AREA COMPUTATIONS USING INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 5.5, 6.1.

Difficulty level: Moderate. The basic computational ideas are of easy to moderate difficulty, but some of the slicing ideas at the end are somewhat hard.

What students should definitely get: The application of integral to computing areas. The idea of slicing and integration of slice lengths.

Executive summary

Words ...

(1) We can use integration to determine the area of the region between the graph of a function f and the x-axis from $x = a$ to $x = b$: this integral is $\int_a^b f(x)\,dx$. The integral measures the signed area: parts where $f \geq 0$ make positive contributions and parts where $f \leq 0$ make negative contributions. The magnitude-only area is given as $\int_a^b |f(x)|\,dx$. The best way of calculating this is to split $[a,b]$ into sub-intervals such that f has constant sign on each sub-interval, and add up the areas on each sub-interval.
(2) Given two functions f and g, we can measure the area between f and g between $x = a$ and $x = b$ as $\int_a^b |f(x) - g(x)|\,dx$. For practical purposes, we divide into sub-intervals so that on each sub-interval one function is bigger than the other. We then use integration to find the magnitude of the area on each sub-interval and add up. If f and g are both continuous, the points where the functions *cross* each other are points where $f = g$.
(3) Sometimes, we may want to compute areas against the y-axis. The typical strategy for doing this is to interchange the roles of x and y in the above discussion. In particular, we try to express x as a function of y.
(4) An alternative strategy for computing areas against the y-axis is to use formulas for computing areas against the x-axis, and then compute differences of regions.
(5) A general approach for thinking of integration is in terms of slicing and integration. Here, integration along the x-axis is based on the following idea: divide the region into vertical slices, and then integrate the lengths of these slices along the horizontal dimension. Regions for which this works best are the regions called *Type I regions*. These are the regions for which the intersection with any vertical line is either empty or a point or a line segment, hence it has a well-defined length.
(6) Correspondingly, integration along the y-axis is based on dividing the region into horizontal slices, and integrating the lengths of these slices along the vertical dimension. Regions for which this works best are the regions called *Type II regions*. These are the regions for which the intersection with any horizontal line is either empty or a point or a line segment, hence it has a well-defined length.
(7) Generalizing from both of these, we see that our general strategy is to choose two perpendicular directions in the plane, one being the direction of our slices and the other being the direction of integration.

Actions ...

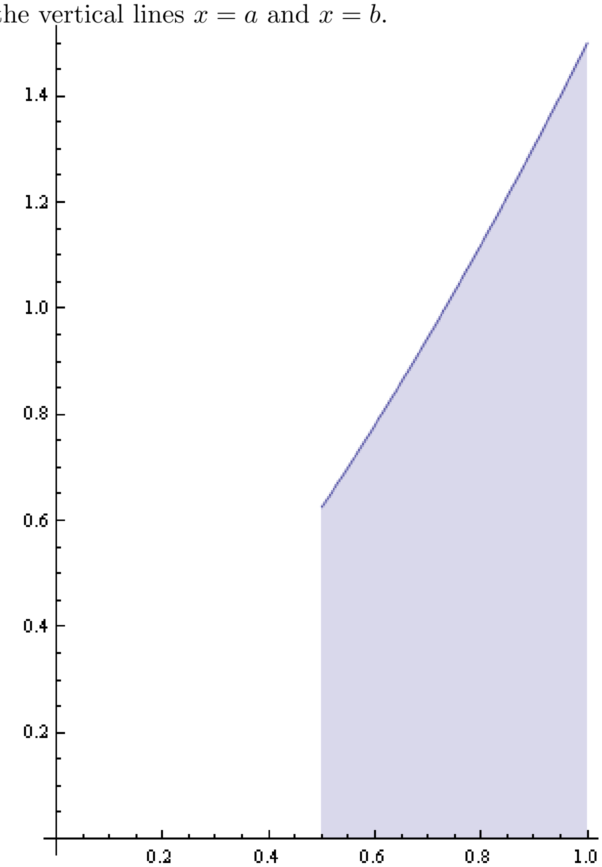
(1) In some situations we are directly given functions and/or curves and are asked to find areas. In others, we are given real-world situations where we need to find areas of regions. Here, we have to find functions and set up the integration problem as an intermediate step.
(2) In all these situations, it is important to draw the graphs in a reasonably correct way. This brings us to all the ideas that are contained in graph drawing. Remember, here we may be interested in simultaneously graphing more than one function. Thus, in addition to being careful about each function, we should also correctly estimate where one function is bigger than the other, and find

(approximately or exactly) the intersection points. (Go over the notes on graph-drawing, and some additional notes on graphing that weren't completely covered in class).

(3) In some situations, we are asked to find the area(s) of region(s) bounded by the graphs of one, two, three, or more functions. Here, we first need to sketch the figure. Then, we need to find the interval of integration, and if necessary, split this interval into sub-intervals, such that on each sub-interval, we know exactly what integral we need to do. For instance, consider the region between the graphs of sin, cos, and the x-axis. Basically, the idea is to find, for all the vertical slices, the upper and lower limits of the slice.

1. Integral and area: against the x-axis

1.1. **Definite integral as the signed area between the graph and the x-axis.** Suppose f is a continuous function on a closed interval $[a, b]$. The graph of f forms a curve in the plane $\mathbb{R}^2$. Consider the signed area between this curve and the x-axis. This is the area of the region bounded by the graph, the x-axis, and the vertical lines $x = a$ and $x = b$.



The basic result of integration is that this area equals the definite integral

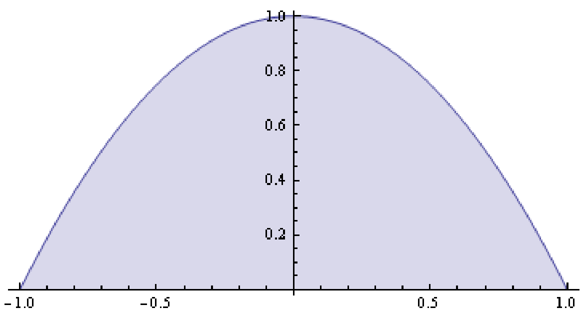
$$\int_a^b f(x)\,dx$$

If $f(x) \geq 0$ for all $x \in [a, b]$, i.e., if the graph is entirely in the upper half-plane (possibly hitting the boundary x-axis), then this integral is nonnegative, and its value is the magnitude of the area. If $f(x) \leq 0$ for all $x \in [a, b]$, i.e., if the graph is entirely in the lower half-plane (possibly hitting the boundary x-axis), then this integral is zero or negative, and its value is the *negative* of the magnitude of the area. If the function has parts where it is positive and parts where it is negative, then the parts where it is positive make positive contributions and the parts where it is negative make negative contributions.

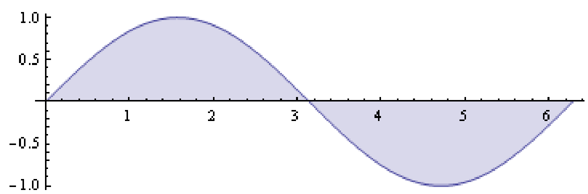
For instance, consider the function $f(x) := 1 - x^2$. We want to find the area between the x-axis and the graph of the part of this function that is above the x-axis.

First, note that the graph is above the x-axis on $(-1,1)$. Thus, in order to find the area, we need to perform the integration:

$$\int_{-1}^{1} (1-x^2)\,dx$$



We can do this integration by finding an antiderivative and evaluating it between limits. We take $x-x^3/3$ as the antiderivative. Evaluating it between limits gives the value $4/3$. Thus, the area of the region we are interested in is $4/3$.



1.2. **Measuring unsigned area.**

Suppose we want to measure the total area between the graph of the sine curve and the x-axis over one period, say $[0,2\pi]$. In other words, we want to compute the integral

$$\int_{0}^{2\pi} \sin x\,dx$$

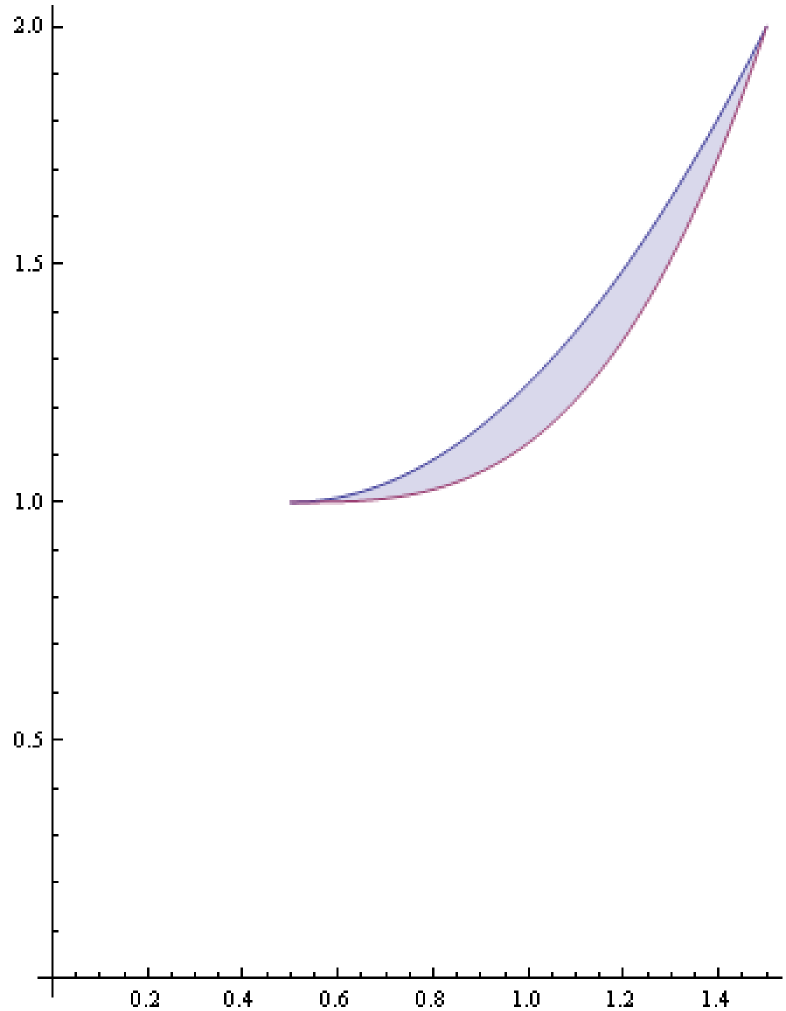
We know that $-\cos$, which is an antiderivative for sin, also has a period of 2π. Hence, its value between limits is zero, so the above integral is zero. Thus, the total *signed area* between the graph of the sine curve and the x-axis is zero. This makes sense graphically. The positive area between the sine curve and the x-axis on the interval $[0,\pi]$ is canceled by a negative area of equal magnitude between the x-axis and the sine curve on the interval $[\pi,2\pi]$. Why are the two areas the same? There are plenty of ways of seeing this geometrically. For instance, we have $\sin(\pi+\theta) = -\sin\theta$ for all angles θ.

Suppose now that, instead of measuring the signed area, we are interested in measuring the unsigned area. The *unsigned area* between the graph of a function f and the x-axis on an interval $[a,b]$ is given by

$$\int_{a}^{b} |f(x)|\,dx$$

Equivalently, we break the interval $[a,b]$ into subintervals such that $f \geq 0$ or $f \leq 0$ on each subinterval. Then we calculate the magnitude of the integral on each subinterval and add these magnitudes.

In the case of the sine function, we can partition $[0,2\pi]$ at π, to get the subintervals $[0,\pi]$ and $[\pi,2\pi]$. On $[0,\pi]$, the integral is $[-\cos x]_0^\pi$, which simplifies to 2. On $[\pi,2\pi]$, the integral is -2, and its magnitude is 2. The total magnitude of the integral is thus $2+2$, and we know that $2+2=4$. Hence, the unsigned area between the graph of sin and the x-axis on $[0,2\pi]$ is 4.



1.3. **Area between two graphs.**

Suppose f and g are two continuous functions. To measure the signed area between the graphs of f and g between the points a and b, we compute the integral

$$\int_a^b [f(x) - g(x)]\, dx$$

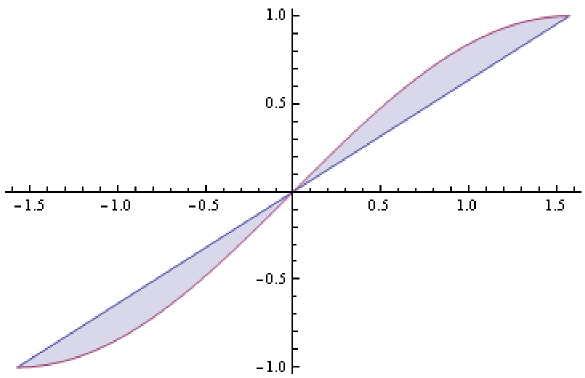
Here, the subintervals where f is bigger than g make positive contributions and the subintervals where g is bigger than f make negative contributions. If we are interested in the unsigned area, whereby we want positive contributions regardless of which function is bigger, we consider the integral

$$\int_a^b |f(x) - g(x)|\, dx$$

To compute this, we break up the interval $[a, b]$ into subintervals based on whether f or g is smaller (the overtaking can happen at points where $f(x) = g(x)$). We then compute the integral of $f - g$ (or $g - f$, depending on which is bigger) on each subinterval and add up the magnitudes.

For instance, consider the unsigned area between the graphs of $f(x) = 2x/\pi$ and $g(x) = \sin x$ on the interval $[-\pi/2, \pi/2]$. We see that $f(x) = g(x)$ at $-\pi/2, 0, \pi/2$. On $(-\pi/2, 0)$, $f(x) > g(x)$, and on $(0, \pi/2)$, $g(x) > f(x)$. Thus, the integral is:

$$\int_{-\pi/2}^0 (2x/\pi - \sin x)\, dx + \int_0^{\pi/2} (\sin x - 2x/\pi)\, dx$$

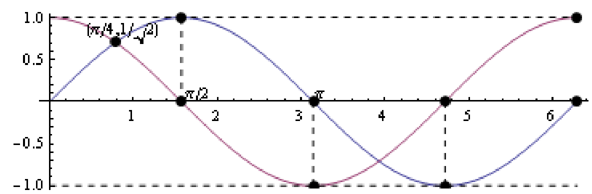


We can calculate and simplify both these integrals. Note that instead of computing indefinite integrals for both separately, we can note that the two functions are negatives of each other, so if we compute an antiderivative for the first, the antiderivative for the second is its negative. We get:

$$[x^2/\pi + \cos x]_{-\pi/2}^{0} + [-\cos x - x^2/\pi]_0^{\pi/2}$$

Both parts are $1 - \pi/4$, and we thus get $2 - \pi/2$. Since π is approximately 3.14, this is approximately 0.43.

Why are the two integrals the same? This can be seen geometrically from the fact that both f and g are odd, so the picture from $-\pi/2$ to 0 is the same as the picture from 0 to $\pi/2$, subjected to a half-turn about the origin. Thus, the magnitude of the two areas is the same.



1.4. **Areas bounded by graphs of different functions.**

Sometimes, the bounding curves for an area come from different functions. In this case, it makes sense to split up the interval of integration into subintervals so that we are dealing with only one function in each subinterval. For instance, consider computing the area of the region between the x-axis and the graphs of sin and cos on the interval $[0, \pi/2]$. On the interval $[0, \pi/4]$, this is the definite integral of the sin function, and on the interval $[\pi/4, \pi/2]$, this is the definite integral of the cos function. The total area is the sum of the values of these two definite integrals.

As we can see, both integrals are $1 - 1/\sqrt{2}$, and the total integral is $2 - \sqrt{2}$, which is approximately 0.59.

Why are the two integrals the same? We can see graphically that the two areas being measured are mirror images of each other about the line $x = \pi/4$. This is because for any angle θ, $\cos(\pi/2 - \theta) = \sin\theta$.

For the second midterm, you are responsible only for the material till this point.

2. Area computations as slicing, and other methods

This way of thinking about area computations will turn out to be useful for the subsequent topic, which is volume computations. It also makes it possible to compute areas of shapes oriented somewhat differently from before.

2.1. **Vertical slicing.**

So far, the situations where we've been computing areas are: area between the graph of a function and the x-axis, area bounded between graph of a function, the x-axis, and two vertical lines, area between the graphs of two functions, area bounded by the graphs of two functions and two vertical lines.

In all these situations, the region Ω whose area we need to compute has the property that the intersection of Ω with any vertical line is either empty or a line segment. Regions of this kind are sometimes called Type I regions. For Type I regions, the general formula for the unsigned area is:

$$\int (\text{Length of the line segment as a function of } x)\, dx$$

This process can be thought of as *vertical slicing*. We are dividing the area that we want to measure into vertical slices, and then integrating the length along the perpendicular axis (which is horizontal).

2.2. **Horizontal slicing.** Horizontal slicing is a lot like vertical slicing, but works for regions where the role of vertical and horizontal is replaced.

Horizontal slicing works for regions Ω which have the property: the intersection of Ω with any horizontal line is either empty or a line segment. Regions of this type are sometimes called Type II regions. The formula for the area of a Type II region is

$$\int (\text{Length of the line segment as a function of } y)\, dy$$

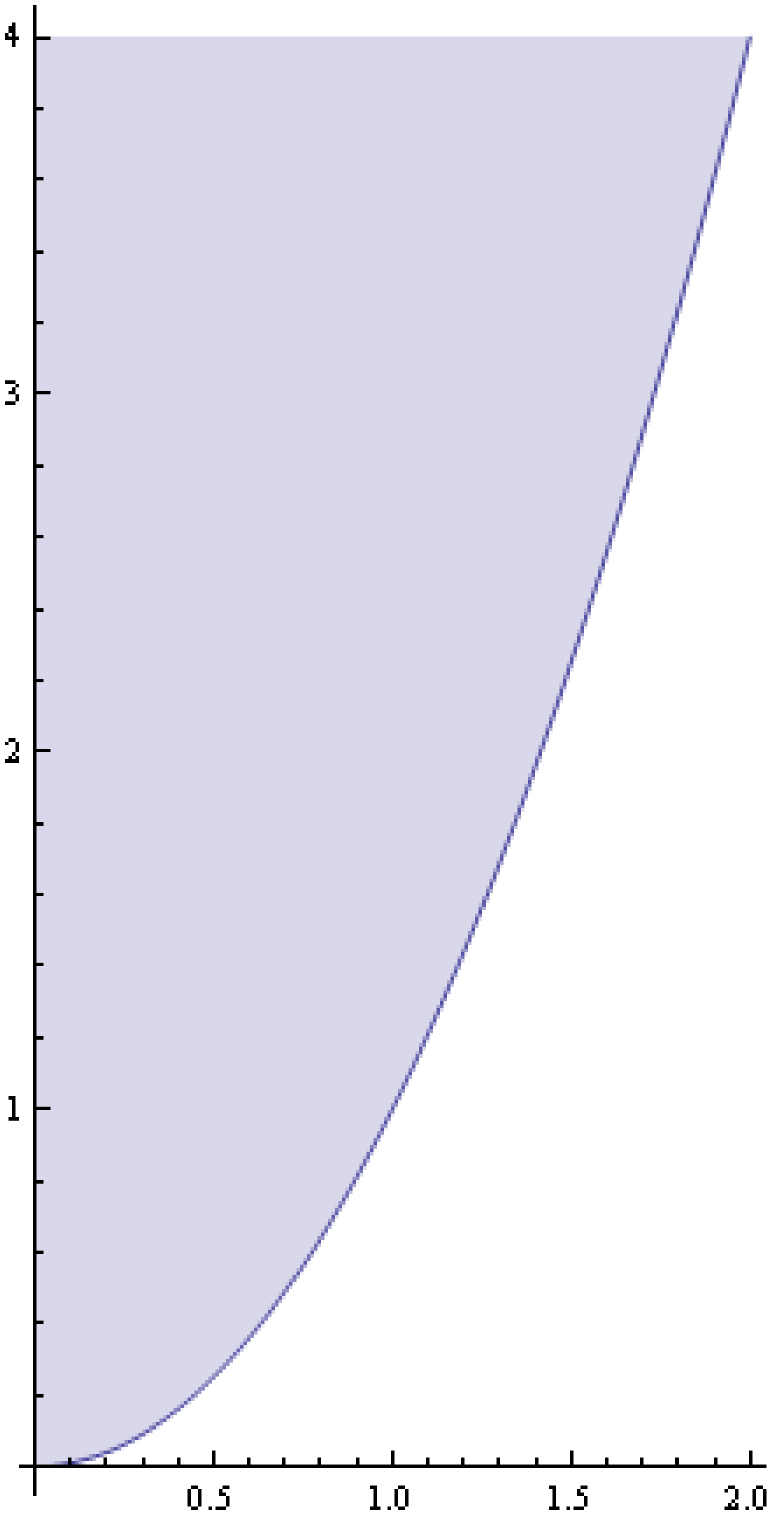
This process can be thought of as *horizontal slicing*. We are dividing the area that we want to measure into horizontal slices, and the integrating the length along the perpendicular axis (which is vertical).

Thus, we have seen two processes of breaking up an area into slices: vertical slicing (where we integrate the lengths along a horizontal axis) and horizontal slicing (where we integrate the lengths along a vertical axis).

Notice that both these procedures are variants of the same basic procedure: choose two mutually perpendicular directions, such that all lines in one direction have intersection with the region that is either empty or a line segment. Then, integrate the length of the line segment along the perpendicular direction.

Note also that the extreme case of both these occurs in rectangles. Here, whether we use horizontal or vertical slicing, we are integrating a constant function.

2.3. **Regions whose area can be computed by integration in multiple ways.** Consider the region bounded by the line $y = 4$, $y = x^2$, and the y-axis. This is both a Type I and a Type II region, so we can determine its area by vertical slicing as well as by horizontal slicing. Let's first compute the area by vertical slicing.



By vertical slicing, the interval is $[0, 2]$, and the lower and upper functions are x^2 and 4 respectively. Thus, the length of the line segment in each vertical slice is $4 - x^2$. The area is thus:

$$\int_0^2 (4 - x^2)\, dx = [4x - (x^3/3)]_0^2 = 8 - 8/3 = 16/3$$

We could also integrate using horizontal slicing. For this, we express x in terms of y. We get $x = \sqrt{y}$, with $y \in [0, 4]$. Measuring the area between this and the y-axis, we get:

$$\int_0^4 \sqrt{y}\, dy = [y^{3/2}/(3/2)]_0^4 = 8/(3/2) = 16/3$$

When we later introduce the concept of *inverse function*, we will notice that what we've just done is moved from integrating one function to integrating its inverse function. We'll also see a relationship between this and integration by parts next quarter.

3. Areas of regions given by inequalities

Suppose a region of the plane is defined by a set of inequalities. In other words, the region is defined as the set of all points in the plane that satisfy a given system of inequalities. How do we find its area?

The first step is to identify the *bounding* lines/curves for this region. The bounding lines are typically the lines given by the case where equality holds instead of inequality. Once we have found these boundary curves, we can then try to use horizontal or vertical slicing to determine the area. In some cases, it makes sense to divide the region into sub-regions so that it is easy to tackle each sub-region separately by slicing.

Another complication is that the boundary curves may not be graphs of functions. Often, they may be graphs of relations, i.e., the set of points (x, y) satisfying $F(x, y) = 0$ for some two-variable function F. In these cases, we try to break it up into functions. We consider some examples.

3.1. **The example of the circular disk.** Consider the region $1 \leq x^2 + y^2 \leq 2$. In other words, we are looking at the set of points (x, y) such that $x^2 + y^2 \in [1, 2]$. We easily see graphically that this region is bounded on the inside by the circle $x^2 + y^2 = 1$ and on the outside by $x^2 + y^2 = 2$. The region is called a *circular annulus*. To find the area of the annulus, we thus need to subtract the area of the disk $x^2 + y^2 \leq 1$ from the area of the disk $x^2 + y^2 \leq 2$.

Now, it so happens that we know formulas for the areas of these disks: they are π and 2π respectively, so the difference of areas is $2\pi - \pi = \pi$. If we did not know these formulas, we would need to break up the circle into graphs of functions $\pm\sqrt{r^2 - x^2}$. Unfortunately, integrating these functions requires a trigonometric substitutions, so illustrating this idea would take us too far afield.