

EXPONENTIATION WITH ARBITRARY BASES, EXPONENTS

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 7.5.

What students should definitely get: The definition of a^b , where $a > 0$ and b is real. The definition of logarithm to positive base. The method of differentiating functions where the exponent (or base of logarithm) itself is variable. Key properties of exponents and logarithms.

EXECUTIVE SUMMARY

Words ...

- (1) For $a > 0$ and b real, we define $a^b := \exp(b \ln a)$. This coincides with the usual definition when b is rational.
- (2) All the laws of exponents that we are familiar with for integer and rational exponents continue to hold. In particular, $a^0 = 1$, $a^{b+c} = a^b \cdot a^c$, $a^1 = a$, and $a^{bc} = (a^b)^c$.
- (3) The exponentiation function is continuous in the exponent variable. In particular, for a fixed value of $a > 0$, the function $x \mapsto a^x$ is continuous. When $a \neq 1$, it is also one-to-one with domain \mathbb{R} and range $(0, \infty)$, with inverse function $y \mapsto (\ln y)/(\ln a)$, which is also written as $\log_a(y)$. In the case $a > 1$, it is an increasing function, and in the case $a < 1$, it is a decreasing function.
- (4) The exponentiation function is also continuous in the base variable. In particular, for a fixed value of b , the function $x \mapsto x^b$ is continuous. When $b \neq 0$, it is a one-to-one function with domain and range both $(0, \infty)$, and the inverse function is $y \mapsto y^{1/b}$. In case $b > 0$, the function is increasing, and in case $b < 0$, the function is decreasing.
- (5) Actually, we can say something stronger about a^b – it is *jointly* continuous in both variables. This is hard to describe formally here, but what it approximately means is that if f and g are both continuous functions, and f takes positive values only, then $x \mapsto [f(x)]^{g(x)}$ is also continuous.
- (6) The derivative of the function $[f(x)]^{g(x)}$ is $[f(x)]^{g(x)}$ times the derivative of its logarithm, which is $g(x) \ln(f(x))$. We can further simplify this to obtain the formula:

$$\frac{d}{dx} \left([f(x)]^{g(x)} \right) = [f(x)]^{g(x)} \left[\frac{g(x)f'(x)}{f(x)} + g'(x) \ln(f(x)) \right]$$

- (7) Special cases worth noting: the derivative of $(f(x))^r$ is $r(f(x))^{r-1} f'(x)$ and the derivative of $a^{g(x)}$ is $a^{g(x)} g'(x) \ln a$.
- (8) Even further special cases: the derivative of x^r is rx^{r-1} and the derivative of a^x is $a^x \ln a$.
- (9) The antiderivative of x^r is $x^{r+1}/(r+1) + C$ (for $r \neq -1$) and $\ln|x| + C$ for $r = -1$. The antiderivative of a^x is $a^x/(\ln a) + C$ for $a \neq 1$ and $x + C$ for $a = 1$.
- (10) The logarithm $\log_a(b)$ is defined as $(\ln b)/(\ln a)$. This is called the logarithm of b to base a . Note that this is defined when a and b are both positive and $a \neq 1$. This satisfies a bunch of identities, most of which are direct consequences of identities for the natural logarithm. In particular, $\log_a(bc) = \log_a(b) + \log_a(c)$, $\log_a(b) \log_b(c) = \log_a(c)$, $\log_a(1) = 0$, $\log_a(a) = 1$, $\log_a(a^r) = r$, $\log_a(b) \cdot \log_b(a) = 1$ and so on.

Actions...

- (1) We can use the formulas here to differentiate expressions of the form $f(x)^{g(x)}$, and even to differentiate longer exponent towers (such as x^{x^x} and 2^{2^x}).
- (2) To solve an integration problem with exponents, it may be most prudent to rewrite a^b as $\exp(b \ln a)$ and work from there onward using the rules mastered earlier. Similarly, when dealing with relative logarithms, it may be most prudent to convert all expressions in terms of natural logarithms and then use the rules mastered earlier.

1. REVIEW AND DEFINITIONS

1.1. Exponents: what we know. Let us consider the expression a^b , with a positive. So far, we have made sense of this expression in the following cases:

- (1) b is a positive integer: In this case, a^b is defined as the product of a with itself b times. This definition is fairly general; in fact, it makes sense even when $a < 0$.
- (2) b is an integer: If b is positive, we use (1). If $b = 0$, we define a^b as 1, and if $b < 0$, we define a^b as $1/a^{|b|}$.
- (3) $b = p/q$ is rational, p, q integers, $q > 0$: In this case, a^b is defined as the unique positive real number c such that $c^q = a^p$. The existence of such a c was not proved rigorously, but it essentially follows by an application of the intermediate value theorem. We proceed as follows: we show that the function $x \mapsto x^q$ is less than a^p for some positive x and greater than a^p for some positive x , and hence, by the intermediate value theorem, it must equal a^p for some positive x . The uniqueness of this is guaranteed by the fact that the function $x \mapsto x^q$ is increasing.

The notion of rational exponent has the added advantage that when the denominator is odd, it can be extended to the negative numbers as well. Also, note that when $b > 0$, we define $0^b = 0$.

So far, exponents satisfy some laws, namely:

$$\begin{aligned} a^0 &= 1 \\ a^{b+c} &= a^b \cdot a^c \\ a^{-b} &= \frac{1}{a^b} \\ a^{bc} &= (a^b)^c \end{aligned}$$

For the rest of this document, where we study arbitrary real exponents b , we restrict ourselves to the situation where the base a of exponentiation is positive.

1.2. And here's how mathematicians would think about it. We're not mathematicians, but since we're doing mathematics, it might help to think about the way mathematicians would view this. A mathematician would begin by defining positive exponents: things like a^b where a is a positive real and b is a positive integer. Then, the mathematician would observe that $a^{b+c} = a^b \cdot a^c$ and $a^{bc} = (a^b)^c$. The mathematician would then ask: is there a way of extending the definition to encompass more values of b while preserving these two laws of exponents? Further, is the way more or less unique or are there multiple different extensions?

It turns out that there is a way, and it is unique, and it is exactly the way I mentioned above. In other words, if we want exponentiation to behave such that $a^{b+c} = a^b \cdot a^c$ and if we have it defined the usual way for positive integers b , we are forced to define it the usual way for all integers b . Further, we are forced to define it the way we have defined it for rational numbers b . The rules constrain us.

1.3. Real exponents: continuity the realistic constraint. We now move to the situation involving real exponents. Unfortunately, the laws of exponents are not enough to force us to a specific definition of a^b for a positive real and b real. However, the laws of exponents, along with *continuity in both a and b* do turn out to be enough to force a specific definition of a^b . To see this, note that we already have defined a^b for b rational, and the rationals are *dense* in the reals, so to figure out the answer a^b for a real value of b , we take rationals c closer and closer to b and consider the limit of a^c .

For instance, to figure out $2^{\sqrt{2}}$, we look at $2, 2^{1.4}, 2^{1.41}$, and so on. Each of these is well-defined, because in each case, the exponent is a rational number. Thus, $2^{1.4}$ is the unique positive solution to $x^5 = 2^7$, and $2^{1.41}$ is the unique positive solution to $x^{100} = 2^{141}$.

However, we would ideally like a clearer description that does not involve this approximation procedure. It turns out that the exponentiation function (obtained as the inverse function to the natural logarithm function) works out.

We define:

$$a^b := \exp(b \ln a) = e^{b \ln a}$$

Note that I use the \exp notation because I want to emphasize that \exp is just the inverse function to \ln ; it does not have an *a priori* meaning as exponentiation. Note that this definition coincides with the usual definition for positive integer exponents, because $a^b = a \cdot a \cdot \dots \cdot a$ b times, and thus we get:

$$\ln(a^b) = b \ln a$$

Applying \exp to both sides gives:

$$a^b = \exp(b \ln a)$$

Similarly, we can show that the definition $a^b := \exp(b \ln a)$ coincides with the usual definition we have for negative integer exponents and for all rational exponents. Thus, this new definition extends the old definition.

Next, we can use the properties of exponents and logarithms to verify:

- (1) With this new definition, the general laws for exponents listed above continue to hold.
- (2) Under this new definition of exponent, a^b is continuous in each variable. In other words, for any fixed a , it is a continuous function of b , and for a fixed b , it is a continuous function of a . Actually, something stronger holds; it is jointly continuous in the two variables. However, joint continuity is a concept that is based on ideas of multivariable calculus, and hence beyond our scope.
- (3) For a fixed $a \neq 1$, the function $x \mapsto a^x$ is a one-to-one function from \mathbb{R} to $(0, \infty)$. When $a > 1$ the function is increasing, and when $a < 1$, the function is decreasing.
- (4) For a fixed $b \neq 0$, the function is a one-to-one function from $(0, \infty)$ to $(0, \infty)$. When $b > 0$, it is an increasing function, and when $b < 0$, it is a decreasing function.

2. MORE PERSPECTIVE: EXPONENTS, LOGARITHMS, AND RADICALS

2.1. Addition, multiplication, and exponentiation. The *inverse operation* corresponding to addition is subtraction, and the *inverse operation* corresponding to multiplication is division. What is the inverse operation corresponding to exponentiation? The answer turns out to be tricky, because there are a couple of nice things about addition and multiplication that are no longer true for exponentiation:

- (1) Addition and multiplication are both *commutative*: We have the remarkable fact that $a + b = b + a$ and $ab = ba$ for any a and b . On the other hand, exponentiation is not commutative. In fact, as we might see some time later, for every $a \in (0, \infty)$, there exist at most two values of $b \in (0, \infty)$ such that $a^b = b^a$, and one of those two values is a . For instance, the only numbers b for which $2^b = b^2$ are $b = 2$ and $b = 4$.
- (2) Addition and multiplication are both *associative*: We have the remarkable fact that $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$. On the other hand, exponentiation is not *associative*, i.e., it is not true in general that $a^{(b^c)} = (a^b)^c$. This is because $(a^b)^c = a^{bc}$, and so the equality would give that $b^c = bc$, which is very rare.

The noncommutativity of exponentiation would mean that there are two notions of inverse operation: a left inverse operation and a right inverse operation. The nonassociativity would mean that these inverse operations behave very differently from subtraction and division, and the analogy cannot be stretched too far.

2.2. Radicals and logarithms. There are two kinds of inverse operations to the a^b operation. The first is to find solutions x to the equation:

$$x^b = c$$

Such a solution is a b^{th} root or b^{th} radical of c , and is given by:

$$x = c^{1/b}$$

It is also denoted as:

$$x = {}^b\sqrt{c}$$

Note that as per our general discussion of the a^b function, we see that this is well-defined and unique if $b \neq 0$.

The other kind of inverse operation we may want is to solve the equation:

$$a^x = c$$

To solve this, we need to use the definition:

$$\exp(x \ln a) = c$$

Taking \ln both sides, we obtain:

$$x \ln a = \ln c$$

Thus, we get that if $a \neq 1$:

$$x = \frac{\ln c}{\ln a}$$

Note that the uniqueness of x corresponds to the fact, observed earlier, that for fixed a , the map $x \mapsto a^x$ is one-to-one.

This solution x is also written as:

$$x = \log_a c$$

This is often read as *logarithm of c to base a* . The map $c \mapsto \log_a(c)$ is called *taking logarithms to base a* . As shown above, it is equivalent to taking natural logarithms and dividing by $\ln a$.

The upshot is that we define:

$$\log_a(c) := \frac{\ln c}{\ln a}$$

where a, c are both positive and $a \neq 1$. In particular, logarithms cannot be taken to base 1. Taking logarithms to base 1 is like division by zero, a forbidden operation.

2.3. Properties of logarithms. We have the following notable properties of logarithms, where we assume that all elements appearing in the base of the logarithm are positive and not equal to 1, and all elements whose logarithm is being taken are positive:

$$\begin{aligned} \log_a(bc) &= \log_a(b) + \log_a(c) \\ \log_a(b^c) &= c \log_a(b) \\ \log_a(1/b) &= -\log_a(b) \\ \log_a(1) &= 0 \\ \log_a(a) &= 1 \\ \log_a(a^r) &= r \\ \log_{1/a}(b) &= -\log_a(b) \\ \log_a(b) \log_b(c) &= \log_a(c) \\ \log_b(a) &= \frac{1}{\log_a(b)} \end{aligned}$$

All of these follow from the definition and the corresponding properties of \ln , which follow from its definition as the antiderivative of the reciprocal function.

2.4. Absolute and relative: natural bases and the bases for their naturality. We can think of logarithm to a given base as measuring a *relative logarithm*. The natural logarithm is the logarithm to base e , which is the *natural logarithm*, in that the base e is the natural choice for a base. This behavior of logarithms is very similar to, for instance, the behavior of refractive indices for pairs of media through which light travels. For any pair of media, we can define a refractive index of the pair, but the *natural base* with respect to which we measure refractive index is vacuum. Natural logarithms play the role of vacuum: the natural base choice.

There are two other common choices of base for logarithms. One is base 10, which has no sound mathematical reason. The reason for taking logarithms to base 10 is because it is easy to compute the logarithm of any number by writing it in scientific notation using a table of logarithm values for numbers from 1 to 10. This allows us to use logarithms to base 10 as a convenient tool for multiplication, a convenience that seems to have been rendered moot in recent times with the proliferation of calculators.

The second natural choice of base for logarithms, which we will talk about next time, is logarithms to base 2. These come up for three reasons, listed below. We will return to some of these reasons in more detail when we study exponential growth and decay next quarter.

- (1) Halving and doubling are operations to which humans relate easily. We measure and record the *half-life* of radioactive substances, talk of the time it takes for a country to *double* its GDP, and routinely hear campaign rhetoric and promotional NGO material that talks of *doubling* and *halving* arbitrary indicators. The reason is not that 2 has any special mathematical or real-world significance (or plausibility, in the case of politicians and NGOs) but rather, that it is easy for people to (believe they) understand. It's a lot less exciting to make a campaign promise to multiply the number of tax breaks or subsidies or scholarships by e , even though $e > 2$.
- (2) The second reason is perhaps more legitimate. In computer science algorithms, it is customary to use *divide-and-conquer strategies* that work by breaking a problem up into two roughly equal subproblems, and solving both of them separately. The amount of time and resources needed to solve problems using such strategies typically involves a logarithm to base 2, since that is the number of times you need to keep dividing the problem into two equal parts until you get to problems of size 1. Thus, logarithms to base 2 frequently pop up in measuring the time and space requirements of algorithms. Similarly, in psychology and the study of human cognition and information processing, logarithms to base 2 play a role if we hypothesize that humans perform complex tasks by using divide-and-conquer strategies.
- (3) The third reason has to do with biology, more specifically with the reproduction strategies of some unicellular organisms. These organisms divide into two organisms. This form of asexual reproduction is termed *binary fission*. Other reproduction strategies or behaviors may also be associated with logarithms to base 2 or 3, because of the discrete nature of numbers of offspring and number of parents.

One way of thinking about this is that logarithms to base 2 are more natural when working with *finite, discrete problems* while logarithms to base e are more common when dealing with *continuous processes*.

3. APPLICATIONS TO DIFFERENTIATION AND INTEGRATION

3.1. Differentiating functions with variables in the exponent. We are now in a position to discuss the general procedure for differentiating the function $f(x)^{g(x)}$, where f is a positive-valued function and g is a real-valued function.

To differentiate, note that:

$$f(x)^{g(x)} = \exp(g(x) \ln(f(x)))$$

We use logarithmic differentiation and simplify to get:

$$(f^g)'(x) = (f^g)(x) \frac{d}{dx} [g(x) \ln(f(x))]$$

This can further be simplified using the product rule:

$$(f^g)'(x) = (f(x))^{g(x)} \left[g'(x) \ln(f(x)) + \frac{g(x)f'(x)}{f(x)} \right]$$

Two special cases are worth noting:

- (1) The case where $g(x)$ is a constant function with value r . In this case, the derivative just becomes $rf(x)^{r-1}f'(x)$. An even further special case is where $g(x) := r$ and $f(x) := x$, in which case we obtain rx^{r-1} . This is the familiar rule for differentiating power functions that we saw, but now, we have established this rule for *all real exponents*, not just for the rational ones.
- (2) The case where $f(x)$ is a constant function with value a . In this case, the derivative is $a^{g(x)}g'(x)\ln a$. In the further special case where $g(x) = x$, we obtain $a^x \ln a$. In other words, the derivative of a^x with respect to x is $a^x \ln a$.

3.2. Key special case formula summary.

$$\begin{aligned} \frac{d}{dx}(x^r) &= rx^{r-1} \\ \frac{d}{dx}(a^x) &= a^x \ln a \\ \int x^r dx &= \begin{cases} x^{r+1}/(r+1) + C, & r \neq -1 \\ \ln|x| + C, & r = -1 \end{cases} \\ \int a^x dx &= \begin{cases} \frac{a^x}{\ln a} + C, & a \neq 1 \\ x + C, & a = 1 \end{cases} \end{aligned}$$

Note: We don't need to put the $|x|$ in the \ln antiderivative if the exponent is irrational because x^r isn't even defined for irrational exponents, so $\ln x + C$ is a valid answer for irrational exponents.

3.3. Differentiating logarithms where both pieces are functions. Consider:

$$\frac{d}{dx} \left[\log_{f(x)}(g(x)) \right]$$

To carry out this differentiate, we first rewrite the logarithm as a quotient of natural logarithms:

$$\frac{d}{dx} \left[\frac{\ln(g(x))}{\ln(f(x))} \right]$$

Note that the base of the logarithm has its \ln in the denominator.

We now use the quotient rule and get:

$$\frac{\ln(f(x))g'(x)/g(x) - \ln(g(x))f'(x)/f(x)}{(\ln(f(x)))^2}$$

4. FUN MISCELLANEA

4.1. Dimension and sense of proportion. [This is optional – will cover only if I have time.]

When you double the lengths, the areas become four times their original value, and the volumes become eight times their original value. More generally, when you scale lengths by a factor of λ , the areas get scaled by a factor of λ^2 , and the volumes get scaled by a factor of λ^3 .

The *exponent* of 2 occurs because area is two-dimensional and the exponent of 3 occurs because volume is three-dimensional.

Suppose there is a certain physical quantity such that, when we scale lengths by a factor of $\lambda \neq 1$, that quantity scales by a factor of μ . What is the dimension of that quantity? It is the value d such that $\lambda^d = \mu$. With our new understanding of logarithms, we can write:

$$d := \log_\lambda(\mu) = \frac{\ln \mu}{\ln \lambda}$$

When we put it this bleakly, it does not seem a foregone conclusion that d must be a positive integer. In fact, there is a whole range of physical objects that have positive measure in *fractal dimension*, i.e., there are quantities that we associate with them whose dimension is not an integer. For instance, there are certain

sets such as Cantor sets that we design in such a way that when we triple the lengths, the size of those objects *doubles*. Thus, the dimension of such an object is:

$$\frac{\ln 2}{\ln 3} \approx \frac{0.7}{1.1} \approx 0.63$$

Similarly, there are sets with the property that when you double lengths they increase by a factor of three times. The dimension of such sets is:

$$\frac{\ln 3}{\ln 2} \approx \frac{1.1}{0.7} \approx 1.57$$