DERIVATIVE AS RATE OF CHANGE, IMPLICIT DERIVATIVES: ROUGH APPROXIMATION OF LECTURE

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Sections 3.4 and 3.7

Difficulty level: Easy to moderate, since most of these should be familiar to you and there are no new subtleties being added here.

What students should definitely get: The notion of derivative as a rate of change, handling word problems that ask for rates of change. The main idea and procedure of implicit differentiation.

What students should hopefully get: The distinction between conceptual and computational, the significance of implicit differentiation, understanding the relative rates concept and its intuitive relationship with the chain rule.

Executive summary

Derivative as rate of change. Words...

(1) The derivative of A with respect to B is the rate of change of A with respect to B. Thus, to determine rates of change of various quantities, we can use the techniques of differentiation.
(2) If there are three linked quantities that are changing together (e.g., different measures for a circle such as radius, diameter, circumference, area) then we can use the chain rule.

Most of the actions in this case are not more than a direct application of the words.

Implicit differentiation. Words...

(1) Suppose there is a curve in the plane, whose equation cannot be manipulated easily to write one variable in terms of the other. We can use the technique of implicit differentiation to determine the derivative, and hence the slope of the tangent line, at different points to the curve.
(2) For a curve where neither variable is expressible as a function of the other, the notion of derivative still makes sense as long as *locally*, we can get y as a function of x. For instance, for the circle $x^2 + y^2 = 1$, y is not a function of x, but if we restrict attention to the part of the circle above the x-axis, then on this restricted region, y is a function of x.
(3) In some cases, even when one variable is expressible as a function of the other, implicit differentiation is easier to handle as it may involve fewer messy squareroot symbols.

Actions ...

(1) To determine the derivative using implicit differentiation, write down the equations of both curves, differentiate both sides with respect to x, and simplify using all the differentiation rules, to get everything in terms of x, y, and dy/dx. Isolate the dy/dx term in terms of x and y, and compute it at whatever point is needed.
(2) This procedure can be iterated to compute higher order derivatives at specific points on the curve where the curve locally looks like a function.

1. Conceptual versus computational

Back in the first lecture, I defined the concept of function. A function is some kind of machine that takes an input and gives an output. And the important thing about functions is that *equal inputs give equal outputs.*

The interesting thing about functions is that this way of thinking about functions is a sort of *black box, hands-off* approach. If you think of the function as this box machine which sucks in an input from one side and spits out the output on the other side, we don't really care *how the black box works.* It doesn't matter what is happening inside, as long as we are guaranteed that equal inputs give equal outputs.

With this abstract concept of function, we defined the notion of limit, which was the $\epsilon - \delta$ definition, and this definition didn't really depend on how you compute f. Then we defined the notion of derivative, which is a particular kind of limit, namely, the limit of the different quotient. And in all this, how to *compute* things wasn't the focus. And simply thinking of things conceptually, we got a lot of insights. We understood what limits mean and we understood what derivatives mean, and we saw the qualitative significance.

Complementing this conceptual understanding of the concepts of functions, limits, continuity, derivatives, and differentiation, there is the computational aspect. The computational aspect tells us how, for functions with specific functional forms or expressions, we can calculate limits and derivatives. And in order to do this, we use general theorems (limits for sums, differences, ...; derivatives for sums, differences ...) and specific tricks and formulas.

What you should remember, though, is that *just because you cannot compute something, doesn't mean that it cannot be understood qualitatively.* So, if you encounter a function and there's no formula to differentiate it, that's not the same as saying that it isn't differentiable. Computation is one tool among many to get a conceptual understanding of ideas.

This is really important because a lot of the places where you'll see these mathematical ideas applied are cases where the functions involved are inherently *unknown* or *unknowable* – there aren't explicit expressions for them. Still, we want to talk about the broad qualitative properties – is the function continuous? Is it differentiable? Is it twice differentiable? Is it increasing or decreasing, is it oscillating? Often, we can answer these qualitative questions without having explicit expressions for the functions.

2. Derivative as a rate of change

Recall that if f is a function, the derivative f' is the *rate of change* of the output of f relative to the input. Or, if we are thinking of two quantities x and y, where y is functionally dependent on x, then the rate of change of y with respect to x is dy/dx. That is the limit of the *difference quotient* $\Delta y/\Delta x$.

This means that if we want to ask the question: *if the rate of change of x is this much, what is the rate of change of y*, we should think of derivatives.

For instance, we know that the area of a circle of radius r is πr^2. We may ask the question: what is the rate of change of the area with respect to the radius? This is the derivative of πr^2 with respect to r, and that turns out to be $2\pi r$.

For instance, if $r = 5$, the rate of change of the area with respect to the radius is 10π.

Now, suppose the radius is changing at the rate of $5m/hr$. That means that every hour, the radius increases by $5m$. What is the rate of increase of the area with respect to time, when the radius is $100m$. Well, here we have three quantities, the area A, the radius r, and the time t. r is a function of t, and $dr/dt = 5m/hr$ and $dA/dr = 2\pi r$. So by the chain rule, we have $dA/dt = (dA/dr)(dr/dt = (2\pi r)(5m/hr)$. And since $r = 100m$, we get $1000\pi m^2/hr$.

3. Implicit differentiation

3.1. **Introduction.** So far, when trying to differentiate one quantity with respect to another quantity, what we do is to write one as a function of the other, and then differentiate that function. This is all very good when we have an explicit expression for the function. Sometimes, however, we do not really have a functional expression for one quantity in terms of the other, but we do know of a *relation* between the two quantities.

Let's think of this a little differently. One importance of differentiation is that it allows us to find tangent lines to curves that arise as the graph of a function. This has some geometric significance, if we are trying to understand the geometry of a curve that arises as the graph of a function. But what about the curves that don't arise from explicit functions? Or, where we don't have explicit functional expressions?

For instance, let's look at the circle of radius 1 centered at the origin. This is given by the equation $x^2 + y^2 = 1$. Note that in this case, y is *not* a function of x, because for many values of x, there are two values of y. For instance, for $x = 0$, we have $y = 1$ and $y = -1$. For $x = 1/2$, we have $y = \sqrt{3}/2$ and $y = -\sqrt{3}/2$. So, y is not a function of x.

However, *locally* y is still a function of x, in the following sense. If you just restrict yourself to the part above the x-axis, then you do get y as a function of x. This is the function $y := \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$. If we restrict ourselves to the part below the x-axis, we consider the function $y := -\sqrt{1 - x^2}$ for $-1 \leq x \leq 1$.

Now, how do we calculate dy/dx? Well, it depends on whether we are interested in the part above the x-axis or in the part below the x-axis. For the part above the x-axis, we have the function $\sqrt{1-x^2}$, and we get that the derivative is:

$$\frac{d(\sqrt{1-x^2})}{dx} = \frac{d(\sqrt{1-x^2})}{d(1-x^2)}\frac{d(1-x^2)}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (2x) = \frac{-x}{\sqrt{1-x^2}}$$

If we are interested in the lower side, we get $x/\sqrt{1-x^2}$.

Now, in this case, we have to split into two cases, and do a painful calculation involving differentiating a square root via the chain rule.

Here's another way of handling this differentiation, that does not involve a messy square root.

We start with the original expression:

$$x^2 + y^2 = 1$$

This is an *identity*, which means that it's true for every point on the curve. When we have an equation that is identically true, it is legitimate to differentiate both sides and still get an identity. Differentiating both sides with respect to x, we get:

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$$

Simplifying and using the chain rule, we get:

$$2x + 2y\frac{dy}{dx} = 0$$

We thus get:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Notice that with this method, we get $-x/y$, which works in *both* cases. When $y = \sqrt{1-x^2}$, we get $-x/\sqrt{1-x^2}$, and when $y = -\sqrt{1-x^2}$, we get $x/\sqrt{1-x^2}$. The method that we used is called *implicit differentiation*.

So the idea of implicit differentiation is that, instead of writing $y = f(x)$ and then differentiating both sides, we differentiate the messy mixed-up expression on both sides with respect to x. Next, we use the various rules (sum rule, difference rule, product rule, quotient rule) to keep splitting things up into smaller and smaller pieces, and in the final analysis, we get everything in terms of x, y, and dy/dx. Then, we try to separate dy/dx completely to one side.

Let's look at another example:

$$\sin(x+y) = xy$$

So, what we do is differentiate both sides:

$$\frac{d(\sin(x+y))}{dx} = \frac{d(xy)}{dx}$$

Now, how would we handle something like $\sin(x+y)$? It is something in terms of $x+y$, so we use the chain rule on the left side, thinking of $v = x+y$ as the intermediate function:

$$\frac{d(\sin(x+y))}{d(x+y)}\frac{d(x+y)}{dx} = x\frac{dy}{dx} + y\frac{dx}{dx}$$

This simplifies to:

$$\cos(x+y)\left[1 + \frac{dy}{dx}\right] = x\frac{dy}{dx} + y$$

Opening up the parentheses, we get:

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

Now, we move stuff together to one side, to get:

$$(\cos(x+y) - x)\frac{dy}{dx} = y - \cos(x+y)$$

And we now isolate dy/dx:

$$\frac{dy}{dx} = \frac{y - \cos(x+y)}{\cos(x+y) - x}$$

3.2. **Implicit differentiation: understood better.** So, in implicit differentiation, what we're doing is, instead of thinking of an explicit functional form, we are using a relation that is true for every point in the curve, then *differentiating both sides*. Next, we keep trying to simplify the expression we have using the various rules until we land up with something that just involves x, y, and dy/dx. Till this point, it's usually smooth sailing. Now, it may be the case that we can *isolate* dy/dx and hence get an expression for it in terms of x and y. If that's the case, then we're in good shape.

Note the following key difference: when y is an explicit function of x, then the expression we get for dy/dx only involves x and does not have the letter y appearing in it. However, in the implicit case, the expression we get for dy/dx involves both x and y together.

3.3. **Higher derivatives using implicit differentiation.** We can also use implicit differentiation to compute second derivatives and higher derivatives. Here's what we do. First, we get the expression for dy/dx. In other words, we write:

$$\frac{dy}{dx} = \text{Some expression in terms of } x \text{ and } y$$

We now differentiate both sides with respect to x. Again, this differentiation is valid because the above relation holds as an identity, and not just as an isolated point.

The left side becomes d^2y/dx^2. For the right side, we again use the same idea: we split as much as possible using the sum rule, product rule, etc. For the expressions that purely involve x, we differentiate the usual way. For the expressions that purely involve y, we differentiate with respect to y and multiply by dy/dx. The upshot is that we get:

$$\frac{d^2y}{dx^2} = \text{Some expression in terms of } x,\ y, \text{ and } \frac{dy}{dx}$$

Now, we plug back the earlier expression for dy/dx in terms of x and y into this expression, and get an expression for d^2y/dx^2.