute value of a product is the product of the absolute values, we get:

$$|a(x - c)| < |a|\delta$$

Okay, so what we *have* is the above, and what we *want* is  $|a(x - c)| < \epsilon$ . And remember, we have the freedom to choose any  $\delta$  that we want. So what value of  $\delta$  do we choose? Well, a little thought should reveal that a simple way of choosing a  $\delta$  that works is to set  $|a|\delta = \epsilon$ . That gives  $\delta = \epsilon/|a|$ . So that's a value of  $\delta$  that works.

**2.5. The winning strategy: proved succinctly. Problem:** Prove that  $\lim_{x \rightarrow c} f(x) = f(c)$  where  $f(x) := ax + b, a \neq 0$ .

**Winning strategy for prover-skeptic game:** Choose  $\delta = \epsilon/|a|$ .

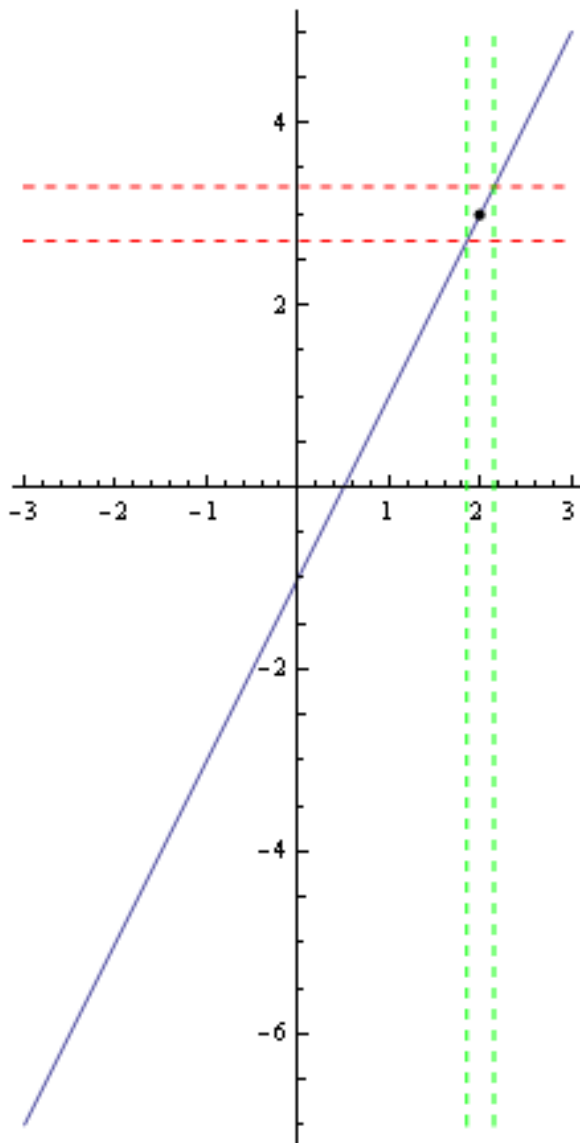
We want to show that, for any  $\epsilon > 0$ , if  $0 < |x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$ , where  $\delta = \epsilon/|a|$ . We do this as follows:

$$\begin{aligned} & |f(x) - f(c)| \\ &= |(ax + b) - (ac + b)| \\ &= |ax - ac| \\ &= |a(x - c)| \\ &= |a||x - c| \\ &< |a|\delta \quad \text{using } |x - c| < \delta \\ &= |a|\frac{\epsilon}{|a|} \quad \text{using } \delta = \epsilon/|a| \\ &= \epsilon \end{aligned}$$

The chain has one strict inequality and the rest all equalities, so we get, overall, that  $|f(x) - f(c)| < \epsilon$ .

So that's it. We've proved that for a linear function, the limit always exists at any point, and moreover, the limit equals the value of the function at the point. As you saw in the informal introduction earlier, that's basically saying that linear functions are continuous.

Here's the  $\epsilon - \delta$  picture for a linear function:



Note that the larger the value of  $|a|$ , the smaller the  $\delta$  needs to be for a given  $\epsilon$ . This makes sense, because the steeper the slope, the more rapidly the function is changing, and hence, the smaller the trap has to be to catch the function.

**2.6. Smaller  $\delta$ s work too.** One other important thing to note is that we have some latitude in choosing  $\delta$ . In other words, the winning strategy isn't always unique. In fact, if, for a given  $\epsilon$ , one value of  $\delta$  works, then any smaller positive value of  $\delta$  also works. I urge you to think about this graphically and also in strategic terms.

**2.7. Principles and practice.** So there are two aspects to everything: there's the *theory* and there's the *practice*. And one of the popular quotes on U of C T-shirts and merchandise is "that's all well and good in practice... but how does it work in theory?" So what we did in this and the previous lecture was take a little glimpse at how limits work in theory. And by these attempts to calculate the  $\delta$  in terms of the  $\epsilon$ , we are basically trying to bridge the gap between theory and practice.

Going from the theory to the practice requires an understanding of inequalities, which you already have. But it doesn't just require that – it requires a better conceptual grasp of *what we have to determine* and *what is given*. The thing here is that in the proof, we have to *work backwards*, in the sense that we have to

first guess what the limit is going to be, then we have to guess how to get a  $\delta$  that works, and then we can check that it works. And this working backwards can be a little tricky because of the implications stuff.

That is why it is very important that you go home and look at the examples we did in class and the other examples in the book. As far as expectations from you are concerned, you are not expected to be able to do the  $\epsilon - \delta$  computations for things other than constant, linear, and quadratic functions, and we've already done the constant and linear cases, and we will now consider general quadratic case. But I suggest you understand the book's examples for a couple of the slightly trickier cases – basically to get an idea of what their goal is and how they're approaching it.

**2.8. General case of quadratic functions: obtaining the formula. NOTE:** On your homework or in the midterm, you cannot assume the  $\epsilon - \delta$  formulas we have derived for linear and quadratic functions. However, knowing these formulas will enable you to skip the *rough work* needed to find the  $\delta$  and move directly to the *fair work* of showing that that  $\delta$  works.

We look here at the general case of a quadratic function:

$$f(x) := ax^2 + bx + c$$

with  $a, b, c \in \mathbb{R}$  and  $a \neq 0$ . Note that if  $a = 0$ , we get a linear or constant function, and we already know how to deal with it.

The letter  $c$  is used up as a variable, so we should not use  $c$  as a limiting point. Let's call the limiting point  $p$  instead. So we claim that, for  $p \in \mathbb{R}$ , we have  $\lim_{x \rightarrow p} f(x) = f(p)$ .

In other words, we need to show that for every  $\epsilon > 0$ , there exists a value  $\delta > 0$  such that for  $0 < |x - p| < \delta$ , we have  $|f(x) - f(p)| < \epsilon$ .

So we again do the usual thing: simplify  $|f(x) - f(p)|$ . We have:

$$f(x) - f(p) = (ax^2 + bx + c) - (ap^2 + bp + c) = a(x^2 - p^2) + b(x - p)$$

Okay, now that's nice, but can we simplify it a little more? Yes, and this is the important idea you should take from this problem: the idea is to factor out the  $(x - p)$  factor from the expression. So we get:

$$f(x) - f(p) = (x - p)(a(x + p) + b)$$

Why did we factor out  $x - p$ ? Think of what we need to do. We need to say that if the absolute value of  $x - p$  is small, then the absolute value of  $f(x) - f(p)$  is small too. And if you go back to the linear example, what we did was to show that  $f(x) - f(p)$  is a scalar multiple of  $x - p$ , i.e., it is  $x - p$  times a constant. In the case of the quadratic function, we do not have a constant for the other factor, so it's going to be a little harder, but this is the right direction.

Okay, now what next? We need to find a  $\delta$  such that if  $0 < |x - p| < \delta$ , then  $|f(x) - f(p)| < \epsilon$ . Now if  $|x - p| < \delta$ , then we have:

$$|f(x) - f(p)| = |x - p||a(x + p) + b| < \delta|a(x + p) + b|$$

Okay, and now we're stuck, because unlike the linear case, the second part isn't a constant, and it involves an  $x$ . But can we bound it by a constant? Well, let's think about this intuitively. We are really interested in situations where  $x - p$  is really small, so  $x$  is really close to  $p$ . So  $x + p = (x - p) + 2p$ , with the  $x - p$  part being small. So:

$$|a(x + p) + b| = |a(x - p) + 2ap + b| \leq |a(x - p)| + |2ap + b| < |a|\delta + |2ap + b|$$

So this is progress, and what exactly is the nature of the progress? Well, what we've found is that that expression which isn't constant is still bounded from above by some constant plus  $|a|\delta$ . So, we get:

$$|f(x) - f(p)| < \delta(|a|\delta + |2ap + b|)$$

And now the original question: given a value  $\epsilon > 0$ , how do we find a value  $\delta > 0$  such that that right side in terms of  $\delta$  is not more than  $\epsilon$ ? Well, you can try solving a quadratic inequality, but that's a pain, so I'll show you a simpler approach.

First, notice that we're really interested in small values of  $\delta$ . So let's assume  $\delta \leq 1$ . Then, we get:

$$|f(x) - f(p)| < \delta(|a| + |2ap + b|)$$

And now, if we have  $\delta \leq \epsilon/(|a| + |2ap + b|)$ , we are in good shape.

So, a value of  $\delta$  that works is  $\min\{1, \epsilon/(|a| + |2ap + b|)\}$ .

What we've done above is obtained a value of  $\delta$ . You should now be able to retrace the steps to confirm that this value of  $\delta$  works.

By the way, this generalizes Example 6 (Section 2.2, Page 68) of the book. In that example,  $a = 1, b = c = 0$ , and  $p = 3$ . So if there are too many symbols in this example and you want a simpler example with fewer symbols, you should look at that example in the book and then, armed with that, come back to master this one.

*Please go through both these very thoroughly; this is very important to understand.*

**2.9. General case of quadratic function: how to present the solution.** We illustrate how the solution would be *presented* for the general case of a quadratic function.

**Problem:** Prove that  $\lim_{x \rightarrow p} f(x) = f(p)$  where  $f(x) := ax^2 + bx + c, a \neq 0$ .

**Winning strategy for prover-skeptic game:** Choose  $\delta = \min\{1, \frac{\epsilon}{|a| + |2ap + b|}\}$ .

**Proof:** We need to show that, for any  $\epsilon > 0$ , if  $0 < |x - p| < \delta$ , then  $|f(x) - f(p)| < \epsilon$ , where  $\delta = \min\{1, \frac{\epsilon}{|a| + |2ap + b|}\}$ .

We have:

$$\begin{aligned} & |f(x) - f(p)| \\ &= |(ax^2 + bx + c) - (ap^2 + bp + c)| \\ &= |a(x^2 - p^2) + b(x - p)| \\ &= |x - p||a(x + p) + b| \\ &= |x - p||a(x - p) - 2ap + b| \\ &\leq |x - p|(|a||x - p| + |2ap + b|) \quad \text{by triangle inequality} \\ &< \delta(|a|\delta + |2ap + b|) \quad \text{using } |x - p| < \delta \\ &\leq \delta(|a| + |2ap + b|) \quad \text{using } \delta \leq 1 \text{ for the inner } \delta \\ &\leq \frac{\epsilon}{|a| + |2ap + b|}(|a| + |2ap + b|) \quad \text{using } \delta \leq \epsilon/(|a| + |2ap + b|) \text{ for the outer } \delta \\ &= \epsilon \end{aligned}$$

Each step of the process involves one of the three signs  $=, <, \leq$ , with one of the steps involving strict inequality. Thus, overall, we obtain that  $|f(x) - f(p)| < \epsilon$ .

Qualitatively, the process can be described as follows:

- Factor the quadratic  $|f(x) - f(p)|$  with one of the factors being  $|x - p|$ .
- Rewrite the other factor as a constant times  $x - p$  plus another constant.
- Now split using the triangle inequality.
- Use  $|x - p| < \delta$  at both places.
- For the inner factor of  $\delta$ , use  $\delta \leq 1$ .
- For the outer factor, use  $\delta \leq \epsilon/(\dots)$ .

**2.10. Concrete case of quadratic.** We consider a concrete example of a quadratic with actual numerical values of  $a, b, c$ , and  $p$ , and walk through what the general steps just described would look like in the concrete example.

Consider the limit proof:

$$\lim_{x \rightarrow 5} (2x^2 + 3x + 17) = 82$$

**Winning strategy:** Take  $\delta = \min\{1, \epsilon/25\}$ . We obtain 25 using the formula  $|a| + |2ap + b|$ , where  $a = 2, p = 5$ , and  $b = 3$ , so we got  $|2| + |2 \cdot 2 \cdot 5 + 3| = |2| + |23| = 25$ .

**Proof:** We want to show that, for any  $\epsilon > 0$ , if  $0 < |x-5| < \delta$ , then  $|(2x^2+3x+17)-(2(5)^2+3(5)+17)| < \epsilon$ , where  $\delta = \min\{1, \epsilon/25\}$ .

We consider:

$$\begin{aligned}
 & |2x^2 + 3x + 17 - (2(5)^2 + 3(5) + 17)| \\
 &= |2x^2 + 3x - 65| \\
 &= |x - 5||2x + 13| \\
 &= |x - 5||2(x - 5) + 23| \\
 &\leq |x - 5|(2|x - 5| + 23) \quad \text{by triangle inequality} \\
 &< \delta(2\delta + 23) \quad \text{using } |x - 5| < \delta \\
 &\leq \delta(2 + 23) \quad \text{using } \delta \leq 1 \\
 &= 25\delta \\
 &\leq 25 \frac{\epsilon}{25} \quad \text{using } \delta \leq \epsilon/25 \\
 &= \epsilon
 \end{aligned}$$

More examples will be done in the relevant review sessions.

### 3. MORE $\epsilon - \delta$ LIMIT COMPUTATIONS

**3.1. Function with left and right definition.** One kind of situation, that we have already seen, is a situation where a function given to us has one definition on the left side of a point and another definition on the right side of the point. Your intuition would tell you that if the left hand limit and the right hand limit exist separately *and are equal*, then the limit exists on the whole as well, and equals both these values.

How can we make this intuition precise in terms of the  $\epsilon - \delta$  game? The idea is to think of it as two games in parallel: the left hand limit game, where I (the prover) have to throw back a  $\delta$  at you that works for  $x$  approaching  $c$  from the left side, and an analogous right hand limit game. The fact that the left hand limit and right hand limit are both equal to  $L$  tells me that I have winning strategies for both games. I now need to combine them into a winning strategy for the two-sided limit game. How do I do this?

Basically, I need to choose  $\delta$  small enough that it's good enough on both the left and the right. The idea is simple: pick  $\delta$  as the *minimum* of the  $\delta$ s that I picked on the left and on the right. This will work for *both* the left side *and* the right side.

See, for instance, the example of the absolute value function (given on Page 67 of the book). At the point 0, the absolute value function has the definition  $-x$  on the left and  $x$  on the right.

So what we first need to do is figure out the left strategy and the right strategy separately. For the function  $f(x) = -x$ , what is the  $\delta$  that works for a given  $\epsilon$ ? Well, you can do the calculations again, but since we already did the general case of a linear function, we can just plug that in and get that  $\delta = \epsilon$  works. Similarly, on the right side, we again get that  $\delta = \epsilon$  works. So in this case, the same strategy works on both sides, so we can use  $\delta = \epsilon$  as the winning strategy for the two-sided limit.

Okay, let's take a more interesting example. Suppose  $f(x) = \begin{cases} 3x, & x < 0 \\ -5x, & x \geq 0 \end{cases}$ . It is an obvious guess that the limit at 0 equals 0. Since both sides are linear functions, we know how to get the strategies for the left side and the right side separately. The strategy for the left side is  $\delta = \epsilon/3$  and the strategy for the right side is  $\delta = \epsilon/5$ . So, what is our overall strategy?

The overall strategy should pick a  $\delta$  small enough that it works for both sides. In this case, the smaller of the numbers  $\epsilon/3$  and  $\epsilon/5$  is  $\epsilon/5$  (remember,  $\epsilon > 0$ ). So, we get a winning strategy by choosing  $\delta = \epsilon/5$ .

**REMINDER:** On your homework or in the midterm, you cannot assume the  $\epsilon - \delta$  formulas we have derived for linear and quadratic functions. However, knowing these formulas will enable you to skip the *rough work* needed to find the  $\delta$  and move directly to the *fair work* of showing that that  $\delta$  works.

Let's now do the general case, with a full proof:

**Claim.** Suppose  $a_1, a_2, b_1, b_2 \in \mathbb{R}$  with  $a_1 \neq 0$  and  $a_2 \neq 0$ . Suppose  $f(x) = \begin{cases} a_1x + b_1, & x < c \\ a_2x + b_2, & x > c \end{cases}$ . Then, if  $L = a_1c + b_1 = a_2c + b_2$ , we have  $\lim_{x \rightarrow c} f(x) = L$ . For any  $\epsilon > 0$ , a  $\delta$  that works is  $\epsilon / \max\{|a_1|, |a_2|\} = \min\{\epsilon/|a_1|, \epsilon/|a_2|\}$  (note that  $\delta$  is positive).

*Proof.* To prove the claim, we need to show that for any  $x$  such that  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ .

We split this into two cases: the case where  $x < c$  and the case where  $x > c$ .

*Case that  $x < c$ :* In this case, we have  $f(x) = a_1x + b_1$ . Thus, we get:

$$|f(x) - L| = |a_1x + b_1 - (a_1c + b_1)| = |a_1(x - c)| = |a_1||x - c| < |a_1|\delta \leq |a_1|(\epsilon/|a_1|) = \epsilon$$

Thus,  $|f(x) - L| < \epsilon$ .

*Case that  $x > c$ :* In this case, we have  $f(x) = a_2x + b_2$ . Thus, we get:

$$|f(x) - L| = |a_2x + b_2 - (a_2c + b_2)| = |a_2(x - c)| = |a_2||x - c| < |a_2|\delta \leq |a_2|(\epsilon/|a_2|) = \epsilon$$

□

**3.2. Weirder functions, mundane ideas.** So far, we have considered the case where the function has one definition on the left and another definition on the right, and how we can combine the winning strategies. Basically, we combine the winning strategies by picking the smaller of the two  $\delta$ s. This allows us to use our strategies for constant, linear, and quadratic functions to tackle functions that are piecewise constant, linear, and quadratic.

The strategy extends to some more weirdly defined functions, where there are multiple definitions, but they are not clearly separated in left-right terms. For instance, consider the function:

$$f(x) = \begin{cases} x^2, & x \text{ rational} \\ x, & x \text{ irrational} \end{cases}$$

What we have done is split the domain of definition into two subsets, but this time the two subsets aren't nicely left and right of some point, they are both scattered all over the place. However, trying to prove limit problems for such functions follows *essentially the same strategy*. For instance, if, for the above function, we are trying to prove that  $\lim_{x \rightarrow 1} f(x) = 1$ , then what we do is to find a winning strategy for the  $x^2$  function, a winning strategy for the  $x$  function, and take the smaller of the  $\delta$ s. And, instead of splitting into cases based on left and right, we split into cases based on – you guessed it – *rational* and *irrational*.

Let's try to formally prove that  $\lim_{x \rightarrow 1} f(x) = 1$ . The first thing we need to do is rough work to figure out the  $\delta$ s that work for the individual functions  $x^2$  and  $x$ . But we've already done this. Recall that for  $x^2$ , we can choose  $\delta = \min\{1, \epsilon/3\}$ , and for  $x$ , we can choose  $\delta = \epsilon$ . So the overall  $\delta$  that we need to choose is  $\min\{1, \epsilon/3, \epsilon\}$ . We can simplify that to  $\min\{1, \epsilon/3\}$ .

To complete the proof, we first consider the case where  $x$  is rational and then consider the case that  $x$  is irrational.

*Proof. Case that  $x$  is rational:* We want to show that if  $0 < |x - 1| < \min\{1, \epsilon/3\}$ , then  $|x^2 - 1| < \epsilon$ . Let's do this:

$$|x^2 - 1| = |x - 1||x + 1| < (\epsilon/3)|x + 1| \leq (\epsilon/3)(|x - 1| + 2) < \epsilon/3(1 + 2) = \epsilon$$

In the second step, we used that  $|x - 1| < \epsilon/3$ , and in the fourth step, we used  $|x - 1| < 1$ .

*Case that  $x$  is irrational:* We want to show that if  $0 < |x - 1| < \min\{1, \epsilon/3\}$ , then  $|x - 1| < \epsilon$ . Let's do this:

$$|x - 1| < \min\{1, \epsilon/3\} \leq \epsilon/3 < \epsilon$$

□

**3.3. Max or min of two functions.** Suppose  $f$  and  $g$  are two functions and we define  $h(x) = \min\{f(x), g(x)\}$ . Suppose, further, that at some point  $c$ , we have  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = L$ . Then, we'll also have  $\lim_{x \rightarrow c} h(x) = L$ .

The reason? Well, the minimum of two functions is a special situation where, at every point, we are picking one of the functions. And we've just discussed that, whenever you get a function by always picking one of two functions, and both of them are approaching the same limit, the new function that you're picking also approaches the same limit. The reason for that is that we can determine the winning strategies for both the  $\epsilon - \delta$  games and then choose the smaller of the  $\delta$ s that we have for both functions.

So, this general strategy works for the minimum. It also works for the maximum.

**3.4. ADDENDUM: A harder problem.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function with the property that, for every  $x \in \mathbb{R}$ , we have  $(f(x))^2 - 3xf(x) + 2x^2 = 0$ . We want to show that  $\lim_{x \rightarrow 0} f(x) = 0$ .

Here's how we do this problem. First, note that the given statement is that:

$$\begin{aligned} (f(x))^2 - 3xf(x) + 2x^2 &= 0 \\ \implies (f(x) - x)(f(x) - 2x) &= 0 \end{aligned}$$

This means that for every value of  $x$ , we either have  $f(x) = x$  or we have  $f(x) = 2x$ . Unlike the previous cases where the domain was split based on left-right or rational-irrational, we do not know exactly how the domain splits up into these two definitions. But we do know that everywhere in the domain, the function behaves like one of these linear functions. And that itself is enough.

Thus, to show that the limit at 0 equals 0, we use the same old trick: we find  $\epsilon - \delta$  winning strategies for the functions  $x$  and  $2x$ , and then we combine these winning strategies by picking the smaller of the  $\delta$ s and show that it works.

So I began by calling this a hard problem but we have somehow overcome the hard part and you should now consider it an easy problem.

#### 4. SKEPTIC'S VICTORY: SHOWING THAT SOMETHING IS NOT THE LIMIT

**4.1. Negating the existence of limit.** Recall the definition of limit: We say that  $\lim_{x \rightarrow c} f(x) = L$  if  $f$  is defined in a neighborhood of  $c$  (except possibly at  $c$ ) and the following holds:

For every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x \in \mathbb{R}$  satisfying  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ . Now, what does it mean to say that it is *not true* that  $\lim_{x \rightarrow c} f(x) = L$ ? It means:

There exists  $\epsilon > 0$  such that for all  $\delta > 0$ , there exists  $x \in \mathbb{R}$  such that  $0 < |x - c| < \delta$  but  $|f(x) - L| \geq \epsilon$ . Note that the quantifiers change roles: *for all* becomes *there exists*, and *there exists* becomes *for all*.

In our game interpretation, this means that the skeptic has a winning strategy. In other words, the skeptic has a strategic choice of  $\epsilon > 0$  such that whatever  $\delta > 0$  the prover tries to use, the prover fails to trap the function within  $\epsilon$  of the claimed limit.

**4.2. What does it mean to say that no limit exists?** What does it mean to say that there is no  $L$  for which  $\lim_{x \rightarrow c} f(x) = L$ ? In other words, what does it mean to say that the limit *does not exist*? It means that:

For all  $L \in \mathbb{R}$ , there exists  $\epsilon > 0$  such that for all  $\delta > 0$ , there exists  $x \in \mathbb{R}$  such that  $0 < |x - c| < \delta$  but  $|f(x) - L| \geq \epsilon$ .

**4.3. An example of showing that no limit exists.** Let's think back to the picture of the function  $f(x) := \sin(1/x)$ . This example was the example we used to realize that we need a certain kind of definition of limit. We noticed that, as  $x \rightarrow 0$ , the value of  $\sin(1/x)$  didn't really approach anything because it was oscillating rapidly between  $-1$  and  $1$ . This led us to define limits in terms of traps; hence the  $\epsilon - \delta$  definition.

We have to now come back full circle and try to explain *why*, as  $x \rightarrow 0$ , there does not exist a limit for  $\sin(1/x)$ . The way to think about this is that the function takes both the values  $1$  and  $-1$  arbitrarily close to  $x = 0$ . We want to somehow make this be in contradiction with the fact that we can set  $\epsilon$ -traps for arbitrarily small  $\epsilon$ . Basically, there are two things going on:



- (1) Suppose  $\lim_{x \rightarrow 0} \sin(1/x) = L$ . Suppose  $\epsilon = 0.1$ . Then, there exists a value of  $\delta$  such that, if  $0 < |x - 0| < \delta$ , then  $|\sin(1/x) - L| < 0.1$ . In other words, we have  $\sin(1/x) \in (L - 0.1, L + 0.1)$ . Now, the interval  $(L - 0.1, L + 0.1)$  has width 0.2, and hence, it cannot contain both the numbers 1 and  $-1$ . Thus, what we have is that there exists a  $\delta > 0$  such that for  $0 < |x| < \delta$ , the function  $\sin(1/x)$  does not take both the values 1 and  $-1$ .
- (2) On the other hand, we have that for every  $\delta > 0$ , the function  $\sin(1/x)$  takes the values  $+1$  and  $-1$  for  $0 < |x| < \delta$ . In fact, it takes *all* values in  $[-1, 1]$ . To see this, note that for  $0 < |x| < \delta$ , the set of possible values for  $1/x$  is  $(-\infty, -1/\delta) \cup (1/\delta, \infty)$ . You can now see from the graph of the sine function that  $\sin(1/x)$  takes all values in  $[-1, 1]$ .

Something similar works for the *Dirichlet function*, except that instead of using  $-1$  and  $1$ , we use  $1$  and  $0$ . And, the main point that needs to be made for the Dirichlet function is that any open interval, no matter how small, contains infinitely many rational numbers and infinitely many irrational numbers.

## 5. STABILITY OF LIMIT

**5.1. Statement of the problem.** Let  $f$  be a function defined on some open interval  $(c - p, c + p)$ . Now change the value of  $f$  at a finite number of points  $x_1, x_2, \dots, x_n$  and call the resulting function  $g$ . Show that if  $\lim_{x \rightarrow c} f(x) = L$ , then  $\lim_{x \rightarrow c} g(x) = L$ .

**5.2. Meaning of the problem.** Here's what the problem means. We start with some function  $f$ . For simplicity you can just imagine  $f$  to be a continuous function – a wiggly wavy curve where the  $x$ -value varies in the interval  $(c - p, c + p)$ . Now, we choose a few points at random and just move the value of the function. So, if say  $f(5) = 7$ , and we don't like that, we just move the value to 9, creating a *hole* at the point 7 here and filling in the point at 9. And we do this for finitely many points.

The new function that we get after we do all these moves, we choose to call  $g$ . Note that although the function  $f$  that we started with was continuous,  $g$  isn't. Of course, it isn't necessary that the function that we start with is continuous either. We could just start with some function that wiggles and waves and jumps and then move a few values here and there and get a new function that wiggles and waves and jumps.

**5.3. What the question asks for, and the intuitive reason it is true.** The question says that if the limit  $\lim_{x \rightarrow c} f(x)$  exists and is equal to  $L$ , then the limit  $\lim_{x \rightarrow c} g(x)$  *also* exists and is equal to  $L$ . In other words, what this is saying is that the notion of limit is *stable* under changes of value at only finitely many points.

Now, the first thing you should notice is that if  $c$  itself is one of the  $x_i$ s (i.e., one of the points where we are changing the value of the function) that should make no difference to the limit. Because, if you recall, the definition of the limit specifically excludes behavior *at the point* where we are taking the limit. The definition says:

For every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \epsilon$ . Note the  $0 <$  part. This basically says that we aren't really imposing any condition when  $x = c$ . Specifically, remember that the set of  $x$  satisfying  $0 < |x - c| < \delta$  is the set  $(c - \delta, c + \delta) \setminus \{c\}$ , which is the same as the set  $(c - \delta, c) \cup (c, c + \delta)$ .

So changing the value of the function at  $c$  doesn't affect anything. And once we've acknowledged this, we'll just assume that if  $c$  is equal to one of the  $x_i$ s, we can just throw out that value of  $x_i$ . In other words, we'll just assume that we retain only those  $x_i$ s that are not equal to  $c$ .

Now, all the *other*  $x_i$ s – the points where we change the value of the function – are far away from  $c$ . What do I mean by that? I mean that we can choose a *small open interval* about  $c$  that excludes all the other  $x_i$ s. (It may be helpful to think of the  $x_i$ s as *bad points* that should be excluded from all decent society for being traitors to their function.)

And the main idea of limit is that it is intensely local – it only matters what is happening really really close to the point.<sup>1</sup> So, if all those other points are far away, the value of the function at those points shouldn't affect the value of the limit.

---

<sup>1</sup>There is a deep mathematical concept related to this called a *germ*, which means the essence of a function really really close to a point, but you don't have to bother about that right now.

**5.4. A reality check: why finitely many?** What is the significance of the fact that we are allowed to change the value of the function at only finitely many points? The thing is that if we are allowed to change the value of the function at infinitely many points, then we can affect the limits at some points. The reason is that when we have infinitely many points, it may not be possible to avoid all of them.

For example, suppose we have a function defined as the constant 5 on  $(-1, 1)$  and we are interested in the limit at 0. Suppose now that we change the value of the function to the constant 7 at the points  $1/2, 1/3, \dots, 1/n, \dots$ . There are two things you should note: first, we have changed the value of the function at infinitely many points, and second, the value of the function at *most* points is still 5. So we haven't really changed the function all that much. However, even this small change is enough to disrupt the limit at 0. Because now, no matter how small an interval I choose about 0, that interval will contain some of those *bad points* – those points where the definition changed.

**5.5. Formalization of the proof in terms of  $\epsilon$  and  $\delta$ .** *Read this only after you feel you have understood the ideas at an intuitive level.*

We begin by throwing out any  $x_i$  that equals  $c$ , because the value of  $f$  or  $g$  at  $c$  is clearly of no relevance to the limits of the functions at  $c$ .

We need to show that, if  $\lim_{x \rightarrow c} f(x) = L$ , then, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|g(x) - L| < \epsilon$ .

We begin by noting that since  $\lim_{x \rightarrow c} f(x) = L$ , plugging in the  $\epsilon - \delta$  definition shows that there exists  $\delta_1 > 0$  such that if  $0 < |x - c| < \delta_1$ , then  $|f(x) - L| < \epsilon$ .

Now, let  $\delta = \min\{\delta_1, |c - x_1|, |c - x_2|, \dots, |c - x_n|\}$ . The intuition here is to pick  $\delta$  small enough so that none of the  $x_i$  are in the interval  $(c - \delta, c + \delta)$ .

Note that  $\delta_1 > 0$ , and since  $c$  is not equal to any of the  $x_i$ s,  $|c - x_i| > 0$ . Thus,  $\delta > 0$ . We claim that this  $\delta$  works.

To see this, we note two things.

First, if  $0 < |x - c| < \delta$ , we also have  $0 < |x - c| < \delta_1$ , and this gives:

$$(1) \quad |f(x) - L| < \epsilon$$

Second, if  $0 < |x - c| < \delta$ , we cannot have  $x$  equal to any of the  $x_i$ s (why? think about this. This is the only part that I haven't justified in the proof). Hence, we get:

$$(2) \quad f(x) = g(x)$$

Combining (1) and (2), we get  $|g(x) - L| < \epsilon$  for  $0 < |x - c| < \delta$ , completing the proof.