DEFINITE INTEGRALS, FUNDAMENTAL THEOREM OF CALCULUS, ANTIDERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 5.3, 5.4

Difficulty level: Hard.

What students should definitely get: Some results leading to and including the fundamental theorem of integral calculus, the definition of antiderivative and how to calculate antiderivatives for polynomials and the sine and cosine functions.

What students should hopefully get: The intuition behind the way differentiation and integration relate; the concept of indeterminacy up to constants when we integrate. The reason for making assumptions such as continuity.

Executive summary

0.1. **Definite integral, antiderivative, and indefinite integral.** Words ..

(1) We have $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
(2) We say that F is an antiderivative for f if $F' = f$.
(3) For a continuous function f defined on a closed interval $[a,b]$, and for a point $c \in [a,b]$, the function F given by $F(x) = \int_c^x f(t)\,dt$ is an antiderivative for f.
(4) If f is continuous on $[a,b]$ and F is a function continuous on $[a,b]$ such that $F' = f$ on (a,b), then $\int_a^b f(x)\,dx = F(b) - F(a)$.
(5) The two results above essentially state that differentiation and integration are opposite operations.
(6) For a function f on an interval $[a,b]$, if F and G are antiderivatives, then $F - G$ is constant on $[a,b]$. Conversely, if F is an antiderivative of f, so is F plus any constant.
(7) The *indefinite integral* of a function f is the collection of all antiderivatives for the function. This is typically written by writing one antiderivative plus C, where C is an arbitrary constant. We write $\int f(x)\,dx$ for the indefinite integral. Note that there are no upper and lower limits.
(8) Both the definite and the indefinite integral are additive. In other words, $\int f(x)\,dx + \int g(x)\,dx = \int f(x) + g(x)\,dx$. The analogue holds for definite integrals, with limits.
(9) We can also pull constants multiplicatively out of integrals.

Actions ...

(1) To do a definite integral, find any one antiderivative and evaluate it between limits.
(2) An important caveat: when using antiderivatives to do a definite integral, it is important to make sure that the antiderivative is defined and continuous everywhere on the interval of integration. (Think of the $1/x^3$ example).
(3) To do an indefinite integral, find any antiderivative and put a $+C$.
(4) To find an antiderivative, use the additive splitting and pulling constants out, and the fact that $\int x^r\,dx = x^{r+1}/(r+1)$.

0.2. **Higher derivatives, multiple integrals, and initial/boundary conditions.** Actions ...

(1) The simplest kind of *initial value problem* (a notion we will encounter again when we study differential equations) is as follows. The k^{th} derivative of a function is given on the entire domain. Next, the *values* of the function and the first $k-1$ derivatives are given at a single point of the domain. We can use this data to find the function. Step by step, we find derivatives of lower orders. First, we integrate the k^{th} derivative to get that the $(k-1)^{th}$ derivative is of the form $F(x) + C$, where C is unknown. We now use the value of the $(k-1)^{th}$ derivative at the given point to find C. Now, we have the $(k-1)^{th}$ derivative. We proceed now to find the $(k-2)^{th}$ derivative, and so on.

(2) Sometimes, we may be interested in finding *all* functions with a given second derivative f. For this, we have to perform an indefinite integration twice. The net result will be a general expression of the form $F(x) + C_1 x + C_2$, where F is a function with $F'' = f$, and C_1 and C_2 are arbitrary constants. In other words, we now have *up to constants or linear functions* instead of *up to constants* as our degree of ambiguity.

(3) More generally, if the k^{th} derivative of a function is given, the function is uniquely determined up to additive differences of polynomials of degree strictly less than k. The number of free constants that can take arbitrary real values is k (namely, the coefficients of the polynomial).

(4) This general expression is useful if, instead of an initial value problem, we have a boundary value problem. Suppose we are given G'' as a function, and we are given the value of G at two points. We can then first find the general expression for G as $F + C_1 x + C_2$. Next, we plug in the values to get a system of two linear equations, that we solve in order to determine C_1 and C_2, and hence G.

1. Statements of main results

1.1. The definite integral: recall and more details. Recall from last time that for a *continuous* (or piecewise continuous where all discontinuities are jump discontinuities) function f on an open interval $[a, b]$, the *integral* of f over the interval $[a, b]$, denoted:

$$\int_a^b f(x)\,dx$$

is a kind of summation for f for all the real numbers from a to b. This integral is also called a *definite integral*. The function is often termed the *integrand*. The number b is termed the *upper limit* of the integration, and the number a is termed the *lower limit* of the integration. The variable x is termed the *variable of integration*.

So far, we have made sense of the expression as described above with $a < b$. We now add in a few details on how to make sense of two other possibilities:

- If $a = b$, then, by *definition*, the integral is defined to be zero.
- If $a > b$, then, by *definition*, the integral is defined as the *negative* of the integral $\int_b^a f(x)\,dx$.

Now, it makes sense to consider the symbol $\int_a^b f(x)\,dx$ without any ordering conditions on a and b. With these definitions, we have, for any $a, b, c \in \mathbb{R}$:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

1.2. Definite integrals do exist for piecewise continuous functions. It is useful to know the following:

(1) For a continuous function f on a closed and bounded interval $[a, b]$, the integral exists and is finite. In fact, the integral over the interval $[a, b]$ is bounded from above by $(b - a)$ times the maximum value of the function and from below by $(b - a)$ times the minimum value of the function. (Both of these exist by the extreme value theorem).

(2) For a piecewise continuous function f on a closed and bounded interval $[a, b]$ such that all the one-sided limits exist and are finite at points of discontinuity, the integral exists and is finite. This follows from the previous part, via the intermediate step of breaking $[a, b]$ into parts such that the restriction of the function to each part is continuous and extends continuously to the boundary of that part.

1.3. The definite integral and differentiation. There is also a clear relationship between the definite integral and differentiation. In some sense, the integral and derivative are inverses (opposites) of each other. Let $[a, b]$ be an interval. Suppose f is a continuous function on $[a, b]$ and $c \in [a, b]$ is any number. Define the following function F on $[a, b]$:

$$F(x) := \int_c^x f(t)\,dt$$

Note the way the function is defined. t is the variable of integration, and F depends on x in the sense that the *upper limit* of the interval of integration is x, whereas the lower limit is fixed at c.

Continuous functions are integrable, as discussed above, so F turns out to be well-defined.

Further, F is continuous on $[a, b]$, differentiable on (a, b), and has derivative

$$F'(x) = f(x) \qquad \text{for all } x \in (a, b)$$

This is Theorem 5.3.5.

1.4. **Concept of antiderivative.** Suppose f is continuous on $[a, b]$. An *antiderivative* for f, or *primitive* for f, or *indefinite integral* for f, is a function G on $[a, b]$ such that:

- G is continuous on $[a, b]$.
- $G'(x) = f(x)$ for all $x \in (a, b)$.

A little while back, we had seen the following result: if F, G are functions on an interval I such that $F' = G'$ for all points in the interior of I, then $F - G$ is a constant function on I. In other words, F and G differ by a constant.

Conversely, if F and G are functions on an interval I with F differentiable on the interior of I, and $F - G$ is constant, then G is also differentiable on I and $F' = G'$ on the interior of I.

Thus, the antiderivative of a function is not unique – we can always add a constant function to one antiderivative to obtain another antiderivative. However, the antiderivative is unique up to differing by constants. In other words, any two antiderivatives differ by a constant.

We are now in a position to state the fundamental theorem of calculus.

Suppose f is a continuous function on the interval $[a, b]$. If G is an antiderivative for f on $[a, b]$, then we have:

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

Also, as we already noted, any two antiderivatives differ by a constant, so if we replace G by another antiderivative, the right side remains the same because both $G(a)$ and $G(b)$ get shifted by the same amount.

For notational convenience, this is sometimes written as:

$$\int_a^b f(t)\,dt = [G(t)]_a^b$$

Here, the right side is interpreted as the difference between the values of $G(t)$ for $t = b$ and $t = a$, which simplifies to $G(b) - G(a)$.

2. Computing antiderivatives and integrals: easy facts

2.1. **Computing some antiderivatives.** We now compute some common expressions for antiderivatives of functions.

(1) If $f(x) = x^r$, and $r \neq -1$, then we can set $G(x) = x^{r+1}/(r+1)$. The factor of $1/(r+1)$ is intended to cancel the factor of $r+1$ that appears as a coefficient when we differentiate x^{r+1}. In particular, if $f(x) = x$, $G(x) = x^2/2$, and if $f(x) = x^2$, $G(x) = x^3/3$. Most importantly, if $f(x) = 1$, then $G(x) = x$.

(2) An antiderivative for sin is $-\cos$ and an antiderivative for cos is sin. Note the sign differences between these formulas and those for the derivative. The derivative of sin is cos but the antiderivative of sin is $-\cos$. The derivative of cos is $-\sin$ and the antiderivative of cos is sin. (We will see more trigonometric antiderivatives later).

2.2. **Linearity of the integral.** The integral is *linear*, in the sense of being additive and allowing for the factoring out of scalars. Specifically:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

Thus, we can pull out scalars and split sums additively when computing integrals, just as we did for derivatives.

2.3. **Linearity of the antiderivative.** The linearity of the integral turns out to be closely related to the linearity of the antiderivative. Of course, it is not precise to say "the" antiderivative, since the antiderivative is defined only up to differences of constants. What we mean is the following:

(1) If F is an antiderivative for f and G is an antiderivative for g, then $F+G$ is an antiderivative for $f+g$.
(2) If F is an antiderivative for f and α is a real number, then αF is an antiderivative for αf.

(These statements are immediate corollaries of the corresponding statements for derivatives).

2.4. **General expression for indefinite integral.** Once we have computed one antiderivative for the integral, the general expression for the indefinite integral is obtained by taking that antiderivative and writing a "+ C" at the end, where C is a freely varying real parameter. What this means is that *every specific choice* of numerical value for C gives yet another antiderivative for the original function.

Note that the letter C is used conventionally, but there is nothing special about this latter. If the situation at hand already uses the letter C in some other context, please use another letter.

For instance:

$$\int (x - \sin x)\,dx = (x^2/2) + \cos x + C$$

2.5. **Getting our hands dirty.** We are now in a position to do some straightforward computations of integrals for polynomials and some basic trigonometric functions. For instance:

$$\int_0^1 (x^2 - x + 1)\,dx$$

We can find an antiderivative for this function, by finding antiderivatives for the individual functions x^2, $-x$, and 1, and then adding up. An antiderivative that works is $x^3/3 - x^2/2 + x$. Now, to calculate the definite integral, we need to calculate the difference between the values of the antiderivative at the upper and lower limit. We write this as:

$$\left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1$$

Next, we do the calculation:

$$\left(\frac{1}{3} - \frac{1}{2} + 1\right) - (0 - 0 + 0) = \frac{5}{6}$$

Thus, the value of the definite integral is $5/6$.

In other words, the signed area between the graph of the function $x^2 - x + 1$ and the x-axis, between the x-values 0 and 1, is $5/6$.

Some people prefer to split the definite integral as a sum first and then compute antiderivatives for each piece. The work would then appear as follows:

$$\int_0^1 (x^2 - x + 1)\,dx = \int_0^1 x^2\,dx - \int_0^1 x\,dx + \int_0^1 1\,dx = [x^3/3]_0^1 - [x^2/2]_0^1 + [x]_0^1 = 1/3 - 1/2 + 1 = 5/6$$

There is no substantive difference in the computations.

3. Higher derivatives and repeated integration

3.1. Finding all functions with given k^{th} derivative. Suppose the second derivative of a function is given. What are all the possibilities for the original function? In order to answer this question, we need to integrate twice. For instance, suppose $f''(x) = \cos x$. Then, we know that:

$$f'(x) = \int \cos x \, dx = (\sin x) + C_1$$

where C_1 is an arbitrary real number.

Integrating again, we get:

$$f(x) = \int f'(x) \, dx = \int [(\sin x) + C_1] \, dx = (-\cos x) + C_1 x + C_2$$

Here, both C_1 and C_2 are arbitrary real numbers. Thus, the family of all possible fs that work is described by two parameters, freely varying over the real numbers.

More generally, if the k^{th} derivative of a function is known, then the original function is known up to additive difference of a polynomial fo degree at most $k - 1$. Each coefficient of that polynomial is a freely varying real parameter, and there are k such coefficients: the constant term, the coefficient of x, and so on till the coefficient of x^{k-1}.

3.2. Degree of freedom and initial/boundary values. One way of thinking of the preceding material is that each time we integrate, we introduce one more degree of freedom. Thus, integrating thrice introduces a total of three degrees of freedom.

In practice, when we are asked to find a function f in the real world, we know the k^{th} derivative of f, but we also have information about the values of f at some points. The two typical ways this information is packaged are:

- *Initial value problem* packaging: Here, the value of f and all its derivatives, up to the $(k-1)^{th}$ derivative, at a single point c are provided, along with the general expression for the k^{th} derivative. For this kind of problem, we can, at each stage of antidifferentiation, determine the value of the constant we get, and thus we get a *single* function at the end.
- *Boundary value problem* packaging: Here, the value of f at k distinct points is specified. To solve this kind of problem, we first find the general expression for f with k unknown constants, then use the values at k distinct points to get a system of k linear equations in k variables, which we then proceed to solve.

4. Subtle issues/additional notes

4.1. Variable of integration – don't reuse! When writing something like $\int_a^b f(t) \, dt$, please remember that the letter t, which is used locally as a variable of integration, *cannot* be used outside the expression.

4.2. Definite integral as a size or norm of function. To completely describe a function f on a closed interval $[a, b]$ requires a lot of work, since it requires specifying the function value at infinitely many points. On the other hand, the value of the integral of f on $[a, b]$, given by $\int_a^b f(x) \, dx$, is a single real number. Since numbers are easier to grasp than functions, we often use the integral of a function on an interval to get an approximate estimate of its size.

More generally, we are often interested in expressions of the form $\int_a^b f(x)g(x) \, dx$ where $g(x)$ plays the role of a weighting function. Usually, we have a bunch of two or three functions g and we are interested in the above integral on $[a, b]$ for each of those gs. We use the collection of two or three numbers we get that way to say profound things about the function f, even without knowing f directly.

4.3. Linear algebra interpretation of antiderivative. (This material is not necessary for this course, but is useful for subsequent mathematics – we'll see it again in 153 and you'll see more of these ideas if you take Math 196/199 or advanced courses in the social and/or physical sciences).

Denote by C^1 the set of functions on $\mathbb{R}$ that are continuously differentiable everywhere. Denote by C^0 the set of continuous functions on $\mathbb{R}$.

First, note that C^0 and C^1 are both *vector spaces* over $\mathbb{R}$. Here's what this means for C^0: the sum of two continuous functions is continuous, and any scalar multiple of a continuous function is continuous. Here's what this means for C^1: the sum of two continuously differentiable functions is continuously differentiable, and any scalar multiple of a continuously differentiable function is continuously differentiable.

Differentiation is a *linear* operator from C^1 to C^0 in the following sense: first, for any $f \in C^1$, f' is an element of C^0. Second, we have the rules $(f+g)' = f' + g'$ and $(\alpha f)' = \alpha f'$. In other words, differentiation respects the vector space structure.

The *kernel* of a linear operator is the set of a functions which go to zero. For any linear operator between vector spaces, the kernel is a subspace.

Our basic result is that the kernel of the differentiation operator is the space of constant functions. Two elements in a vector space have the same image under a linear operator iff their difference is in the kernel of that operator. In our context, this translates to the statement that two functions have the same derivative iff their difference is a constant function.

Our second basic result is that any function in C^0 arises as the derivative of something in C^1. This something can be computed using a definite integral.

Thus, the kernel of the differentiation operator is a copy of the real line inside C^1, given by the scalar functions. For any element of C^0, the set of elements of C^1 which map to it is a line inside C^1 parallel to the line of constant functions.

Instead of looking at functions on the entire real line, we can also restrict attention to functions on an open interval inside the real line – qualitatively, all our results hold.