CONCAVITY, INFLECTIONS, CUSPS, TANGENTS, AND ASYMPTOTES

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 4.6, 4.7.

Difficulty level: Moderate to hard. If you have seen these topics in AP Calculus, then moderate difficulty; if you haven't, then hard.

What students should definitely get: The definitions of *concave up*, *concave down*, and *point of inflection*. The strategies to determine *limits at infinity*, *limits valued at infinity*, *vertical tangents*, *cusps*, *vertical asymptotes*, and *horizontal asymptotes*.

What students should hopefully get: The intuitive meanings of these concepts, important examples and boundary cases, the significance of concavity in determining local extrema, the use of higher derivative tests. Important tricks for calculating limits at infinity.

Executive summary

0.1. Concavity and points of inflection. Words ...

(1) A function is called *concave up* on an interval if it is continuous and its first derivative is continuous and increasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be positive except possibly at isolated points, where it can be zero. (Think x^4, whose first derivative, $4x^3$, is increasing, and the second derivative is positive everywhere except at 0, where it is zero).
(2) A function is called *concave down* on an interval if it is continuous and its first derivative is continuous and decreasing on the interval. If the function is twice differentiable, this is equivalent to requiring that the second derivative be negative except possibly at isolated points, where it can be zero.
(3) A *point of inflection* is a point where the sense of concavity of the function changes. A point of inflection for a function is a point of local extremum for the first derivative.
(4) Geometrically, at a point of inflection, the tangent line to the graph of the function *cuts through* the graph.

Actions ...

(1) To determine points of inflection, we first find critical points for the first derivative (which are points where this derivative is zero or undefined) and then use the first or second derivative test at these points. Note that these derivative tests are applied to the first derivative, so the first derivative here is the second derivative and the second derivative here is the third derivative.
(2) In particular, if the second derivative is zero and the third derivative exists and is nonzero, we have a point of inflection.
(3) A point where the first two derivatives are zero could be a point of local extremum or a point of inflection. To find out which one it is, we either use sign changes of the derivatives, or we use higher derivatives.
(4) Most importantly, the second derivative being zero does *not* automatically imply that we have a point of inflection.

0.2. Tangents, cusps, and asymptotes. Words...

(1) We say that f has a horizontal asymptote with value L if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. Sometimes, both might occur. (In fact, in almost all the examples you have seen, the limits at $\pm\infty$, if finite, are both equal).
(2) We say that f has a vertical asymptote at c if $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$. Note that in this case, it usually also happens that $f'(x) \to \pm\infty$ on the relevant side, with the sign

the same as that of $f(x)$'s approach if the approach is from the left and opposite to that of $f(x)$'s approach if the approach is from the right. However, this is not a foregone conclusion.

(3) We say that f has a vertical tangent at the point c if f is continuous (and finite) at c and $\lim_{x \to c} f'(x) = \pm\infty$, with the *same sign of infinity* from both sides. If f is increasing, this sign is $+\infty$, and if f is decreasing, this sign is $-\infty$. Geometrically, points of vertical tangent behave a lot like points of inflection (in the sense that the tangent line cuts through the graph). Think $x^{1/3}$.

(4) We say that f has a vertical cusp at the point c if f is continuous (and finite) at c and $\lim_{x \to c^-} f'(x)$ and $\lim_{x \to c^+} f'(x)$ are infinities of opposite sign. In other words, f takes a sharp about-turn at the x-value of c. Think $x^{2/3}$.

(5) We say that f is asymptotic to g if $\lim_{x \to \infty} f(x) - g(x) = \lim_{x \to -\infty} f(x) - g(x) = 0$. In other words, the graphs of f and g come progressively closer as $|x|$ becomes larger. (We can also talk of one-sided asymptoticity, i.e., asymptotic only in the positive direction or only in the negative direction). When g is a *nonconstant linear function*, we say that f has an *oblique asymptote*. Horizontal asymptotes are a special case, where one of the functions is a constant function.

Actions...

(1) Finding the horizontal asymptotes involves computing limits as the domain value goes to infinity. Finding the vertical asymptotes involves locating points in the domain, or the boundary of the domain, where the function limits off to infinity. For both of these, it is useful to remember the various rules for limits related to infinities.

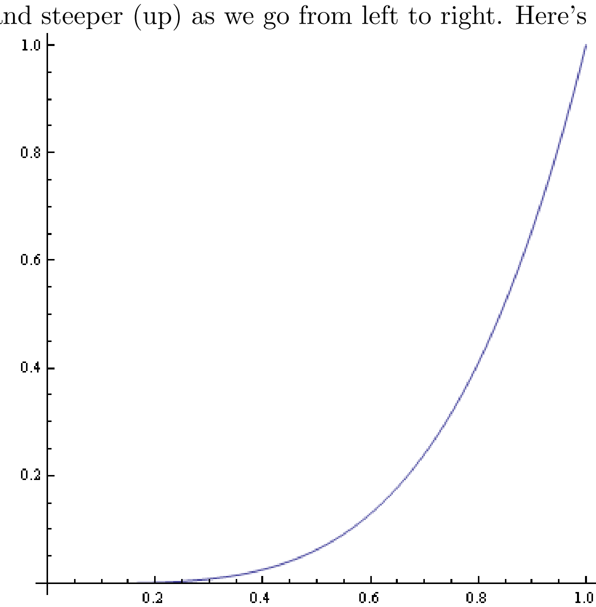
(2) Remember that for a vertical tangent or vertical cusp at a point, it is necessary that the function be continuous (and take a finite value). So, we not only need to find the points where the derivative goes off to infinity, we also need to make sure those are points where the function is continuous. Thus, for the function $f(x) = 1/x$, $f'(x) \to -\infty$ on both sides as $x \to 0$, but we do *not* obtain a vertical tangent – rather, we obtain a vertical asymptote.

1. Concavity and points of inflection

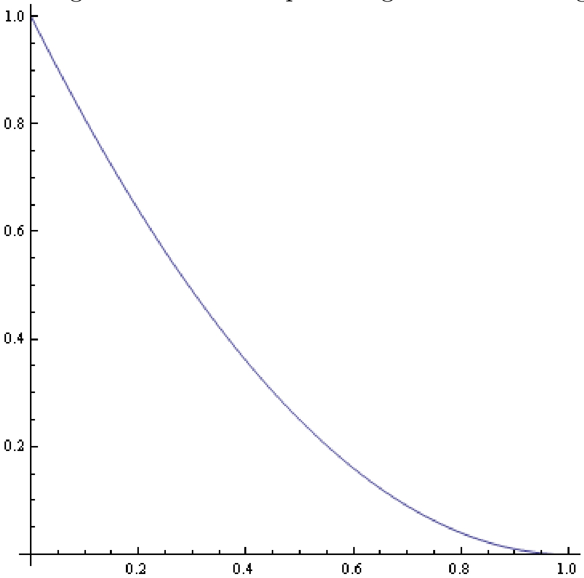
1.1. **Concavity.** *Concave up* means that the derivative of the function (which measures its rate of change) is itself increasing. Formally, a function f differentiable on an open interval I is termed *concave up* on I if f' is increasing on I. I hope you remember the definition of an *increasing function*: it means that for two points $x_1, x_2 \in I$, with $x_1 < x_2$, we have $f'(x_1) < f'(x_2)$.

Here's three points:

(1) If f itself is increasing (so that f' is positive), then being concave up means that f is increasing *at an increasing rate*. In other words, the slope of the tangent line to the graph of f becomes steeper and steeper (up) as we go from left to right. Here's a typical picture:



(2) If f itself is decreasing (so that f' is negative), then being concave up means that f is decreasing *at a decreasing rate*. In other words, the slope of the tangent line to the graph is negative, but it is becoming less and less steep as we go from left to right. Here's a typical picture:

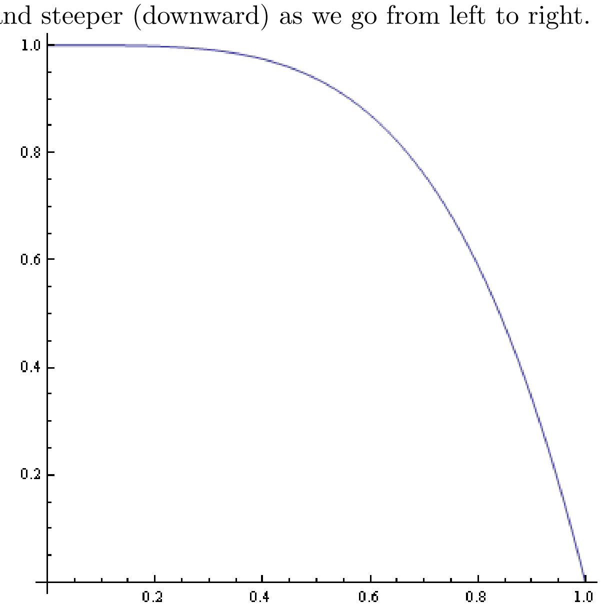


(3) If f is *twice differentiable*, i.e., f' is differentiable, then we can deduce whether f' is increasing by looking at f''. Specifically, if f' is continuous on I, and $f'' > 0$ everywhere on I except at a few isolated points, then f is concave up throughout.

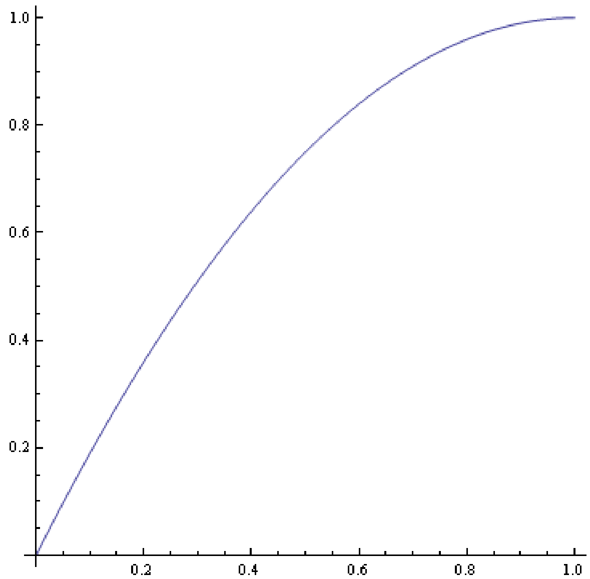
Similarly, if f is differentiable on an open interval I, we say that f is *concave down* on I if f' is *decreasing* on the interval I. I hope you remember the definition of a *decreasing function*: it means that for two points $x_1, x_2 \in I$, with $x_1 < x_2$, we have $f'(x_1) > f'(x_2)$.

Here's three points:

(1) If f itself is decreasing (so that f' is negative), then being concave down means that f is decreasing *at an increasing rate*. In other words, the slope of the tangent line to the graph of f becomes steeper and steeper (downward) as we go from left to right.



(2) If f itself is increasing (so that f' is positive), then being concave down means that f is increasing *at a decreasing rate*. In other words, the slope of the tangent line to the graph is positive, but it is becoming less and less steep as we go from left to right.



(3) If f is *twice differentiable*, i.e., f' is differentiable, then we can deduce whether f' is increasing by looking at f''. Specifically, if f' is continuous on I, and $f'' < 0$ everywhere on I except at a few isolated points, then f is concave down throughout.

1.2. **Points of inflection.** A point of inflection is a point c in the interior of the domain of a differentiable function (i.e., the function is defined and differentiable on an open interval containing that point) such that the function is concave in one sense to the immediate left of c and concave in the other sense to the immediate right of c.

Another way of thinking of this is that points of inflection of a function are points where the derivative is increasing to the immediate left and decreasing to the immediate right, or decreasing to the immediate left and increasing to the immediate right. In other words, it is a point of local maximum or a point of local minimum for the derivative of the function.

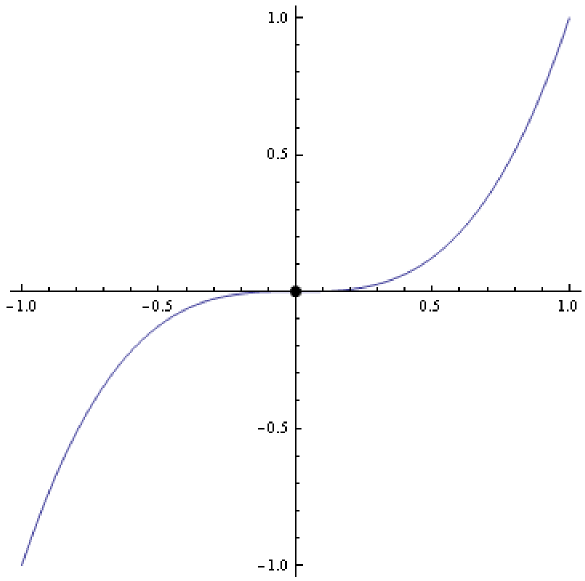
Recall that earlier, we noted that for a point of local maximum or a point of local minimum, either the derivative is zero or the derivative does not exist. Since everything we're talking about now is related to f', we have that for a point of inflection, either $f'' = 0$ or f'' does not exist.

So the upshot: concave up means the derivative is increasing, concave down means the derivative is decreasing, point of inflection means the sense in which the derivative is changing changes at the point.

1.3. **A point of inflection where the first two derivatives are zero.** We now consider one kind of point of inflection: where the first derivative and the second derivative are both zero. Let's begin with the example.

Consider the function $f(x) := x^3$. Recall first that since f is a cubic function, it has odd degree, so as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, we also have $f(x) \to \infty$. Further, if we compute $f'(x)$, we get $3x^2$. Note that the function $3x^2$ is positive for $x \neq 0$, and is 0 at $x = 0$. So, from our prior discussion of increasing and decreasing functions, we see that f is increasing on $(-\infty, 0]$ and then again on $[0, \infty)$. And since the point 0 is common to the two intervals, f is in fact increasing everywhere on $(-\infty, \infty)$.

If you remember, this was an important and somewhat weird example because, although $f'(0) = 0$, f does *not* attain a local extreme value at 0. This is because the derivative of f is positive *on both sides* of 0.



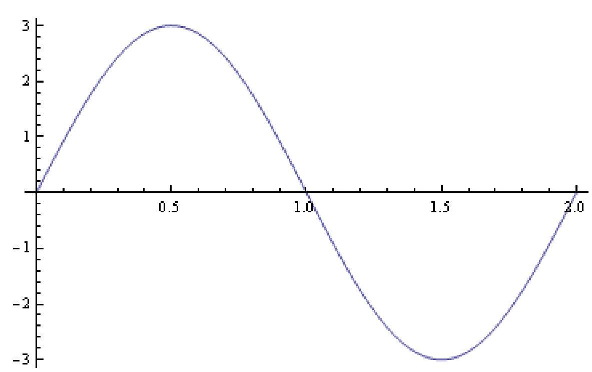
This was the picture we had from our earlier analysis. But now, with the concepts of concave up, concave down, and points of inflection, we can get a better understanding of what's going on. Specifically, we see that the second derivative f'' is $6x$, which is negative for $x < 0$, zero for $x = 0$, and positive for $x > 0$. Thus, f is concave down for $x < 0$, $x = 0$ is a point of inflection, and f is concave up for $x > 0$.

So, here's the picture: for $x < 0$, f is negative, f' is positive, and f'' is again negative. Thus, the graph of f is below the x-axis (approaching 0), it is going upward, and it is going up at a decreasing rate. So, as $x \to 0$, the graph becomes flatter and flatter.

For $x > 0$, f is positive, f' is positive, and f'' is again positive. Thus, the graph of f is above the x-axis (starting from the origin), it is going upward, and it is going up at an increasing rate. The graph starts out from flat and becomes steeper and steeper.

So, $x = 0$ is a *no sign change* point for f', which is why it is not a point of local maximum or local minimum. This is because on both sides of 0, f' is positive. What happens is that it is going down from positive to zero and then up again from zero to positive. But on a related note, because f' itself dips down to zero and then goes back up, the point 0 is a point of local minimum for f', so it is a point of inflection for f.

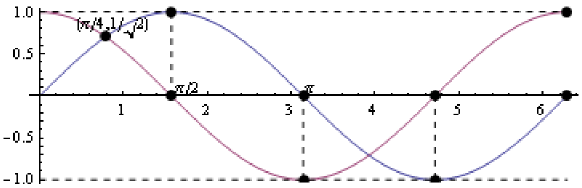
The main thing you should remember is that when we have a critical point for a function, where the derivative is zero, but it is neither a point of local maximum nor a point of local minimum, then it is *likely to be* a point of inflection. In other words, this idea of something that is increasing (or decreasing) and momentarily stops in its tracks, is the picture of neither a local maximum nor a local minimum but a point of inflection.



1.4. **Points of inflection where the derivative is not zero.**

Let's review the graph of the sine function. The sine function starts with $\sin(0) = 0$, goes up from 0 to $\pi/2$, where it reaches the value 1, then drops down to 0, drops down further to -1 at $3\pi/2$, and then turns back up to reach 0 at 2π. And this pattern repeats periodically.

So far, you've taken me on faith about the way the graph curves. But we can now start looking at things in terms of concave up and concave down.

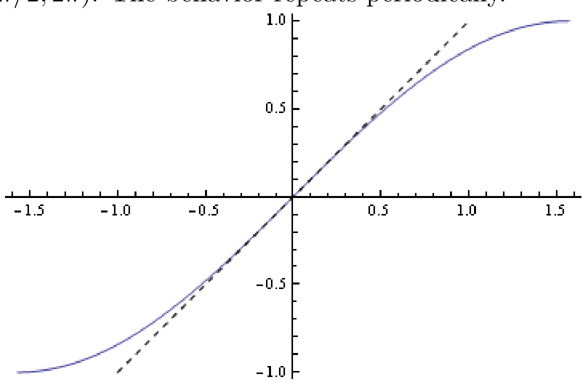


The derivative of the sin function is the cos function. Let's graph the cos function. This starts with the value 1 at $x = 0$, goes down to zero at $x = \pi/2$, dips down to -1 at $x = \pi$, goes back up to 0 at $x = 3\pi/2$, and then goes up to 1 at $x = 2\pi$.

We see that cos is positive from 0 to $\pi/2$, and sin is increasing on that interval. cos is negative from $\pi/2$ to $3\pi/2$, and sin is decreasing on that interval. cos is again positive from $3\pi/2$ to 2π, and sin is again increasing on that interval.

Next, we want to know where sin is concave up and where it is concave down. And for this, we look at the second derivative of sin, which is the function $-\sin$. As you know, the graph of this function is the same as the graph of sin, but flipped about the x-axis. This means that where sin is positive, its second derivative is negative, and where sin is negative, its second derivative is positive.

So, from the interval between 0 and π, sin is concave down and on the interval between π and 2π, sin is concave up. Breaking the interval down further, sin is increasing and concave down on $(0, \pi/2)$, decreasing and concave down on $(\pi/2, \pi)$, decreasing and concave up on $(\pi, 3\pi/2)$, and increasing and concave up on $(3\pi/2, 2\pi)$. The behavior repeats periodically.



Now, let's concentrate on the points of inflection. Note that the sense of concavity changes at multiples of π – at the point 0, the function changes from concave up to concave down. At the point π, the function changes from concave down to concave up. Another way of thinking about this is that just before π, the function is decreasing at an increasing rate – it is becoming progressively steeper. But from π onwards, it starts decreasing at a decreasing rate, in the sense that it starts becoming less steep. So π is the point where the way the tangent line is turning starts changing.

1.5. **A graphical characterization of inflection points.** Inflection points are graphically special because they are points where the way the tangent line is turning changes sense. There's a related characterization. If you draw the tangent line through an inflection point, the tangent line *cuts through* the curve. Equivalently, the curve *crosses* the tangent line. This is opposed to any other point, where the curve *locally* lies to one side of the tangent line.

For instance, for the cube function $f(x) := x^3$, the tangent line is the x-axis, and the curve crosses the x-axis at $x = 0$. We see something similar for the tangent lines at the points of inflection 0 and π for the sin function.

1.6. **Third and higher derivatives: exploration.** (I may not get time to cover this in class).

A while ago, we had developed criteria to determine whether a critical point is a point where a local extreme value is attained. We discussed two tests that could be used: the *first derivative test* and the *second derivative test*. The first derivative test said that if f' changes sign across the critical point, it is a point where a local extreme value occurs: a local maximum if the sign change is from positive on the left to negative on the right, and a local minimum if the sign change is from negative on the left to positive on the right.

The second derivative test was a test specially suited for functions that are twice differentiable at the critical point. This test states that if the second derivative at a critical point is negative, the function attains a local maximum, and if the second derivative is positive, the function attains a local minimum. This leaves one case open: what happens if the second derivative at the critical point is zero?

In this case, things are inconclusive. We might have a point of local maximum, a point of local minimum, an inflection point, or none of the above. How do we figure this out? I will give two general principles of alternation, and then we will look at some examples:

(1) If c is a point of inflection for f' and and $f'(c) = 0$, then c is a point of local extremum for f. If the point of inflection is a change from concave up to concave down, we get a local maximum and if the change is from concave down to concave up, we get a local minimum.
(2) If c is a point of local maximum or minimum for f', then c is a point of inflection for f. Local maximum implies a change from concave up to concave down and local minimum implies a change from concave down to concave up.

Let's illustrate this with the function $f(x) := x^5$ and the point $c = 0$. Let's also assume you knew nothing except differentiation and applying the derivative tests. We have $f'(x) = 5x^4$, $f''(x) = 20x^3$, $f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$, and $f^{(5)}(x) = 120$. At $c = 0$, $f^{(5)}$ is the first nonzero derivative.

Now, 0 is a point at which $f^{(4)} = 0$ and $f^{(5)} > 0$. Thus, by the second derivative test, 0 is a point of local minimum for $f^{(3)}$. So, 0 is a point of inflection for $f^{(2)}$, by point (2) above. Thus, 0 is a point of local minimum for f', by point (1) above. Thus, 0 is a point of inflection for f, by point (2) above.

So, the upshot of this is the alternating behavior between derivatives.

1.7. **Higher derivative tests.** The discussion above gives a practical criterion to simply use evaluation of derivatives to determine whether a critical point, where a function is infinitely differentiable, is a point of local maximum, point of local minimum, point of inflection, or none of these.

Suppose f is an infinitely differentiable function around a critical point c for f.. Let k be the smallest integer for which $f^{(k)}(c) \neq 0$ and let L be the nonzero value of the k^{th} derivative. Then:

(1) If k is odd, then c is a point of inflection for f and hence neither a point of local maximum nor a point of local minimum.
(2) If k is even and $L > 0$, then c is a point of local minimum for f.
(3) If k is even and $L < 0$, then c is a point of local maximum for f.

For instance, for the power function $f(x) := x^n, n \geq 2$. 0 is a critical point and f is infinitely differentiable. In this case, $c = 0$, $k = n$, and $L = n! > 0$. Thus, if n is even, then f does attain a local minimum at 0. if n is odd, 0 is a point of inflection. In this simple situation, we could have deduced this directly from the first derivative test – for n even, the first derivative changes sign fron negative to positive at 0, because $n - 1$ is odd. For n odd, $n - 1$ is even, so the first derivative has positive sign on both sides of 0. However, the good news is that this general method is applicable for other situations where the first derivative test is harder to apply.

1.8. **Notion of concave up and concave down for one-sided differentiable.** So far, we have defined the notion of concave up and concave down on an interval assmuing the function is differentiable everywhere on the interval. In higher mathematics, a somewhat more general definition is used, and this makes sense for functions that have one-sided derivatives everywhere.

Note: Please, please, please, please make sure you understand this clearly: we can calculate the left-hand derivative and right-hand derivative using the formal expressions *only* after we have checked that the function is continuous from that side! If there is a piecewise description of the function and it is not continuous from one side where the definition is changing, then the corresponding one-sided derivative is *not*, repeat *not* defined.

Suppose f is a function on an interval $I = (a, b)$ such that both the left-hand derivative and the right-hand derivative of f are defined everywhere on I. Note that *both* one-sided derivatives being defined at every point in particular means that the function is continuous at each point, and hence on I. However, f need not be differentiable at every point, because it is possible that the left-hand derivative and right-hand derivative differ at different points.

We then say that:

(1) f is concave up if, at every point, the right-hand derivative is greater than or equal to the left-hand derivative, and both one-sided derivative are increasing functions on $\mathbb{R}$.
(2) f is concave down if, at every point, the right-hand derivative is less than or equal to the left-hand derivative, and both one-sided derivatives are decreasing functions on $\mathbb{R}$.

For instance, first consider the function f on $(0, \infty)$:

$$f(x) := \{ \begin{array}{ll} x^2, & 0 < x \leq 1 \\ x^3, & 1 < x \end{array}$$

Before we proceed further, we check/note that the function is continuous at 1. Indeed it is. Hence, to calculate the left-hand derivative and right-hand derivative at 1, we can formally differentiate the expressions at 1. We obtain that the left-hand derivative at 1 is $2 \cdot 1 = 2$ and the right-hand derivative at 1 is $3 \cdot 1^2 = 3$. We thus obtain the following piecewise definitions for the left-hand derivative and right-hand derivative:

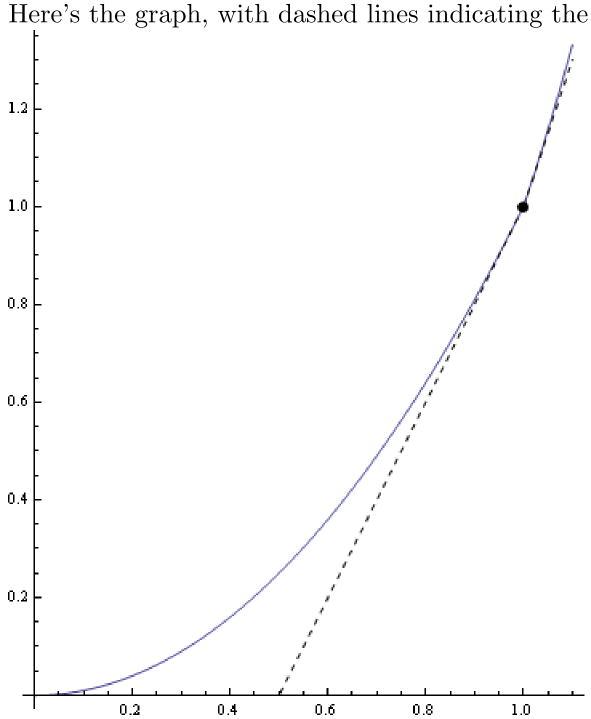
$$\text{LHD of } f \text{ at } x = \{ \begin{array}{ll} 2x, 0 < x \leq 1 \\ 3x^2, & 1 < x \end{array}$$

and:

$$\text{RHD of } f \text{ at } x = \{ \begin{array}{ll} 2x, 0 < x < 1 \\ 3x^2, & 1 \leq x \end{array}$$

The derivative is undefined at 1. Note that both one-sided derivatives are increasing everywhere, and at the point 1, where the function is not differentiable, the right-hand derivative is bigger. Thus, the function is concave up on $(0, \infty)$.

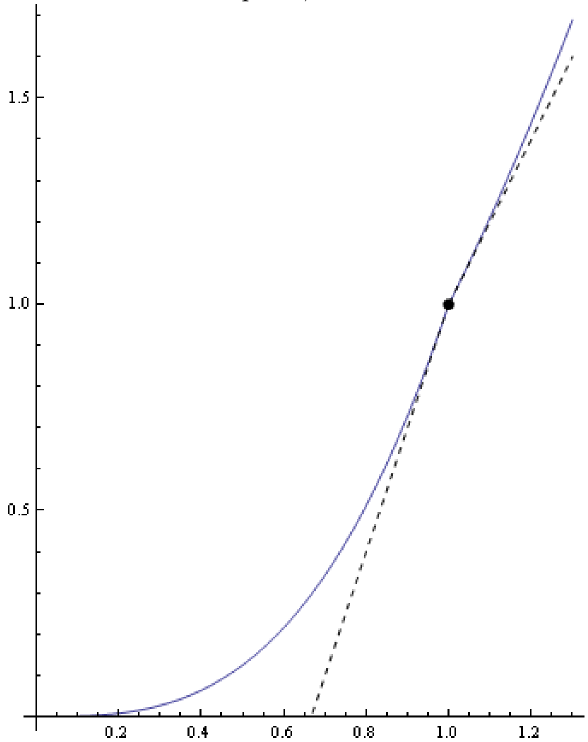
Here's the graph, with dashed lines indicating the one-sided derivatives:



On the other hand, consider the function g on $(0, \infty)$:

$$g(x) := \{ \begin{array}{ll} x^3, & 0 < x \leq 1 \\ x^2, & 1 < x \end{array}$$

The function g is continuous and has one-sided derivatives everywhere. Also note that on the intervals $(0,1)$ and $(1,\infty)$, g is concave up. However, at the critical point 1 where g' is undefined, the right-hand derivative is smaller than the left-hand derivative. Thus, the function is not concave up overall on $(0,\infty)$, because at the critical point, its rate of increase takes a plunge for the worse. Here's the picture of g:



1.9. **Graphical properties of concave functions.** Here are some properties of the graphs of functions that are concave up, which are particularly important in the context of optimization. You should be able to do suitable role changes and obtain corresponding properties for concave down functions. In all the points below, f is a continuous function on an interval $[a,b]$ and is concave up on the interior (a,b).

(1) The only possibilities for the increase-decrease behavior of f are: increasing throughout, decreasing throughout, or decreasing first and then increasing.
(2) In particular, f either has exactly one local minimum or exactly one endpoint minimum, and this local or endpoint minimum is also the absolute minimum.
(3) Also, f cannot have a local maximum in its interior. It has exactly one endpoint maximum, and this is also the absolute maximum.
(4) For any two points x_1, x_2 in the domain of f, the part of the graph of f between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies below the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.
(5) If we assume that f is differentiable on (a,b), the tangent line through any point $(x, f(x))$ for $a < x < b$ does *not* intersect the curve at any other point. In the more general notion where f has one-sided derivatives, both the left and right tangent line satisfy this property.

For concave down functions, the role of minimum and maximum gets interchanged, and in point (3) above, the graph is now above the chord rather than below.

1.10. **Addendum: concave and convex.** The book uses the terminology *concave up* and *concave down*, but it's worth knowing that in much of mathematics as well as applications of mathematics, the term *convex* is used for concave up and the term *concave* is used for concave down. However, there is some confusion about this since some people use *concave* for concave up and *convex* for concave down.

2. Infinity and asymptotes

2.1. **Limits to infinity and vertical asymptotes.** We have already discussed what it means to say $\lim_{x\to c} f(x) = +\infty$, but here's a friendly reminder. It means that as x comes closer and closer to c (from either side), $f(x)$ goes above every finite value and does not then come back down. You can similarly understand what it means to say that $\lim_{x\to c} f(x) = -\infty$. You should also be able to understand the *one-sided versions* of these concepts: $\lim_{x\to c^-} f(x) = \infty$, $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^-} f(x) = -\infty$, and $\lim_{x\to c^+} f(x) = -\infty$.

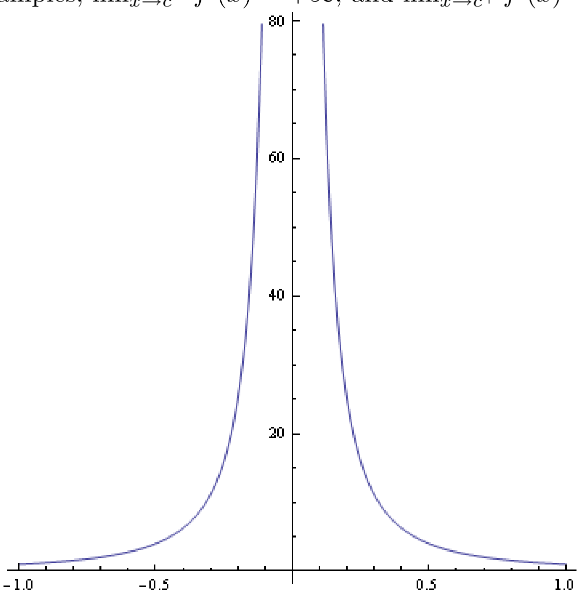
Now, here's a little guesswork question: if $\lim_{x\to c^-} f(x) = \infty$, what can you say about $\lim_{x\to c^-} f'(x)$? As a general rule, nothing, but in most of the situations that we see, it turns out that $f'(x)$ also approaches $+\infty$[1]. The reason is easy to see graphically: for $f(x)$ to head to $+\infty$ as x approaches a finite value, f needs to climb faster and faster and faster.

Similarly, it is usually the case that if $\lim_{x\to c^-} f(x) = -\infty$, then $\lim_{x\to c^-} f'(x) = -\infty$. Also, if $\lim_{x\to c^+} f(x) = \infty$, then it is likely that $\lim_{x\to c^+} f'(x) = -\infty$ (because it has to drop very quickly down from infinity) and if $\lim_{x\to c^+} f(x) = -\infty$, then $\lim_{x\to c^+} f'(x) = \infty$ (because it has to rise very quickly back up from $-\infty$). Again, these things are not always true, but for most of the typical examples you'll see, they will be.

Now here's the meaning of *vertical asymptote*. If $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$, then the line $x = c$ is termed a *vertical asymptote* for f. This is because the graph of f is approaching the vertical line $x = c$. In some sense, if we think of $f(c) = +\infty$ or $-\infty$ as the case may be, the vertical line becomes the tangent line to the curve at that infinite point.

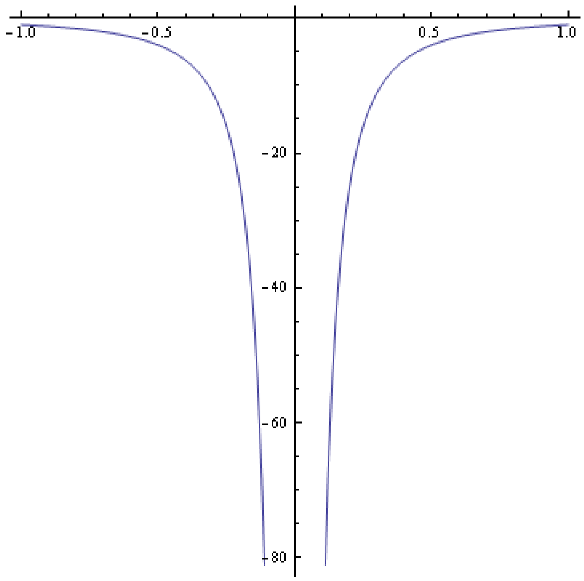
Some of the typical situations worth noting are:

(1) $\lim_{x\to c} f(x) = +\infty$ from both sides. An example of this is the function $f(x) = 1/x^2$ with $c = 0$. The vertical asymptote is the y-axis, i.e., the line $x = 0$. In this case, and in most other representative examples, $\lim_{x\to c^-} f'(x) = +\infty$, and $\lim_{x\to c^+} f'(x) = -\infty$.

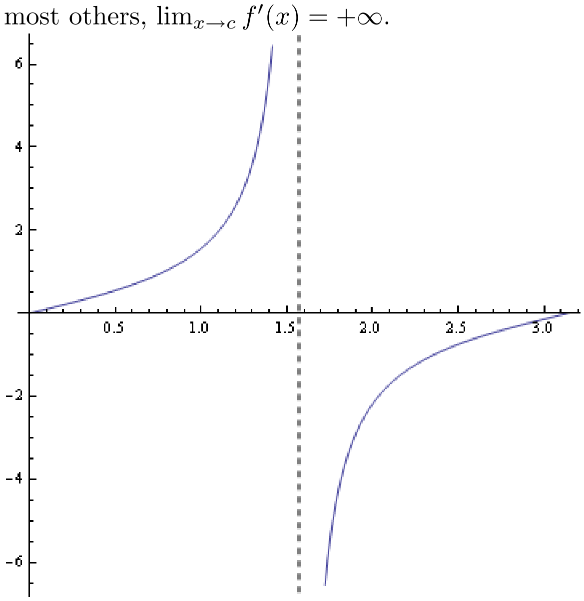


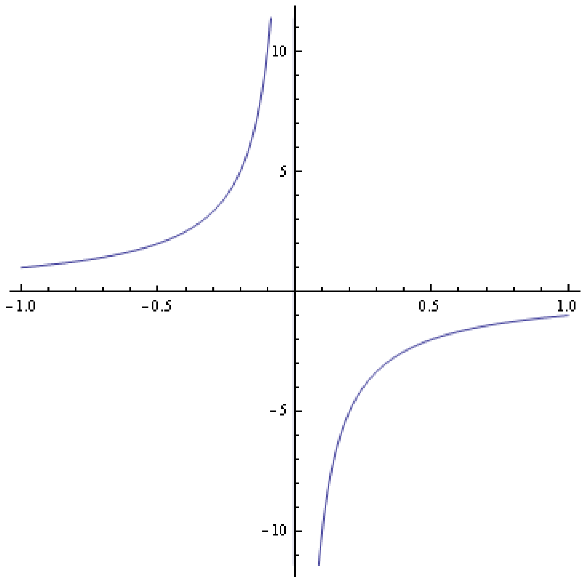
(2) $\lim_{x\to c} f(x) = -\infty$ from both sides. An example of this is the function $f(x) = -1/x^2$ with $c = 0$. The vertical asymptote is the y-axis, i.e., the line $x = 0$. In this case, and in most other representative examples, $\lim_{x\to c^-} f(x) = -\infty$, and $\lim_{x\to c^+} f(x) = +\infty$.

[1] More precisely, it turns out that if f' is continuous and *does* approach something, that something must be $+\infty$. However, there are weird examples where it oscillates

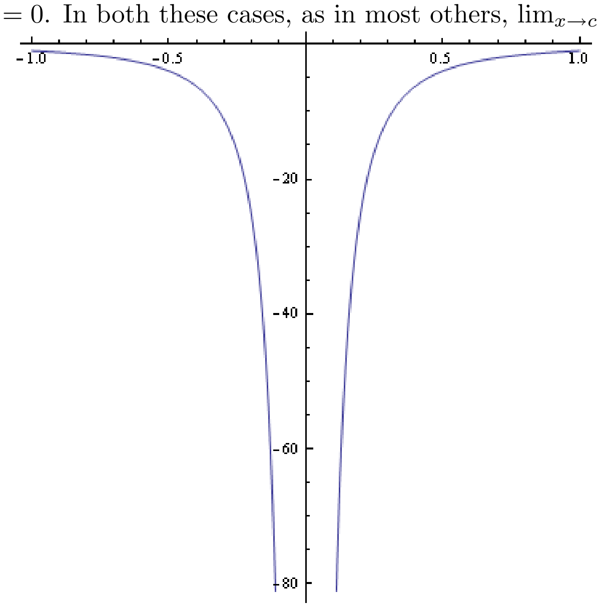


(3) $\lim_{x\to c^-} f(x) = \infty$ and $\lim_{x\to c^+} f(x) = -\infty$. Examples include $f = \tan$ at $c = \pi/2$ (vertical asymptote $x = \pi/2$) and $f(x) = -1/x$ at $c = 0$ (vertical asymptote $x = 0$). In both these cases, as in most others, $\lim_{x\to c} f'(x) = +\infty$.





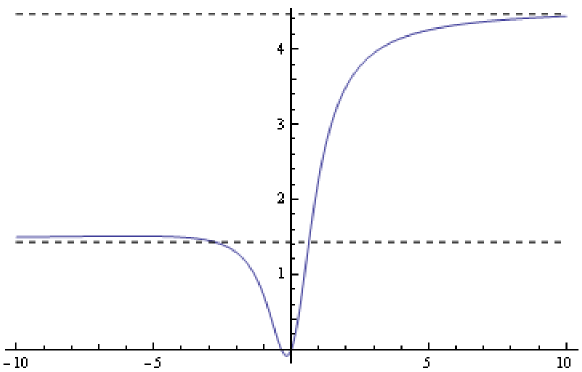
(4) $\lim_{x\to c^-} f(x) = -\infty$ and $\lim_{x\to c^+} f(x) = \infty$. Examples include $f = \cot$ at $c = 0$ and $f(x) := 1/x$ at $c = 0$. In both these cases, as in most others, $\lim_{x\to c} f'(x) = -\infty$.

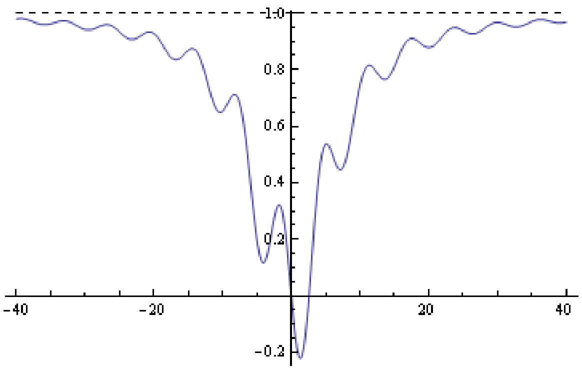


2.2. **Horizontal asymptotes.** Horizontal asymptotes are horizontal lines that the graph comes closer and closer to, just as vertical asymptotes are vertical lines that the graph comes closer and closer to.

We saw that vertical asymptotes arose when the *range value* was approaching $\pm\infty$ for a finite limiting value of the domain. Horizontal asymptotes arise where the *domain value* approaches $\pm\infty$ for a finite limiting value of the range.

Explicitly, if $\lim_{x\to\infty} f(x) = L$ (with L a finite number), then the line $y = L$ is a horizontal asymptote for the graph of f, because as $x \to \infty$, the graph comes closer and closer to this horizontal line. Similarly, if $\lim_{x\to-\infty} f(x) = M$, then the line $y = M$ is a horizontal asymptote for the graph of f. Thus, a function whose domain extends to infinity in both directions could have zero, one, or two horizontal asymptotes.

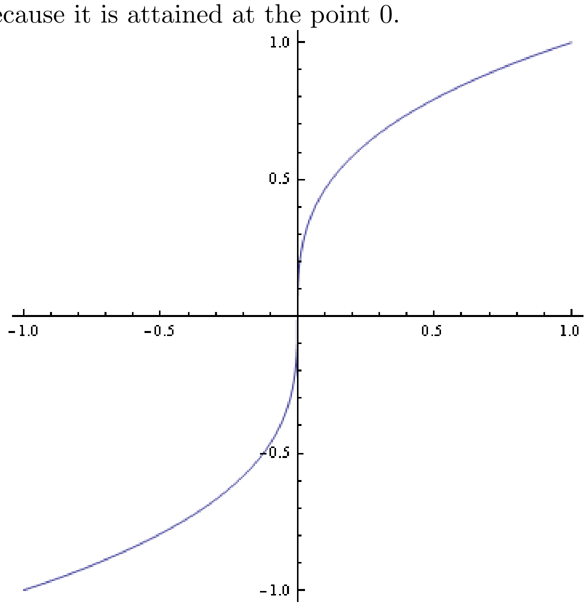




We will discuss some of the computational aspects of vertical and horizontal asymptotes in the problem sessions. Later in the lecture, and in the addendum, we look at some computational tips and guidelines over and above what is there in the book.

2.3. **Vertical tangents.** A vertical tangent to the graph of a function f occurs at a point $(c, f(c))$ if f is continuous but not differentiable at c, and $\lim_{x \to c} f'(x) = +\infty$ or $\lim_{x \to c} f'(x) = -\infty$. *It is important that the sign of infinity in the limit is the same from both the left and the right side.*

An example is the function $f(x) := x^{1/3}$ at the point $c = 0$. The function is continuous at 0. The derivative functions is $(1/3)x^{-2/3}$, and the limit of this as $x \to 0$ (from either side) is $+\infty$. Graphically, what this means is that the tangent is vertical. In this case, the vertical tangent coincides with the y-axis, because it is attained at the point 0.



Points of vertical tangent are points of inflection, as we can see from the $x^{1/3}$ example. Recall that the horizontal tangent case of the point of inflection was typified by x^3, and the general slogan was that the function slows down for an instant to speed zero. For vertical tangents, we can think of it as the function speeding up instantaneously to speed infinity before returning to the realm of finite speed.

It is important to note that the situation of a vertical tangent requires that the function itself be defined and continuous, and hence finite-valued, at the point. Thus, for instance, the function $f(x) := 1/x$ satisfies $\lim_{x\to 0} f'(x) = -\infty$ but does *not* have a vertical tangent at zero because the function is undefined at zero.

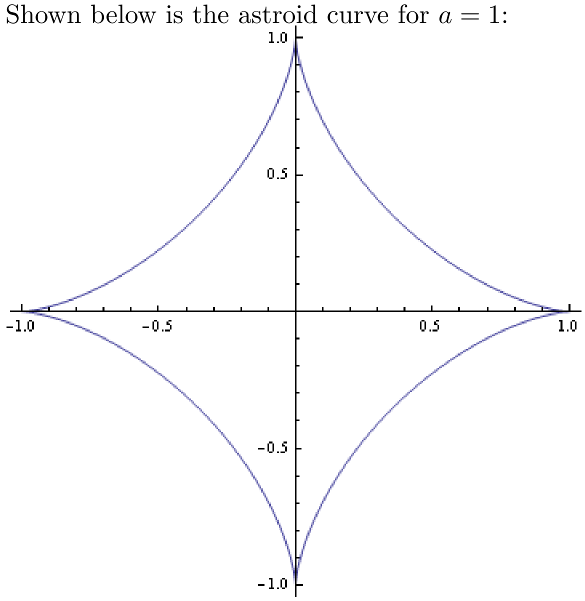
2.4. **Vertical cusps.** A vertical cusp in the graph of f occurs at a point c if f is continuous at c, and both one-sided limits of f' at c are infinities of *opposite* sign. There are two possibilities:

(1) The left-hand limit of the derivative is $+\infty$ and the right-hand limit of the derivative is $-\infty$. Then, $(c, f(c))$ is a point of local maximum. An example is $f(x) := -x^{2/3}$ and $c = 0$ What happens in this situation is the the graph has a sharp peak (picking up to speed infinity) at the point c, after which it rapidly starts dropping.

(2) The left-hand limit of the derivative is $-\infty$ and the right-hand limit of the derivative is $+\infty$. In this case we get a local minimum. An example is $f(x) := x^{2/3}$ and $c = 0$.

There is a special curve called the *astroid curve* (I had planned to put this on the homework, but it went on the chopping block when I needed to trim the homeworks to size), given by the equation $x^{2/3} + y^{2/3} = a^{2/3}$. This curve is not the graph of a function, since every value of x in $(-a, a)$ has two corresponding values of y. Nonetheless, the curve is a good illustration of the concept of cusps: there are two vertical cusps at the points $(0, a)$ and $(0, -a)$ respectively, and two horizontal cusps at the points $(a, 0)$ and $(-a, 0)$ respectively.

Shown below is the astroid curve for $a = 1$:



Note that for the graph of a function, the only kind of cusp that can occur is a vertical cusp, because a horizontal or oblique cusp would result in the curve intersecting a vertical line at multiple points, which would contradict the meaning of a function.

It is important to note that the situation of a vertical cusp requires that the function itself be defined and continuous, and hence finite-valued, at the point. Thus, for instance, the function $f(x) := 1/x^2$ satisfies $\lim_{x\to 0^-} f'(x) = +\infty$ and $\lim_{x\to 0^+} f'(x) = -\infty$ but does *not* have a vertical tangent at zero because the function is undefined at zero.

3. Computational aspects

3.1. **Computing limits at infinity: a review.** We review the main results that you have probably seen and add some more:

(1) $(\to \infty)(\to \infty) = \to \infty$. In other words, if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c} f(x)g(x) = \infty$. c could be finite or $\pm\infty$ here, and we could take one-sided limits instead.

(2) $(\to \infty)(\to -\infty) =\to -\infty$, and $(\to -\infty)(\to -\infty) =\to \infty$.
(3) $(\to a)(\to \infty) =\to \infty$ if $a > 0$ and $(\to a)(\to \infty) =\to -\infty$ if $a < 0$. Similarly, $(\to a)(\to -\infty) =\to -\infty$ if $a > 0$ and $(\to a)(\to -\infty) =\to \infty$ if $a < 0$.
(4) The previous point can be generalized somewhat: $(\to \infty)$, times a function that eventually has a positive lower bound (even if it keeps oscillating), is also $\to \infty$. Analogous results hold for negative upper bounds.
(5) $(\to 0)(\to \infty)$ is an indeterminate form: it is not clear what it tends to without doing more work.
(6) $(\to \infty) + (\to \infty) =\to \infty$.
(7) $(\to \infty) + (\to a) =\to \infty$ where a is finite. More generally, $\to \infty$ plus anything that is bounded from below is also $\to \infty$.
(8) $(\to \infty) - (\to \infty)$ and $(\to \infty) + (\to -\infty)$ are indeterminate forms.

Apart from this, the main facts you need to remember are that if $a > 0$, then $\lim_{x\to\infty} x^a = \infty$ and $\lim_{x\to\infty} x^{-a} = \lim_{x\to-\infty} x^{-a} = 0$. Note that this holds regardless of whether a is an integer.

When a is an odd integer or a rational number with odd numerator and odd denominator, $\lim_{x\to-\infty} x^a = -\infty$. When a is an even integer or a rational number with even numerator and odd denominator, $\lim_{x\to-\infty} x^a = \infty$.

Also worth noting: $\lim_{x\to 0^+} x^{-a} = \infty$ for $a > 0$ and $\lim_{x\to 0^-} x^{-a} = \lim_{x\to-\infty} x^a$, which is computed by the rule above.

We can use these facts to explain most of the limits involving polynomial and rational functions. Earlier, we had noted that when calculating the limit of a polynomial, it is enough to calculate the limit of its leading monomial. Let's now see why.

Consider the function $f(x) := x^7 - 5x^5 + 3x + 2$. Then, we can write $f(x) = x^7\left[1 - 5x^{-2} + 3x^{-6} + 2x^{-7}\right]$. The expression on the inside is 1 plus various negative powers of x. Each of those negative powers of x goes to 0 as $x \to \infty$. So, we obtain:

$$\lim_{x\to\infty}[1 - 5x^{-2} + 3x^{-6} + 2x^{-7}] = 1$$

We also have $\lim_{x\to\infty} x^7 = \infty$. Thus, the limit of the product is ∞.

Let's now consider an example of a rational function:

$$\frac{9x^3 - 3x + 2}{103x^2 - 17x - 99}$$

Earlier, we had discussed that when computing such limits at $\pm\infty$, we can simply calculate thel imits of the leading terms and ignore the rest. We now have a better understanding of the rationale behind this. Formally:

$$\begin{aligned}
&\lim_{x\to\infty} \frac{9x^3 - 3x + 2}{103x^2 - 17x - 99} \\
&= \lim_{x\to\infty} \frac{x^3(9 - 3x^{-2} + 2x^{-3})}{x^2(103 - 17x^{-1} - 99x^{-2})} \\
&= \lim x \to \infty x \lim_{x\to\infty} \frac{9 - 3x^{-2} + 2x^{-3}}{103 - 17x^{-1} - 99x^{-2}} \\
&= \lim_{x\to\infty} x \cdot \frac{9}{103} \\
&= +\infty
\end{aligned}$$

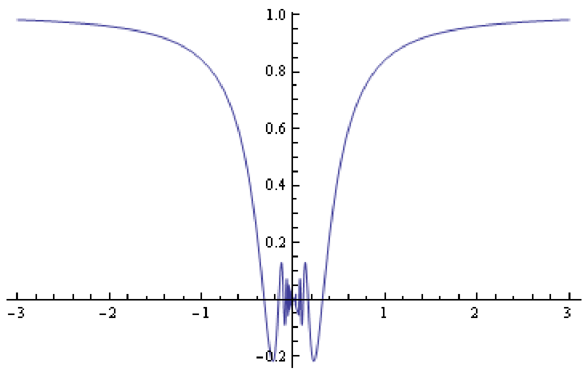
More generally, we see that if the degree of the numerator is greater than the degree of the denominator, the fraction approaches $\pm\infty$ as $x \to \pm\infty$, with the sign depending on the signs of the leading coefficients and the parity (even versus odd) of the exponents.

If the numerator and denominator have equal degree, the limit is a finite number. For $x \to \pm\infty$, it is the ratio of the leading coefficients (notice that it is the same on both sides). This is the case where we get horizontal asymptotes. In this case, the horizontal asymptotes on both ends coincide.

For instance:

$$\lim_{x \to -\infty} \frac{2x^2 - 3x + 5}{23x^2 - x - 1}$$
$$= \lim_{x \to -\infty} \frac{x^2(2 - 3x^{-1} + 5x^{-2})}{x^2(23 - x^{-1} - x^{-2})}$$
$$= \lim_{x \to -\infty} \frac{x^2}{x^2} \lim_{x \to -\infty} \frac{2 - 3x^{-1} + 5x^{-2}}{23 - x^{-1} - x^{-2}}$$
$$= 1 \cdot \frac{2}{23}$$
$$= \frac{2}{23}$$

Finally, when the degree of the numerator is less than the degree of the denominator, then the fraction tends to 0 as $x \to \infty$ and also tends to 0 as $x \to -\infty$. Thus, in this case, we get the x-axis as the horizontal asymptote on both sides.



3.2. **The $1/x$ substitution trick.**

Consider the limit:

$$\lim_{x \to \infty} x \sin(1/x)$$

This limit cannot be computed by plugging in values, because $x \to \infty$ and $1/x \to 0$, so $\sin(1/x) \to 0$, and we get the indeterminate form $(\to \infty)(\to 0)$. The approach we use here is to set $t = 1/x$. As $x \to \infty$, $t \to 0^+$. Since $t = 1/x$, we get $x = 1/t$. Plugging in, we get:

$$\lim_{t \to 0^+} \frac{\sin t}{t}$$

This limit is 1, as we know well.

Note that with this general substitution, limits to infinity correspond to right-hand limits at 0 for the reciprocal and limits at $-\infty$ correspond to left-hand limits at 0 for the reciprocal. If there is a two-sided limit at 0 for the reciprocal, the limits at $\pm\infty$ are the same. In fact, in the $x \sin(1/x)$ example, the limits at ∞ and $-\infty$ are both 1 since $\lim_{t \to 0} \sin t/t = 1$.

3.3. **Difference of square roots.** Consider the limit:

$$\lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x})$$

There are many ways to compute this limit, but the easiest is to use the general $1/x$ substitution trick. Let $t = 1/x$. Then the above limit becomes:

$$\lim_{t \to 0^+} \frac{\sqrt{t+1} - 1}{\sqrt{t}}$$

This is an indeterminate form (specifically, a 0/0 form). However, we can do the rationalization trick and rewrite this as:

$$\lim_{t \to 0^+} \frac{t}{\sqrt{t}(\sqrt{t+1}+1)}$$

The t and $\sqrt{t}$ cancel to give a $\sqrt{t}$ in the numerator, and we can evaluate and find the limit to be 0. A similar approach can be used to handle, for instance, a difference of cube roots.

3.4. **Combinations of polynomial and trigonometric functions.** We illustrate using some examples:

(1) Consider the function $f(x) := x + 25\sin x$. As $x \to \infty$, this is the sum of a function that tends to infinity and a function that oscillates. The oscillating component, however, has a finite lower bound, and hence, $\lim_{x\to\infty} f(x) = \infty$. Similarly, $\lim_{x\to-\infty} f(x) = -\infty$.

(2) Consider the function $f(x) := x\sin x$. As $x \to \infty$, this is the product of a function that goes to ∞ and a function that oscillates between -1 and 1. The oscillating part causes the sign of the whole expression to shift, and so as $x \to \infty$, $f(x)$ is oscillating with an ever-increasing magnitude of oscillation. A similar observation holds for $x \to -\infty$.

(3) Consider the function $f(x) := x(3 + \sin x)$ As $x \to \infty$, this is the product of a function that tends to ∞ and a function that oscillates between 2 and 4. The important point here is that the latter oscillation has a *positive lower bound*, so the product still tends to ∞.

(4) Consider the function $f(x) := x\sin(1/x)$. As $x \to \infty$, this is the product of a function that tends to ∞ and a function that tends to 0, so it is an indeterminate form. We already discussed above how this particular indeterminate form can be handled.